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ON $-P \cdot P$ OF SURFACE SINGULARITIES

TOMOHIRO OKUMA

1. INTRODUCTION

Let $(X, x)$ be a normal surface singularity over the complex number field $\mathbb{C}$ and $f : (M, A) \to (X, x)$ a resolution of the singularity $(X, x)$. Let $K$ be the canonical divisor on $M$. Let $A = \bigcup_{i=1}^{k} A_{i}$ be the decomposition of the exceptional set $A$ into irreducible components. Assume that $f$ is the minimal good resolution, i.e., $f$ is the smallest resolution for which $A$ consists of non-singular curves intersecting among themselves transversally, with no three through one point. It is well known that there exists a unique minimal good resolution.

**Definition 1.1.** By [12, Theorem A.1], $K + A$ admits a unique Zariski-decomposition $P + N$, $P, N \in \sum_{i=1}^{k} \mathbb{Q}A_{i}$, where

1. $(K + A) \cdot A_{i} = (P + N) \cdot A_{i}$ for all $i$.
2. $P$ is $f$-nef, i.e., $P \cdot A_{i} \geq 0$ for all $i$.
3. $N$ is effective.
4. $P \cdot N = 0$.

Then we define the invariant $P^{2}$ by $P^{2} := P \cdot P$.

The $P \cdot P$ is a topological invariant and its fundamental properties are stated in [15]. It is expected that $P^{2}$ has many of nice properties of the invariant $K \cdot K$ studied by Laufer [8]. The upper semicontinuity of $-P^{2}$ in a family of surface singularities follows from that of the $L^{2}$-plurigenera $\delta_{m}$ (cf. [2]), since the following equality holds (see [15, Introduction]):

$$-P \cdot P/2 = \limsup_{m \to \infty} \delta_{m}/m^{2}.$$  

In this note, we prove the following.

**Theorem.** Let $\pi : X \to T$ be a deformation of a normal Gorenstein surface singularity such that $T$ is a neighborhood of the origin of $\mathbb{C}$. Let $P^{2}_{t}$ be the invariant of the fiber $X_{t}, t \in T$. Then the following conditions are equivalent:

1. $\pi$ admits the simultaneous log-canonical model.
2. $P^{2}_{t}$ is constant.
2. Preliminaries

Let $X$ be a normal variety over $\mathbb{C}$ of dimension $d \geq 2$, and $X_{\text{sing}}$ the singular locus of $X$. Let $f: Y \to X$ be a birational morphism of normal varieties and $E = f^{-1}(X_{\text{sing}})_{\text{red}}$ the largest reduced exceptional divisor on $Y$. For a $\mathbb{Q}$-Cartier divisor $D$ on $X$, we denote by $f^! D$ the sum of the divisors $E$ and the strict transform of $D$ under the morphism $f$. The morphism $f: Y \to X$ is called a good resolution of the pair $(X, D)$, if $Y$ is nonsingular and the support of $f^! D$ is a divisor with only simple normal crossings.

Definition 2.1 (cf. [7], [13]). Let $B$ be a reduced divisor on $X$. The divisor $K_X + B$ is said to be log-canonical if the following conditions are satisfied:

1. $K_X + B$ is a $\mathbb{Q}$-Cartier divisor.
2. There exists a good resolution $f: Y \to X$ of $(X, B)$ such that

$$K_Y + f^! B = f^*(K_X + B) + \sum a_i E_i$$

for $a_i \in \mathbb{Q}$ with the condition that $a_i \geq 0$, where the $E_i$ are the exceptional prime divisors.

Definition 2.2 (cf. [7], [13]). Let $f: Y \to X$ be a partial resolution with the exceptional divisor $E = f^{-1}(X_{\text{sing}})_{\text{red}}$. Then the morphism $f: Y \to X$ is called a log-canonical model of $X$, if the divisor $K_Y + E$ is log-canonical and $K_Y + E$ is $f$-ample.

Theorem 2.3 (cf. [6], [13]). Let $X$ be a normal variety of dimension $d \leq 3$. Then there exists the log-canonical model $f: Y \to X$ of $X$. In fact, the following morphism gives the log-canonical model:

$$\text{Proj} \left( \bigoplus_{n \geq 0} f_* \mathcal{O}_Y(n(K_Y + E)) \right) \to X,$$

where $f: Y \to X$ is a partial resolution with $E = f^{-1}(X_{\text{sing}})_{\text{red}}$ such that the divisor $K_Y + E$ is log-canonical.

