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The proximinality of the center of a quasicentral C*-algebra

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Abstract. Let A be a quasicentral C*-algebra and Z(A) its center. If the maximal ideal space of Z(A) is σ–compact and paracompact, then Z(A) is a proximinal subspace of A.

1. Banach 空間Xの部分空間Yは、Xの各点からYへの距離を実現するYの点があるとき、proximinal であると言う。有限次元部分空間は常に proximinal であるが、一般の部分空間が常に proximinal であるとは限らない。今XをC*-環と言う具体的な Banach 空間に限って話しを進める。Akermann - Perderson - Tomiyama [1] はC*-環の任意の閉両側イデアルは proximinal であることを示した。また Olech [8] は単位的可換C*-環の任意の同じ単位元を含むC*-部分環は proximinal であることを示した。C*-環の中心への距離は内部微分の作用素ノルムと関連して、大事な概念であるが、Somerset [11] はC*-環の中心の proxinimality に興味を持ち、単位的なある種のC*-環の中心は proximinal であることを示した。最近彼は [12] で任意の単位的C*-環の中心は proximinal であることを示した。彼の手法は Glimm イデアルを用いて中心への距離をはかり、Stampfli [13] の作用素 Pythagorean relation をうまく利用して、Michael [7] の continuous selection theorem に持ち込むと言うものであった。しかし非単位的な場合は、一般には彼の手法をそのまま使うことはできない。そこで彼の手法を利用出来そうな非単位的C*-環はないかと考えた。あるとき、Archbold [1] によって導入された擬中心的C*-環が、もしかしたらそのようなC*-環であろうと直観した。これはどんな原始イデアルも中心を含まないようなC*-環を言う (cf. [14])。我々は Somerset の手法を利用出来るように工夫し、先ず擬中心的C*-環の場合も、中心への距離が彼と同様な計算式で与えられることを示す。次に、その計算式及び Cohen [3] の factorization theorem を用い、彼の手法に沿って次の結果を示す。

定理。擬中心的C-環の中心は、その極大イデアル空間が σ–コンパクト且つパラコンパクトであるとき、proximinal である。*

これはC*-環が単位的であれば Somerset の定理そのものであることに注意する。またこれらの問題はもっと一般の Banach module で考えるべきではないかと考えている (cf. [15])。

2. 以後特に断わらない限りAを擬中心的C*-環、Z(A)をその中心、$\Phi_{Z(A)}$ をZ(A)の極大イデアル空間とする。各 $\varphi\in\Phi_{Z(A)}$ に対して、$\hat{e}_\varphi(\varphi)=1,\ 0\le\hat{e}_\varphi\le 1$ なる $e_\varphi\in Z(A)$ を一つ選んでおく。ここに ^ は Gelfand 変換を表す。$\tilde{A}=A+C\cdot 1$ をAに単位元を添加して得られるC*-環とする。このとき、容易な観察から、その中心 $Z(\tilde{A})$ は $Z(A)+C\cdot 1$ に等しいことが分かる。各 $\varphi\in\Phi_{Z(A)}$ に対して、

$\tilde{\varphi}(x+\lambda\cdot 1)=\varphi(x)+\lambda$ $(x+\lambda\cdot 1\in\tilde{A})$ ，更に $\tilde{0}(x+\lambda\cdot 1)=\lambda$ $(x+\lambda\cdot 1\in\tilde{A})$ とおいて、写像：$\varphi\rightarrow\tilde{\varphi}$ は $\Phi_{Z(A)}\cup\{0\}$ $(\subseteq A^*)$ から $\Phi_{Z(\tilde{A})}$ への同相写像を与える。それ故 $\Phi_{Z(\tilde{A})}$ は $\Phi_{Z(A)}$ の 1 点コンパクト化と考えられる。次の定理は Cohen の Factori zation Theorem と呼ばれるもので、本論の主要な手段の一つである。

3. Theorem (Cohen [3]). Let B be a Banach algebra with a left approximate identity bounded by $K\geq 1$ and let X be a left Banach B-module. Then for every $z\in X_e$ and $\varepsilon>0$ there exist elements $a\in B$ and $y\in X$ such that $z=ay, \|a\|\leq K, y\in\overline{Bz}, \|y-z\|<\varepsilon$, where X_e is the closed linear subspace of X spanned by BX, which is called the essential part of X.

