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On Natural Criteria in Set-Valued Optimization*

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Abstract

We introduce some natural criteria of a minimization programming problem whose objective function is a set-valued map. For such criteria, we define some semicontinuities and prove certain theorems with respect to existence of solutions of the problem. Also, we investigate certain duality problem for the set-valued minimization problem.

1. Natural Criteria of Set-Valued Optimization

First, we define our set-valued minimization problem (SP). Let X be a topological space, S a nonempty subset of X, $(Y, \leq_K)$ an ordered topological vector space with an ordering convex cone K, and F a map from X to 2^Y with $F(x) \neq \emptyset$ for each $x \in S$. Our set-valued minimization problem is the following:

$$\text{(SP)} \qquad \text{Minimize} \quad F(x) \qquad \text{subject to} \quad x \in S.$$

To define notions of solutions for our problem, we introduce some relations between two nonempty sets which like the order relation in topological vector spaces; though the number types of such relations is six, we treat two important relations of them, see [9].

Definition 1.1. (*l*-**Inequality &** *u*-**Inequalities)**
For nonempty subsets A, B of Y,

$$A \leq^l B \iff \mathrm{cl}(A+K) \supset \mathrm{cl}(B+K);$$

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$$A \leq^u B \iff \mathrm{cl}(A-K) \subset \mathrm{cl}(B-K).$$

In these cases, A is said to be smaller than B with l-inequality(resp. u-inequality) if $A \leq^l B$(resp. $A \leq^u B$). Also, a subset A of Y is said to be a l-closed(resp. u-closed) if $A+K$(resp. $A-K$) is closed set of Y.

Note that $\mathrm{cl}(A+K) \supset \mathrm{cl}(B+K)$ is equivalent to $\mathrm{cl}(A+K) \supset B$ and also $\mathrm{cl}(A-K) \subset \mathrm{cl}(B-K)$ is equivalent to $A \subset \mathrm{cl}(B-K)$. When A and B are l-closed set, $A \leq^l B$ if and only if $A+K \supset B$, and $A \leq^l B$ and $B \leq^l A$ implies that $\mathrm{Min}\, A = \mathrm{Min}\, B$. If A and B are u-closed set, $A \leq^u B$ if and only if $A \subset B-K$, and $A \leq^u B$ and $B \leq^u A$ implies that $\mathrm{Max}\, A = \mathrm{Max}\, B$.

By using the set relations above, we introduce two types criteria of minimal solutions. In this paper, when we consider l-minimal solution, we assume that F is l-closed map, that is $F(x)$ is l-closed for each $x \in X$ for simple consideration. Also we assume similar assumption when we consider u-minimal solution.

Definition 1.2. (Minimal Solutions)

- $x_0 \in S$ is said to be *l-minimal solution* of (SP) if
  $F(x) \leq^l F(x_0)$ and $x \in S$ implies $F(x_0) \leq^l F(x)$;
- $x_0 \in S$ is said to be *u-minimal solution* of (SP) if
  $F(x) \leq^u F(x_0)$ and $x \in S$ implies $F(x_0) \leq^u F(x)$.

These concepts above are natural definitions for our set-valued optimization (SP) since the criteria is based on comparisons between the objective set-values of F.

Example. (Vector-Valued Game)

We consider a vector-valued two-person game; A and B are nonempty sets, $(Y, \leq_K)$ is an ordered vector space, and f is a map from $A \times B$ to Y. Assume that player 1 chooses first and player 2 chooses next. Player 1 chooses a and player 2 chooses b, $f(a,b)$ is the loss for player 1. When player 2 is cooperative toward player 1, player 1 may choose a l-minimal solution of the following set-valued optimization problem (VG):

$$\text{(VG)} \qquad \begin{array}{ll} \text{Minimize} & f(a,B) \\ \text{subject to} & a \in A. \end{array}$$

When player 2 is non-cooperative, to be exact, player 2 wills player 1's loss, then player 1 should choose a u-minimal solution of (VG).

2. Natural Semicontinuity of Set-Valued Maps

To consider existence of solutions of (SP) for our solutions, remember classical results with respect to existence of solution of some minimization problems:

(i) Let Z be a topological space, D a compact set in Z, and f a lower semicontinuous real-valued function on D. Then, f attains its minimum on D.