3. The plurigenera

In this section, we describe basic facts concerning plurigenera of normal isolated singularities needed later.

Definition 3.1 (cf. [9], [16]). Let $(X, x)$ be a normal isolated singularity and $f: (M, A) \to (X, x)$ a good resolution of the singularity $(X, x)$. We define the log-plurigenera...
\{\lambda_m(X, x)\}_{m \in \mathbb{N}} \text{ and the } L^2\text{-plurigenera } \{\delta_m(X, x)\}_{m \in \mathbb{N}} \text{ by }

\lambda_m(X, x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_*\mathcal{O}_M(m(K_M + A)) \quad \text{and}
\delta_m(X, x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_*\mathcal{O}_M(m(K_M + A) - A), \text{ respectively.}

The definition does not depend on the choice of the good resolution.

**Lemma 3.2.** Let \(X\) be a normal variety and \(B\) a reduced divisor on \(X\) such that \(K_X + B\) is log-canonical. Let \(f: Y \to X\) be a good resolution of the pair \((X, B)\) with \(B_Y := f^!B\). Then we have \(f_*\mathcal{O}_Y(m(K_Y + B_Y)) = \mathcal{O}_X(m(K_X + B)).\)

**Proof.** It is clear that \(f_*\mathcal{O}_Y(m(K_Y + B_Y)) \subset \mathcal{O}_X(m(K_X + B)).\) We assume that \(X\) is affine, and we show that \(f_*\mathcal{O}_Y(m(K_Y + B_Y)) \supset \mathcal{O}_X(m(K_X + B)).\)

Let \(r\) be the index of the divisor \(K_X + B\) and \(m\) a positive integer which divides by \(r\). By assumption, we have that \(m(K_Y + B_Y) \geq f^!(m(K_X + B))\). Hence we obtain that

\[H^0(\mathcal{O}_Y(m(K_Y + B_Y))) \supset H^0(f^*\mathcal{O}_X(m(K_X + B))) = H^0(\mathcal{O}_X(m(K_X + B))).\]

For any positive integer \(m\) and any element \(\omega\) in \(H^0(\mathcal{O}_X(m(K_X + B)))\), we obtain that \(v_{E_i}(\omega^r) \geq -mr\) for all exceptional prime divisor \(E_i\) on \(Y\), where \(v_{E_i}\) is the valuation associated to the prime divisor \(E_i\). Hence \(\omega\) belongs to \(H^0(\mathcal{O}_Y(m(K_Y + B_Y)))\).

\[\square\]

**Corollary 3.3.** Let \((X, x)\) be a normal isolated singularity and \(f: Y \to X\) a partial resolution with \(E = f^{-1}(x)_{\text{red}}\) such that \(K_Y + E\) is log-canonical. Then we have

\[\lambda_m(X, x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_*\mathcal{O}_Y(m(K_Y + E)).\]

Let \(\pi: X \to T\) be a deformation of a normal Gorenstein surface singularity \((X_0, x) = \pi^{-1}(0)\), where \(T\) is a neighborhood of the origin of \(\mathbb{C}\). Put \(X_t := \pi^{-1}(t)\). Then we define the \(m\)-th log-plurigenera and \(m\)-th \(L^2\)-plurigenera of \(X_t\) by

\[\lambda_m(X_t) := \sum_{p \in (X_t)_{\text{sing}}} \lambda_m(X_t, p) \quad \text{and} \quad \delta_m(X_t) := \sum_{p \in (X_t)_{\text{sing}}} \delta_m(X_t, p).\]

Let \(\psi_t: M_t \to X_t\) be the minimal good resolution of the singularities and \(K_t\) the canonical divisor on \(M_t\). Let \(A_{t,p}\) be the connected component of the exceptional set \(A_t\) on \(M_t\) which blows down to \(p \in (X_t)_{\text{sing}}\). Let \(P_{t,p} + N_{t,p}\) be the Zariski decomposition of \(K_t + A_{t,p}\). Here, \(P_{t,p}\) and \(N_{t,p}\) are \(\mathbb{Q}\)-divisor supported in \(A_{t,p}\). We define the \(\mathbb{Q}\)-divisor \(P_t\) on \(M_t\) by \(P_t := \sum_{p \in (X_t)_{\text{sing}}} P_{t,p}\). We put \(P_t^2 := -P_t \cdot P_t\) and define the function \(\mathcal{P}: T \to \mathbb{Q}\) by \(\mathcal{P}(t) = -P_t^2\). From [15, Theorem 1.6], [11, Remark 2.7] and Introduction, we obtain the following.
Theorem 3.4. For any $m \in \mathbb{N}$,