4. 各 $\varphi\in\Phi_{Z(A)}$, に対して、$G_\varphi=G_\varphi(A)$ を Glimm ideal, つまり、$Ker\ \varphi$ の生成する A の両側イデアルとする。今 A を left Banach $Ker\ \varphi$-module とみて、その essential part は Theorem 2 から $G_\varphi=\overline{G_\varphi}=(Ker\ \varphi)A$ となっている。

Lemma. $e_\varphi+G_\varphi$ is the identity element of A/G_φ .

Proof. Let $x\in A$. Since A is quasicentral, it folows that there exist $z\in Z(A)$ and $a\in A$ such that $x=za$ by tTheorem 2. Therefore $e_\varphi z-z\in Ker\ \varphi\subseteq G_\varphi$ and hence $e_\varphi x-x=(e_\varphi z-z)a\in G_\varphi$. This means that $e_\varphi+G_\varphi$ is the identity element of A/G_φ . Q. E. D.

5. 次は [9, Theorem 2.7. 5] の直接の結果である。但し Prim A は A の原始イデアル全体のつくる構造空間を表す。

Lemma . The mapping：

$$\begin{array}{ccc} \text{Prim } A & \rightarrow & \Phi_{Z(A)} \\ \Psi & & \Psi \\ P & \rightarrow & \varphi_P \end{array} \qquad (Ker\varphi_P=P\cap Ker\ \varphi)$$

is continuous and surjective.

In particular, $\bigcap_{\varphi\in\Phi_{Z(A)}} G_\varphi=\{0\}$ and henc、$\|x\|=\sup_{\varphi\in\Phi_{Z(A)}}\|x+G_\varphi\|$ for all $x\in A$.

6. $\hat{A}$ を A のゼロでない既約表現の同値類のクラスとする（Jacobson 位相から導かれる位相を入れたものを、A のスペクトラムと呼ぶ。）$\pi\in\hat{A}$ に対して、$\tilde{\pi}(x+\lambda\cdot 1)=\pi(x)+\lambda I_{H_\pi}$, $\omega(x+\lambda\cdot 1)=\lambda I_C$ とおいて、$(\tilde{A})^{\wedge}=\{\tilde{\pi}:\pi\in\hat{A}\}\cup\{\omega\}$ である。

7. Lemma. $\lambda e_\varphi+G_{\tilde{\varphi}}=\lambda\cdot 1+G_{\tilde{\varphi}}$ $(\lambda\in C, \varphi\in\Phi_{Z(A)})$.

Proof. Let $\lambda\in C$ and $\varphi\in\Phi_{Z(A)}$. Then $\tilde{\varphi}(\lambda e_\varphi)=\varphi(\lambda e_\varphi)=\lambda=\tilde{\varphi}(\lambda\cdot 1)$ and hence $\lambda e_\varphi-\lambda\cdot 1\in Ker\ \tilde{\varphi}\subseteq G_{\tilde{\varphi}}$. Q. E. D.