(ii) Let Z be a complete metric space, $f : Z \to \mathbf{R} \cup \{\infty\}$ a lower semicontinuous and proper function which is bounded from below. Then there exists $z_0 \in Z$ such that $f(z) \geq f(z_0) - \epsilon d(z, z_0)$ for all $z \in Z$. (Ekeland's variational theorem, [1])

(iii) Let Z be a Banach space, C a closed convex cone in Z, $C \subset \{z \in Z | \langle z, z^* \rangle + \epsilon ||z|| \geq 0\}$ for some $z^* \in Z^*$, which is the topological dual space of Z, $\epsilon > 0$, and D a nonempty closed subset of Z such that z^* is bounded from below on D. Then, $\operatorname{Min} D \neq \emptyset$. (Phelps' extreme theorem, [1])

We can find that some of theorems are concerned with concept of the lower semicontinuity of real-valued functions. For set-valued maps, we know the usual lower semicontinuity; a set-valued map F from X to Y is said to be lower semicontinuous at x_0 if for any $y \in F(x_0)$ and for any net $\{x_\lambda\}$ with $x_\lambda \to x_0$, there exists a net of elements $y_\lambda \in F(x_\lambda)$ converging to y. However, this notion is a generalization of the continuity of real-valued functions, then it is not a generalization of the lower semicontinuity and not suitable for our purpose to use this definition. Therefore, in this section, we define some lower semicontinuities of set-valued maps which are generalizations of the lower semicontinuities of real-valued functions and based on our natural criteria. Remember the lower semicontinuities of real-valued functions; a real-valued function f on a topological space X is said to be lower semicontinuous on a subset S of X if one of the following is satisfied:

(A) for each $x_0 \in S$ and for any $\epsilon > 0$, there exists a neighborhood U of the null vector in X such that $x \in x_0 + U$ implies that $f(x_0) - \epsilon < f(x)$;

(B) for each $x_0 \in S$, if a net $\{x_\lambda\}$ satisfies $x_\lambda \to x_0$ then $f(x_0) \leq \underline{\lim}_\lambda f(x_\lambda)$;

(C) for $\alpha \in \mathbf{R}$, $\mathcal{L}(\alpha) = \{x \in S | f(x) \leq \alpha\}$ is closed.

We introduce our lower semicontinuities as generalizations the above. To this end, we define the *upper limit* and the *lower limit* of $\{A_\lambda\}$, see [2].

Definition 2.1. ($\operatorname{Lim\,inf}_\lambda A_\lambda$ & $\operatorname{Lim\,sup}_\lambda A_\lambda$)
For $\{A_\lambda\} \subset 2^Y$, $(\Lambda, <)$: a directed set,

$$\operatorname*{Lim\,inf}_{\lambda} A_\lambda = \text{the set of limit points of } \{a_\lambda\}, a_\lambda \in A_\lambda;$$

$$\operatorname*{Lim\,sup}_{\lambda} A_\lambda = \text{the set of cluster points of } \{a_\lambda\}, a_\lambda \in A_\lambda.$$

In general, $\operatorname{Lim\,inf}_\lambda A_\lambda \subset \operatorname{Lim\,sup}_\lambda A_\lambda$ and if equality holds, it is said to be $\{A_\lambda\}$ converges to the set. From the above notation, condition $f(x_0) \leq \underline{\lim}_\lambda f(x_\lambda)$ is presented by $\{f(x_0)\} \leq^l \operatorname{Lim\,sup}_\lambda(f(x_\lambda) + \mathbf{R}_+)$ or $\{f(x_0)\} \leq^u \operatorname{Lim\,sup}_\lambda(f(x_\lambda) - \mathbf{R}_+)$. From this, to define several kinds of lower semicontinuity, we use notion Lim sup.

Definition 2.2. **(l-type Lower Semicontinuity)**
A set-valued map F is said to be

- l-type **(A)** lower semicontinuous at $x_0 \in S$ if
  for any net $\{x_\lambda\}$ with $x_\lambda \to x_0$ and for any open set U with $U \leq^l F(x_0)$, there exists $\hat{\lambda}$ such that $\hat{\lambda} < \lambda$ implies $U \leq^l F(x_\lambda)$;

- l-type **(B)** lower semicontinuous at $x_0 \in S$ if
  for any net $\{x_\lambda\}$ with $x_\lambda \to x_0$, $F(x_0) \leq^l \operatorname{Lim\,sup}_\lambda (F(x_\lambda) + K)$;

- l-type **(C)** lower semicontinuous on S if
  for any l-closed subset A of Y, $\mathcal{L}^l(A) = \{x \in S | F(x) \leq^l A\}$ is closed.