(3.1) \[ \lambda_m(X_t) = \mathcal{P}(t)m^2/2 + P_t \cdot K_t m/2 + b_t(m) \quad \text{and} \]
(3.2) \[ \delta_m(X_t) = \mathcal{P}(t)(m - 1)^2/2 - P_t \cdot K_t (m - 1)/2 + b'_t(m), \]
where $b_t$ and $b'_t$ are bounded functions. Furthermore, the function $\mathcal{P}$ is upper semicontinuous.

4. SOME INVARIANTS WHICH DEPEND ON A DEFORMATION

In this section, we fix the following notation. Let $\pi: X \to T$ be a deformation of a normal Gorenstein surface singularity $(X_0, x) = \pi^{-1}(0)$, where $T$ is a neighborhood of the origin of $\mathbb{C}$. Then $X$ is a three-dimensional Gorenstein variety. Therefore, for any $t \in T$, we have the isomorphism $O_{X_t}(mK_X) \cong O_X_t(mK_{X_t})$. We denote by $Y_t$ the fiber $f^{-1}(t)$ and put $f_t := f|_{Y_t}$. Let $f: Y \to X$ be the log-canonical model of $X$ with $E = f^{-1}(X_{\text{sing}})^{\text{red}}$. We define the sheaves by $\mathcal{I}_m := f_*O_Y(m(K_e + E))$ and $\mathcal{Q}_m := O_X(mK_X)/\mathcal{I}_m$ for any $m \in \mathbb{N}$. We put $T^* := T \setminus \{0\}$. We assume that $T$ is sufficiently small.

Let $\mathcal{C}(t)$ be the residue field of $t \in T$, i.e., $\mathcal{C}(t) = O_{T,t}/\mathcal{M}_t$ where $\mathcal{M}_t$ is the maximal ideal. We use the symbol $\otimes \mathbb{C}(t)$ instead of $\otimes_{\mathcal{O}_T} \mathbb{C}(t)$. By Nakayama's Lemma, we obtain that

(4.1) \[ \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t) \leq \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0), \]
where the equality holds if and only if $\mathcal{Q}_m$ is a torsion free $O_T$-module. Let $\mathcal{I}_{m,0}$ be the image of the homomorphism $\mathcal{I}_m \otimes \mathbb{C}(0) \to O_{X_0}(mK_{X_0})$.

The following Lemmas are proved by an argument similar to that in [4, §1].

Lemma 4.1. The following conditions are equivalent.

1. The equality in (4.1) holds.
2. $\mathcal{Q}_m$ is a torsion free $O_T$-module.
3. $\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0}$.

Lemma 4.2. For any $t \in T^*$, the restriction $f_t: Y_t \to X_t$ is the log-canonical model of $X_t$. Moreover, for each $m \in \mathbb{N}$, there exists a closed analytic subset $S_m$ of $T$ containing the origin such that $\lambda_m(X_t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t)$, for all $t \in T \setminus S_m$.

Let $\psi: (M, A) \to (X_0, x)$ be a good resolution. For every $m \in \mathbb{N}$, we put $A_m := \psi_*O_M(m(K_M + A))$ and define the invariant $\epsilon_m$ and $\theta_m$ by

\[ \epsilon_m := \dim_{\mathbb{C}} A_m/(\mathcal{I}_{m,0} \cap A_m) \]
\[ \theta_m := \dim_{\mathbb{C}} \mathcal{I}_{m,0}/ (A_m \cap \mathcal{I}_{m,0}). \]
Then we have the diagram

\[
\begin{array}{c}
\mathcal{A}_m \cap \mathcal{I}_{m,0} \longrightarrow \mathcal{I}_{m,0} \\
\downarrow \quad \downarrow \\
\mathcal{A}_m \longrightarrow \mathcal{O}_{X_0}(mK_{X_0}).
\end{array}
\]

From (4.1) and Lemma 4.2, we have the following inequality for every \( m \in \mathbb{N} \):

\[
(4.2) \quad \lambda_m(X_i) \leq \lambda_m(X_0) + \epsilon_m - \theta_m.
\]