8. Lemma. $\|x+G_\varphi\|=\|x+G_{\tilde{\varphi}}\|$ for all $x\in A$ and $\varphi\in\Phi_{Z(A)}$.

Proof. Let $x \in A$ and $\varphi \in \Phi_{Z(A)}$. Note that $G_\varphi \neq A$. In fact, if $G_\varphi = A$, then $Z(A) \subseteq G_\varphi$. By Lemma 5, there exists $P \in \text{Prim } A$ such that $P \cap Z(A) = Ker\ \varphi$ and hence $Ker\ \varphi \subseteq P$, so $A = G_\varphi \subseteq P$, a contradiction. Therefore there exists an element $\rho \in (A / G_\varphi)^\wedge$ such that $\left|\rho(x + G_\varphi)\right| = \left|x + G_\varphi\right|$ (see [4, Lemma 3.3.6). But there exists a unique element $\pi \in \hat{A}$ such that $G_\varphi \subseteq Ker\ \pi$ and $\rho(a + G_\varphi) = \pi(a)$ for all $a \in A$. We assert that $G_{\tilde{\varphi}} \subseteq Ker\ \tilde{\pi}$. In fact, for each $z \in Z(A)$, we can find a uniue complex number $f(z)$ such that $\pi(z) = f(z)I_{H_\pi}$, since π is irreducible. Then f is a homomorphism. Also $f \neq 0$. If not, then $Z(A) \subseteq Ker\ \pi$ and this contradicts the quasicentrality of A. Thus $f \in \Phi_{Z(A)}$. Moreover, $Ker\ f \subseteq Ker\ \pi \cap Z(A) \in \text{Prim } Z(A)$ and so $Ker\ f = Ker\ \pi \cap Z(A)$. But $Ker\ \varphi \subseteq G_\varphi \subseteq Ker\ \pi$ and so $Ker\ \varphi = Ker\ \pi \cap Z(A)$. Thus $Ker\ \varphi = Ker\ f$, so $\varphi = f$. Therefore if $z + \lambda \cdot 1 \in Ker\ \tilde{\varphi}$, then

$$\tilde{\pi}(z + \lambda \cdot 1) = \pi(z) + \lambda I_{H_\pi} = \varphi(z) I_{H_\pi} + \lambda I_{H_\pi} = \tilde{\varphi}(z + \lambda \cdot 1) I_{H_\pi} = 0,$$

so that $z + \lambda \cdot 1 \in Ker\ \tilde{\pi}$. Then $Ker\ \tilde{\varphi} \subseteq Ker\ \tilde{\pi}$ and hence $G_{\tilde{\varphi}} \subseteq Ker\ \tilde{\pi}$. It follows that

$$\left|x + G_\varphi\right| = \left|\pi(x)\right| = \left|\tilde{\pi}(x)\right| = \left|x + Ker\ \tilde{\pi}\right| \leq \left|x + G_{\tilde{\varphi}}\right|.$$

On the other hand, since $G_\varphi = (Ker\ \varphi)A \subseteq (Ker\ \tilde{\varphi})\tilde{A} = G_{\tilde{\varphi}}$, it follows that $\left|x + G_\varphi\right| \geq \left|x + G_{\tilde{\varphi}}\right|$. Q. E. D.

9. Lemma. Let $x \in A$ and $\lambda \in C$. Then the mapping : $\varphi \to \left|(x + \lambda e_\varphi) + G_\varphi\right|$ is upper semi-continuous on $\Phi_{Z(A)}$.

Proof. Let $x \in A$, $\lambda \in C$ and $\alpha > 0$, and set

$$G(\alpha, x, \lambda) = \{\varphi \in \Phi_{Z(A)} : \left|(x + \lambda e_\varphi) + G_\varphi\right| < \alpha\}.$$

$$\tilde{G}(\alpha, x, \lambda) = \{\psi \in \Phi_{Z(\tilde{A})} : \left|(x + \lambda \cdot 1) + G_\psi\right| < \alpha\}.$$

Then $\tilde{G}(\alpha, x, \lambda)$ is an open subset of $\Phi_{Z(\tilde{A})}$ by [11, Proposition 1.1]. Also if $\varphi \in \Phi_{Z(A)}$, then

$$\begin{aligned} \left|(x + \lambda \cdot 1) + G_{\tilde{\varphi}}\right| &= \left|(x + G_{\tilde{\varphi}}) + (\lambda \cdot 1 + G_{\tilde{\varphi}})\right| \\ &= \left|(x + G_{\tilde{\varphi}}) + (\lambda e_\varphi + G_{\tilde{\varphi}})\right| \text{ (by Lemma 7)} \\ &= \left|(x + \lambda e_\varphi) + G_{\tilde{\varphi}}\right| \\ &= \left|(x + \lambda e_\varphi) + G_\varphi\right| \text{ (by Lemma 8)}. \end{aligned}$$

This implies that $(G(\alpha, x, \lambda))^\sim = \tilde{G}(\alpha, x, \lambda) \setminus \{\tilde{0}\}$ and hence $G(\alpha, x, \lambda)$ is an open subset of $\Phi_{Z(A)}$. Thus the mapping : $\varphi \to \left|(x + \lambda e_\varphi) + G_\varphi\right|$ is upper semi-continuous on $\Phi_{Z(A)}$.