A set-valued map F is said to be l-type (A) (resp. (B)) lower semicontinuous on S if it is l-type (A) (resp. (B)) lower semicontinuous at each point of S.

These concepts are generalizations of lower semicontinuity of real-valued functions, however, the following concept is more weaker than the lower semicontinuity.

Definition 2.3. **(l-type Demi-Lower Semicontinuity)**
A set-valued map F is said to be *l-type demi-lower semicontinuous* at $x_0 \in S$ if for each net $\{x_\lambda\}$ with $x_\lambda \to x_0$ and $\lambda < \lambda'$ implies $F(x_{\lambda'}) \leq^l F(x_\lambda)$, $F(x_0) \leq^l \operatorname{Lim\,sup}_\lambda (F(x_\lambda) + K)$. A set-valued map F is said to be l-type demi-lower semicontinuous on S if it is l-type demi-lower semicontinuous at each point of S.

Now we can see some characterization with respect to these lower semicontinuities.

Proposition 2.1. We have the following:

(i) l-type **(A)** l.s.c. on $S \Rightarrow l$-type **(B)** l.s.c. on S;
(ii) l-type **(B)** l.s.c. on $S \Rightarrow l$-type **(C)** l.s.c. on S;
(iii) l-type **(C)** l.s.c. on $S \Rightarrow l$-type demi-l.s.c. on S.

Also, if Y is finite dimensional and F is locally bounded then, l-type **(A)**, **(B)**, and **(C)** lower semicontinuities are equivalent.

Now, we investigate u-type lower semicontinuity of set-valued maps.

Definition 2.4. **(u-type Lower Semicontinuity)** A set-valued map F is said to be

- u-type **(A)** lower semicontinuous at x_0 if
  for any net $\{x_\lambda\}$ with $x_\lambda \to x_0$ and for any open set U with $F(x_0) \cap U \neq \emptyset$, for any λ, there exists $\lambda' > \lambda$ such that $(F(x_\lambda) - K) \cap U \neq \emptyset$;

- u-type **(B)** lower semicontinuous at x_0 if
  for any net $\{x_\lambda\}$ with $x_\lambda \to x_0$, $F(x_0) \leq^u \operatorname{Lim\,sup}_\lambda (F(x_\lambda) - K)$;

- u-type **(C)** lower semicontinuous on S if for any subset A of Y, $\mathcal{L}^u(A) = \{x|F(x) \leq^u A\}$ is closed.

A set-valued map F is said to be u-type (A) (resp. (B)) lower semicontinuous on S if it is u-type (A) (resp. (B)) lower semicontinuous at each point of S.

Definition 2.5. (u-type Demi-Lower Semicontinuity) A set-valued map F is said to be u-type demi-lower semicontinuous at x_0 if for any net $\{x_\lambda\}$ with $x_\lambda \to x_0$ and $\lambda < \lambda'$ implies $F(x_{\lambda'}) \leq^u F(x_\lambda)$, $F(x_0) \leq^u \operatorname{Lim\,sup}_\lambda(F(x_\lambda) - K)$. A set-valued map F is said to be u-type demi-lower semicontinuous on S if it is u-type demi-lower semicontinuous at each point of S.

Proposition 2.2. (u-type Lower Semicontinuity) We have the following:

(i) u-type **(B)** l.s.c. on $S \Rightarrow u$-type **(C)** l.s.c. on S;
(ii) u-type **(C)** l.s.c. on $S \Rightarrow u$-type demi-l.s.c. on S.

Also, if Y is finite dimensional and F is locally bounded then, u-type **(A)**, **(B)**, and **(C)** lower semicontinuities are equivalent.

3. Existence Theorems for Two Types Semicontinuities of Set-Valued Maps

Theorem 3.1. (Existence of l-type Solutions 1)
Let X be a topological space and Y an ordered topological vector space. If S is a nonempty compact subset of X and $F : S \to 2^Y$ is a l-type demi-lower semicontinuous. set-valued map, then there exists a l-minimal solution of (SP).