**Lemma 4.3.** There exist \( a, b \in \mathbb{Q} \) such that \( \epsilon_m \leq am + b \).

**Proof.** First, we show that \( \psi_* \mathcal{O}_M(mK_M + (m-1)A) \subset \mathcal{I}_{m,0} \cap \mathcal{A}_m \). Let \( \omega \) be a section of \( \psi_* \mathcal{O}_M(mK_M + (m-1)A) \). By [2, Theorem 2.1], there exists a section \( \omega' \) of \( f_* \mathcal{O}_Y(mK_Y + (m-1)E) \) of which the image in \( \mathcal{O}_{X_0}(mK_{X_0}) \) is \( \omega \). Since \( f_* \mathcal{O}_Y(mK_Y + (m-1)E) \subset \mathcal{I}_m \), we see that \( \omega \) belongs to \( \mathcal{I}_{m,0} \). Hence we obtain the inclusion. Then the inclusion implies that

\[
\epsilon_m \leq \dim_{\mathbb{C}} \mathcal{A}_m/\psi_* \mathcal{O}_M(mK_M + (m-1)A)) = \delta_m(X_0, x) - \lambda_m(X_0, x).
\]

From Theorem 3.4, we obtain the assertion. \( \square \)

In [14], Tomari and Watanabe proved their main theorem by using Izumi’s results on the analytic orders [5]. We need their useful arguments. The following lemma is the version due to Ishii.

**Lemma 4.4 (Ishii [3, Lemma 1.5]).** Let \((W, \omega)\) be a \(d\)-dimensional normal isolated singularity and \(h: W_1 \to W\) a resolution of the singularity which factors through the blowing up by the maximal ideal of the singular point. Let \( F = \bigcup_{i=1}^k F_i \) be the exceptional divisor on \( W_1 \), where the \( F_i \) are irreducible components. Then there exist positive numbers \( \beta \in \mathbb{R} \) and \( b \in \mathbb{N} \) such that:

For an \( \mathcal{O}_W \)-ideal \( J = h_* \mathcal{O}_{W_1}(- \sum_{i=1}^k a_i F_i) \) with \( a_i > b \) for some \( i \), the inequalities \( \dim_{\mathbb{C}} \mathcal{O}_W / J \geq \beta(a_i)^d \) \( (i = 1, \ldots, k) \) hold.

**Lemma 4.5.** If \( \theta_r \neq 0 \) for some \( r \in \mathbb{N} \), then there exists a positive integer \( c \in \mathbb{R} \) such that \( \theta_{mr} \geq cm^2 \) for all \( m \in \mathbb{N} \).

**Proof.** Assume \( \theta_r \neq 0 \). By Lemma 3.2, we may assume that \( \psi: (M, A) \to (X_0, x) \) is a good resolution of the singularity which factors through the blowing up by the maximal ideal of the singular point. Let \( \omega \) be a section of \( \mathcal{I}_{r,0} \) which does not belong to \( \mathcal{A}_r \). We define a homomorphism \( \varphi_m: \mathcal{O}_{X_0} \to \mathcal{I}_{mr,0} \) by \( \varphi_m(s) = s\omega^m \) for every \( m \in \mathbb{N} \). We denote by \( J_m \) the inverse image \( \varphi_m^{-1}(\mathcal{A}_{mr} \cap \mathcal{I}_{mr,0}) \). Then we have the injection

\[
\mathcal{O}_{X_0}/J_m \to \mathcal{I}_{mr,0}/\mathcal{A}_{mr} \cap \mathcal{I}_{mr,0}.
\]
We put \( a_i := \min \{ v_i(\omega) + r, 0 \} \), where \( v_i \) is the valuation at an irreducible component \( A_i \) of \( A \). Then \( J_m = \psi_* \mathcal{O}_M(\sum m a_i A_i) \). By the choice of \( \omega \), there exists a component \( A_i \) such that \( a_i < 0 \). By Lemma 4.4, there exists \( c \in \mathbb{R} \) such that \( \theta_{mr} \geq cm^2 \) for any \( m \in \mathbb{N} \). \( \square \)

**Corollary 4.6.** If \( \mathcal{P}(t) \) is constant, then \( \theta_m = 0 \) for all \( m \in \mathbb{N} \).

**Proof.** It follows from Theorem 3.4, (4.2) and lemmas above. \( \square \)

5. **Main Theorem**

In this section, we prove the main theorem. We use the same notation as in the preceding section.