Q. E. D.

10. Lemma. If A is quasicentral, then $G_\varphi \cap Z(A) = Ker\ \varphi$ for each $\varphi \in \Phi_{Z(A)}$.

Proof. Assume that A is quasicentral and let $\varphi \in \Phi_{Z(A)}$. Choose $P \in \text{Prim } A$ such that $G_\varphi \subseteq P$. Then $G_\varphi \cap Z(A) \subseteq P \cap Z(A) \neq Z(A)$ by the quasicentrality of A. But since $Ker\ \varphi \subseteq G_\varphi \cap Z(A)$, it follows that $Ker\ \varphi = G_\varphi \cap Z(A)$.

11. Lemma. Let $x \in A$ and $\alpha > 0$. Then $\{\varphi \in \Phi_{Z(A)} : \left| x + G_\varphi \right| \geq \alpha\}$ is compact.

Proof. Let $x \in A$ and $\alpha > 0$. Set $K = \{\varphi \in \Phi_{Z(A)} : \left| x + G_\varphi \right| \geq \alpha\}$. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a decreasing net of relatively closed non-empty subsets of K. For each $\lambda \in \Lambda$, set

$$J_\lambda = \bigcap_{\varphi \in F_\lambda} G_\varphi .$$

Also for each $\lambda \in \Lambda$, take an element φ_λ of F_λ and then

$$\left| x + J_\lambda \right| \geq \left| x + G_{\varphi_\lambda} \right| \geq \alpha . \qquad (*)$$

Set $J = \overline{\bigcup_{\lambda \in \Lambda} J_\lambda}$. Since $\{J_\lambda : \lambda \in \Lambda\}$ is a increasing net of closed two-sided ideals of A, it follows that J is a closed two-sided ideal of A. Also (*) implies that $\left| x + J \right| \geq \alpha$. Since $\alpha > 0$, we have $x \notin J$ and hence A / J is non-zero C*-algbera. By [4, Lemma 3.3.6], we can find $\rho \in (A / J)^\wedge$ such that $\left| \rho(x + J) \right| = \left| x + J \right|$. But there exists a unique element $\pi \in \hat{A}$ such that $J \subseteq Ker\ \pi$ and $\rho(a + J) = \pi(a)$ for all $a \in A$. Hence

$$\left| x + Ker\ \pi \right| = \left| \pi(x) \right| = \left| \rho(x + J) \right| = \left| x + J \right| \geq \alpha .$$

Choose $\varphi \in \Phi_{Z(A)}$ such that $Ker\ \pi \cap Z(A) = Ker\ \varphi$. Since $G_\varphi \subseteq Ker\ \pi$, it follows that $\left| x + G_\varphi \right| \geq \alpha$ and so $\varphi \in K$. But since

$$\begin{aligned} Ker\ \varphi &= Ker\ \pi \cap Z(A) \\ &\supseteq J \cap Z(A) \\ &\supseteq J_\lambda \cap Z(A) \\ &= \bigcap_{\psi \in F_\lambda} G_\psi \cap Z(A) \\ &= \bigcap_{\psi \in F_\lambda} Ker\ \psi \ \text{(by Lemma 10)} \end{aligned}$$

for all $\lambda \in \Lambda$, it follows that $\varphi \in \overline{F_\lambda}$ for all $\lambda \in \Lambda$. Hence $\varphi \in \bigcap_{\lambda \in \Lambda} F_\lambda$ because each F_λ is relatively closed in K. We thus obtain that K is compact. Q. E. D.

12. Let X be a normed space and Y a subspace of X. For each $x \in X$, set

$$\pi_Y(x) = \{y \in Y : d(x, Y) \equiv \inf_{u \in Y} \left| x - u \right| = \left| x - y \right|\}.$$

We say that Y is proximinal if $\pi_Y(x) \neq \phi$ for all $x \in A$. We also say that Y is Chebychev if $\pi_Y(x)$ consists of a single point for each $x \in A$.