In the rest of the paper, let Y^* be the topological dual space of Y, $K^+ = \{y^* \in Y^*|\langle y^*, k\rangle \geq 0, \forall k \in K\}$, and θ^* the null vector of Y^*.

Theorem 3.2. (Existence of l-type Solutions 2)
Let (X, d) be a complete metric space, Y an ordered locally convex space with the cone K. Also, F be a map from X to 2^Y satisfying the following conditions:

- there exists $y^* \in K^+ \setminus \{\theta^*\}$ such that
  - $\inf \langle y^*, F(\cdot)\rangle : S \to \mathbf{R}$
  - $F(x_1) \leq^l F(x_2), x_1, x_2 \in S \Rightarrow \inf \langle y^*, F(x_2)\rangle - \inf \langle y^*, F(x_1)\rangle \geq d(x_2, x_1)$
- $F : S \to 2^Y$ is l-type **(C)** lower semicontinuous.

Then, there exists a l-minimal solution of (SP).

Theorem 3.3. (Existence of u-type Solutions 1)
Let X be a topological space and Y an ordered topological vector space. If S is a nonempty compact subset of X and $F : S \to 2^Y$ is a u-type demi-lower semicontinuous. set-valued map, then there exists a u-minimal solution of (SP).

Moreover, we can show the following theorem in similar way of Theorem 3.2..

Theorem 3.4. (Existence of u-type Solutions 2)
Let (X, d) be a complete metric space, Y an ordered locally convex space with the cone K. Also, F be a map from X to 2^Y satisfying the following conditions:

- there exists $y^* \in K^+ \setminus \{\theta^*\}$ such that
  - $\sup \langle y^*, F(\cdot)\rangle : S \to \mathbf{R}$
  - $F(x_1) \leq^u F(x_2), x_1, x_2 \in S \Rightarrow \sup \langle y^*, F(x_2)\rangle - \sup \langle y^*, F(x_1)\rangle \geq d(x_2, x_1)$
- $F : S \to 2^Y$ is u-type **(C)** lower semicontinuous.

Then, there exists a u-minimal solution of (SP).

4. Duality Problem for Set-Valued Optimization

In this section, we introduce a duality problem for our l-type set-valued minimization problem (SP) with $S = \{x \in X | G(x) \leq \theta\}$, and we show some properties between these problems. First, we redefine our set-valued problem (SP) and its dual problem (DP):

$$
\begin{array}{lll}
\text{(SP)} & l\text{-Minimize} & F(x) \\
 & \text{subject to} & G(x) \leq \theta \\
 & & \\
\text{(SD)} & l\text{-Maximize} & \Phi(T) \\
 & \text{subject to} & T \in \mathcal{L}^+(Y, Z)
\end{array}
$$

where

- X : a nonempty set;
- $(Y, \leq_K)$, $(Z, \leq_L)$: ordered vector spaces with an ordering cones K, L, respectively;
- $F : X \to 2^Z$, $G : X \to 2^Y$;
- $\mathcal{L}(Y, Z) \equiv \{T : Y \to Z \mid T \text{ is linear}\}$,
  $\mathcal{L}^+(Y, Z) \equiv \{T \in \mathcal{L}(Y, Z) \mid T(K) \subset L\}$;
- $\Phi : \mathcal{L}(Y, Z) \to 2^Z$ defined by
  $\Phi(T) \equiv l\text{-}\operatorname{Min}\{F(x) + T(y) \mid (x, y) \in \operatorname{Graph}(G)\}$.

Proposition 4.1. (Weak Duality) Let x be a feasible solution of (SP), T a feasible solution of (SD), and (x_1, y_1) an element of Graph(G) satisfying $F(x_1) + T(y_1) \in \Phi(T)$. Then,

$$F(x) \leq^l F(x_1) + T(y_1) \text{ implies } \begin{cases} F(x_1) \leq^l F(x) \\ T(y_1) = \theta. \end{cases}$$

Corollary 4.1. Let x be a feasible solution of (SP) and T a feasible solution of (SD). Then $F(x) = F(x) + T(\theta) \in \Phi(T)$.

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