13. Let T be an operator of the Banach space B(H) consisting of all bounded linear operators on a Hilbet space H and set

$$W_0(T) = \{\lambda \in C : (T\xi_n, \xi_n) \to \lambda \text{ where } \left| \xi_n \right| = 1 \text{ and } \left| T\xi_n \right| \to \left| T \right|\}.$$

We call $W_0(T)$ the maximal numerical range of T.

Theorem (Stampfli [13]). The following three conditions are equivalent:

(1) $0 \in W_0(T)$.

(2) $\|T\|^2 + |\lambda|^2 \leq \|T + \lambda I_H\|^2$ for all $\lambda \in C$.

(3) $\|T\| \leq \|T + \lambda I_H\|$ for all $\lambda \in C$.

In particular, CI_H is a Chebyshev subspace of B(H) and

$$\|T - \pi_{CI_H}(T)\|^2 + \|\lambda I_H - \pi_{CI_H}(T)\|^2 \leq \|T - \lambda I_H\|^2$$

for all $\lambda \in C$.

14. Theorem. Let A be a quasicentral C*-algebra. If $x \in A$, then

$$d(x, Z(A)) = \sup_{\varphi \in \Phi_{Z(A)}} \|(x + G_\varphi) - \pi_{C(e_\varphi + G_\varphi)}(x + G_\varphi)\|.$$

Proof. Let $x \in A$ and α the value of the right side above. Then

$$\begin{aligned}\|x - z\| &\geq \|x + G_\varphi - (z + G_\varphi\| \\ &= \|x + G_\varphi - \hat{z}(\varphi)(e_\varphi + G_\varphi)\| \\ &\geq \|x + G_\varphi - \pi_{C\cdot(e_\varphi + G_\varphi)}(x + G_\varphi)\|\end{aligned}$$

for all $z \in Z(A)$ *and* $\varphi \in \Phi_{Z(A)}$. Hence $d(x, Z(A)) \geq \alpha$. To show the converse inequality, let $\varepsilon > 0$ and set $K = \{\varphi \in \Phi_{Z(A)} : \|x + G_\varphi\| \geq \alpha + \varepsilon\}$. We consider two cases:

(i) $K = \phi$. Since $\|x + G_\varphi\| < \alpha + \varepsilon$ for all $\varphi \in \Phi_{Z(A)}$, it follows that

$$d(x, Z(A)) \leq \|x\| = \sup_{\varphi \in \Phi_{Z(A)}} \|x + G_\varphi\| \leq \alpha + \varepsilon$$

and hence $d(x, Z(A)) \leq \alpha$ as $\varepsilon \downarrow 0$.

(ii) $K \neq \phi$. By Lemma 11, K is a non-empty compact subset of $\Phi_{Z(A)}$. Let φ be any element of K. Then there exists a unique scalar λ_φ such that $\pi_{C(e_\varphi + G_\varphi)}(x + G_\varphi) = \lambda_\varphi(e_\varphi + G_\varphi)$ since $C\,(e_\varphi + G_\varphi)$ is a Chebychev subspace of A / G_φ by Theorem 13. Set $z_\varphi = \lambda_\varphi e_\varphi$ and so $\|(x - z_\varphi) + G_\varphi\| \leq \alpha$. Also put

$$W_\varphi = \{\psi \in \Phi_{Z(A)} : \|(x - z_\varphi) + G_\psi\| < \alpha + \varepsilon\}.$$

Then $\varphi \in W_\varphi$ and W_φ is a open subset of $\Phi_{Z(A)}$ by Lemma 9. Thus W_φ is an open neighbourhood of φ. Take a relative compact open neighbourhood U_φ of φ such that $U_\varphi \subseteq W_\varphi$. Since K is compact, there exist elements $\varphi_1, \dots, \varphi_n \in K$ such that $\bigcup_{i=1}^{n} U_{\varphi_i} \supseteq K$. Let $\{f_1, \dots, f_n, f_\infty\}$ be a partition of the identity for the convering $\{U_{\varphi_1}, \dots, U_{\varphi_n}, \Phi_{Z(A)} \setminus K\}$. Since each U_{φ_i} is relative compact, it follows that f_{φ_i} vanishes at infinity and hence there is an element $u_i \in Z(A)$ such that $f_i = \hat{u}_i$. Set

$$z = u_1 z_{\varphi_1} + \dots + u_n z_{\varphi_n}.$$

For any element ψ of $\Phi_{Z(A)}$ we have

$$\left|x + G_\psi - (z + G_\psi)\right| = \left|\sum_{i=1}^{n} f_i(\psi)(x - z_{\varphi_i}) + G_\psi + f_\infty(\psi)x + G_\psi\right|$$
$$\leq \sum_{i=1}^{n} f_i(\psi)\left|(x - z_{\varphi_i}) + G_\psi\right| + f_\infty(\psi)\left|x + G_\psi\right|.$$

If $\psi \in K$, then

$$\left|x + G_\psi - (z + G_\psi)\right| = \sum_{\psi \in U_{\varphi_i}} f_i(\psi)\left|(x - z_{\varphi_i}) + G_\psi\right| \leq \alpha + \varepsilon.$$

If also $\psi \notin K$, then

$$\left|x + G_\psi - (z + G_\psi)\right| \leq \sum_{\psi \in U_{\varphi_i}} f_i(\psi)\left|(x - z_{\varphi_i}) + G_\psi\right| + f_\infty(\psi)\left|x + G_\psi\right|$$
$$\leq \sum_{\psi \in U_{\varphi_i}} f_i(\psi)\,(\alpha + \varepsilon) + f_\infty(\psi)\,(\alpha + \varepsilon)$$
$$\leq \alpha + \varepsilon.$$

This implies that $\left|x - z\right| \leq \alpha + \varepsilon$ and hence $d(x, Z(A)) \leq \left|x - z\right| \leq \alpha$ as $\varepsilon \downarrow 0$. Q. E. D.

15. Theorem (Michael [7]). Let Ω be a paracompact T_1-space and X a Banach space. Then every lower semi-continuous carrier for Ω to the family of non-empty, closed convex subsets of X admits a continuous selection.

16. Theorem. Let A be a quasicentral C*-algebra and Z(A) its center. Suppose that Z(A) satisfies the following two conditions: (i) $\Phi_{Z(A)}$ is paracompact. (ii) there exists an element $v \in Z(A)$ such that $\hat{v}(\varphi) > 0$ for all $\varphi \in \Phi_{Z(A)}$. Then Z(A) is a proximinal subspace of A.

Proof. Suppose $\Phi_{Z(A)}$ is paracompact and there exists an element $v \in Z(A)$ such that $\hat{v}(\varphi) > 0$ for all $\varphi \in \Phi_{Z(A)}$. Let $x \in A$ and set $\alpha = d(x, Z(A))$. We can without loss of generality assume that $\alpha = 1$. By Theorem 3, we can find elements $u \in Z(A)$ and $a \in A$ such that $x = ua$ and $\left|u\right| \leq 1$. For each $\varphi \in \Phi_{Z(A)}$, there exists a unique scalar λ_φ such that $\pi_{C(e_\varphi + G_\varphi)}(x + G_\varphi) = \lambda_\varphi(e_\varphi + G_\varphi)$ since $C\,(e_\varphi + G_\varphi)$ is a Chebychev subspace of A / G_φ by Theorem 13. Then $\left|(x - \lambda_\varphi e_\varphi) + G_\varphi\right| \leq 1$ by Theorem 14. Also

$$\left|\lambda_\varphi\right| \leq \left|x + G_\varphi\right| = \left|(u + G_\varphi)(a + G_\varphi)\right| = \left|\hat{u}(\varphi)(e_\varphi a + G_\varphi)\right| \leq \left|\hat{u}(\varphi)\right|\left|a\right|. \tag{1}$$

The first inequality follows from Theorem 13. Set

$$C_\varphi = \{\lambda \in C : \left|(x - \lambda e_\varphi) + G_\varphi\right| \leq 1 \text{ and } \left|\lambda\right| \leq \left|a\right|\left|\hat{u}(\varphi)\right| + \hat{v}(\varphi)\}.$$

Then each C_φ is a non-empty, closed, convex subset of C. We prove that the set-valued map : $\varphi \rightarrow C_\varphi$ is lower semi-continuous on $\Phi_{Z(A)}$. Let U be any open subset of C and set $\Phi = \{\varphi \in \Phi_{Z(A)} : C_\varphi \cap U \neq \phi\}$. To show that Φ is an open subset of $\Phi_{Z(A)}$, let φ_0 be any element of Φ. Choose $\lambda_0 \in C_{\varphi_0} \cap U$ and take an open ball $U(\lambda_0; \varepsilon)$ of radius

$\varepsilon\,(0<\varepsilon<1)$ centered on λ_0 such that $U(\lambda_0;\,\varepsilon)\subseteq U$. Set

$$\Phi_0=\{\varphi\in\Phi_{Z(A)}:\left|(x-\lambda_0 e_\varphi)+G_\varphi\right|<1+\tfrac{\varepsilon^3}{8}\ \text{and}\ \left|\lambda_0\right|<\left|a\right|\left|\hat{u}(\varphi)\right|+(1+\tfrac{\varepsilon}{2})\hat{v}(\varphi)\}\,.$$

Then $\varphi_0\in\Phi_0$ since $\varepsilon>0$ and $\hat{v}(\varphi_0)>0$, and Φ_0 is open by Lemma 9. Let $\psi\in\Phi_0$ and put $\beta=1-\left|(x-\lambda_\psi e_\psi)+G_\psi\right|$ and so $0\le\beta\le 1$ by Theorem 14. Also, Theorem 13 implies that

$$\left|(x+G_\psi)-\lambda_\psi(e_\psi+G_\psi)\right|^2+\left|\lambda_0-\lambda_\psi\right|^2\le\left|(x+G_\psi-\lambda_0(e_\psi+G_\psi)\right|^2<\left(1+\tfrac{\varepsilon^3}{8}\right)^2,$$

and hence

$$\left|\lambda_0-\lambda_\psi\right|^2<\left(1+\tfrac{\varepsilon^3}{8}\right)^2-(1-\beta)^2=\tfrac{\varepsilon^3}{4}+\tfrac{\varepsilon^6}{64}+2\beta-\beta^2\ . \qquad (2)$$

We consider two cases:

(i) $\beta<\frac{\varepsilon^2}{4}$. It follows form (2) that $\left|\lambda_0-\lambda_\psi\right|^2<\frac{\varepsilon^3}{4}+\frac{\varepsilon^6}{64}+\frac{\varepsilon^2}{2}<\frac{49}{64}\varepsilon^2$ and hence $\lambda_\psi\in U(\lambda_0;\,\varepsilon)$. Also we have $\left|(x-\lambda_\psi e_\psi)+G_\psi\right|\le 1$ by Theorem 14 and $\left|\lambda_\psi\right|\le\left|\hat{u}(\psi)\right|\left|a\right|$ by (1), so that $\lambda_\psi\in C_\psi$. Then $\psi\in\Phi$.

(ii) $\beta\ge\frac{\varepsilon^2}{4}$. Set $\mu=(1-\frac{\varepsilon}{2})\lambda_0+\frac{\varepsilon}{2}\lambda_\psi$. It follows from (2) that

$$\left|\lambda_0-\mu\right|^2=\tfrac{\varepsilon^2}{4}\left|\lambda_0-\lambda_\psi\right|^2\le\tfrac{\varepsilon^2}{4}\left(\tfrac{\varepsilon^3}{4}+\tfrac{\varepsilon^6}{64}+2\beta-\beta^2\right)<\tfrac{\varepsilon^2}{4}\left(\tfrac{1}{4}+\tfrac{1}{64}+1\right)=\tfrac{81}{256}\varepsilon^2,$$

and hence $\mu\in U(\lambda_0;\,\varepsilon)$. Also we have

$$\begin{aligned}\left|(x-\mu e_\psi)+G_\psi\right|&\le(1-\tfrac{\varepsilon}{2})\left|(x-\lambda_0 e_\psi)+G_\psi\right|+\tfrac{\varepsilon}{2}\left|(x-\lambda_\psi e_\psi)+G_\psi\right|\\&\le(1-\tfrac{\varepsilon}{2})\,(1+\tfrac{\varepsilon^3}{8})+\tfrac{\varepsilon}{2}(1-\beta)\ \ (\text{since}\ \psi\in\Phi_0)\\&<(1-\tfrac{\varepsilon}{2})\,(1+\tfrac{\varepsilon^3}{8})+\tfrac{\varepsilon}{2}(1-\tfrac{\varepsilon^2}{4})\\&=1-\tfrac{\varepsilon^4}{16}<1\,.\end{aligned}$$

Moreover,

$$\begin{aligned}\left|\mu\right|&\le(1-\tfrac{\varepsilon}{2})\left|\lambda_0\right|+\tfrac{\varepsilon}{2}\left|\lambda_\psi\right|\\&<(1-\tfrac{\varepsilon}{2})\left(\left|a\right|\left|\hat{u}(\psi)\right|+(1+\tfrac{\varepsilon}{2})\hat{v}(\psi)\right)+\tfrac{\varepsilon}{2}\left|a\right|\left|\hat{u}(\psi)\right|\ \ (\text{since}\ \psi\in\Phi_0\ \text{and by (1)})\\&=\left|a\right|\left|\hat{u}(\psi)\right|+(1-\tfrac{\varepsilon^2}{4})\,\hat{v}(\psi)\\&<\left|a\right|\left|\hat{u}(\psi)\right|+\hat{v}(\psi)\,.\end{aligned}$$

Then $\mu\in C_\psi$ and hence $\psi\in\Phi$.

This shows that Φ is an open subset of $\Phi_{Z(A)}$ and hence the set-valued map : $\varphi\rightarrow C_\varphi$ is lower semi-continuous on $\Phi_{Z(A)}$. Since $\Phi_{Z(A)}$ is paracompact, it follows from Theorem 15 that we can find a continuous complex-valued function f on $\Phi_{Z(A)}$ such that $f(\varphi)\in C_\varphi$ for all $\varphi\in\Phi_{Z(A)}$. Since $\left|f(\varphi)\right|\le\left|a\right|\left|\hat{u}(\varphi)\right|+\hat{v}(\varphi)$ for all $\varphi\in\Phi_{Z(A)}$, the function f vanishes at infinity and so $f = \hat{z}$ for some $z \in Z(A)$. Moreover,

$$\begin{aligned} |x - z| &= \sup_{\varphi \in \Phi_{Z(A)}} |(x + G_\varphi) - (z + G_\varphi)| \\ &= \sup_{\varphi \in \Phi_{Z(A)}} |(x + G_\varphi) - (\hat{z}(\varphi)e_\varphi + G_\varphi)| \\ &= \sup_{\varphi \in \Phi_{Z(A)}} |(x - f(\varphi)e_\varphi) + G_\varphi| \\ &\leq 1 \ \text{(since } f(\varphi) \in C_\varphi \text{ for all } \varphi \in \Phi_{Z(A)}). \end{aligned}$$

Therefore we have $|x - z| = d(x, Z(A))$. Q. E. D.

17. Remarks. (i) If $\Phi_{Z(A)}$ is connected, then $\Phi_{Z(A)}$: paracompact $\Leftrightarrow$ $\Phi_{Z(A)}$: σ–compact.

(ii) If $\Phi_{Z(A)}$ is σ–compact, then there exists an element $v \in Z(A)$ such that $\hat{v}(\varphi) > 0$ for all $\varphi \in \Phi_{Z(A)}$.

(iii) If Z(A) is separable, then $\Phi_{Z(A)}$ is paracompact and there exists an element $v \in Z(A)$ such that $\hat{v}(\varphi) > 0$ for all $\varphi \in \Phi_{Z(A)}$.

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