**Definition 5.1.** Let \( f : Y \to X \) be the log-canonical model of \( X \) with the exceptional divisor \( E \). We call \( f \) the simultaneous log-canonical model, SLC model for short, if the restriction \( f_t : Y_t \to X_t \) is the log-canonical model of \( X_t \) and \( K_{Y_t} + E_t \) is log-canonical for any \( t \in T \).

**Definition 5.2.** For any \( m \in \mathbb{N} \), we define the function \( \Lambda_m : T \to \mathbb{Z} \) by \( \Lambda_m(t) := \lambda_m(X_t) \).

The following Lemma is proved by an argument similar to that in Lemma 4.5.

**Lemma 5.3.** Let \( g : (X', B) \to (X_0, x) \) be a partial resolution such that \( K_{X'} + B \) is log-canonical. Let \( D \) be a reduced divisor on \( X' \) such that \( 0 \leq D \leq B \). For every \( m \in \mathbb{N} \), we define the invariant \( \nu_m(X'; B, D) \) by

\[
\nu_m(X'; B, D) = \dim_{\mathcal{O}} g_* \mathcal{O}_M(m(K_{X'} + B))/g_* \mathcal{O}_M(m(K_{X'} + D)).
\]

If \( \nu_r(X'; B, D) \neq 0 \) for some \( r \in \mathbb{N} \), then there exists a positive integer \( c \in \mathbb{R} \) such that \( \nu_{mr}(X'; B, D) \geq cm^2 \) for all \( m \in \mathbb{N} \).

**Proposition 5.4.** Assume that there exists the SLC model of the deformation \( \pi : X \to T \). Then the function \( \Lambda_m \) is constant for \( m >> 0 \).

**Proof.** Let \( f : Y \to X \) be the SLC model of the deformation \( \pi \). Since \( K_Y + E \) is \( f \)-ample, \( R^1 f_* \mathcal{O}_Y(m(K_Y + E)) = 0 \) for \( m >> 0 \). From the exact sequence (cf. [10])

\[
0 \to f_* \mathcal{O}_Y(m(K_Y + E)) \to f_* \mathcal{O}_Y(m(K_Y + E)) \to f_* \mathcal{O}_Y(m(K_{Y_0} + E_0))
\]

\[
\to R^1 f_* \mathcal{O}_Y(m(K_Y + E)),
\]

we have \( f_* \mathcal{O}_{Y_0}(m(K_{Y_0} + E_0)) = \mathcal{I}_m \otimes \mathcal{C}(0) \) for \( m >> 0 \). Since \( f_* \mathcal{O}_{Y_0}(m(K_{Y_0} + E_0)) \) is a submodule of \( \mathcal{O}_{X_0}(mK_{X_0}) \), we have the equality \( \mathcal{I}_m \otimes \mathcal{C}(0) = \mathcal{I}_{m,0} \). Then Lemma 4.1 and Lemma 4.2 imply that

\[
\lambda_m(X_t) = \dim_{\mathcal{C}} \mathcal{Q}_m \otimes \mathcal{C}(0).
\]
We denote by $B$ the exceptional set on $Y_0$. Since $E_0 \leq B$, we obtain the equality 
\[
\dim \mathbb{C} \mathcal{Q}_m \otimes \mathbb{C}(0) = \lambda_m(X_0, x) + \nu_m(Y_0; B, E_0).
\]
Since $\mathcal{P}(t)$ is upper semicontinuous, $\nu_m(Y_0; B, E_0) = 0$ by the lemma above.

**Lemma 5.5.** $\mathcal{Q}_m$ is a torsion free $\mathcal{O}_{\mathcal{T}}$-module for any $m \in \mathbb{N}$, if $\mathcal{P}$ is constant.

**Proof.** We assume that there exists a section $\omega \in \mathcal{O}_X(rK_X) \setminus \mathcal{I}_r$ of which the image in $\mathcal{Q}_r$ is a torsion element. Then there exists an exceptional prime divisor $F$ on $Y$ lying over $X_0$ such that $v_F(\omega) < -r$. We note that $F$ is a projective surface. Let $\mathcal{I}_F$ be the $\mathcal{O}_Y$-ideal of the subvariety $F$, and let $L_m := m(K_Y + E)$. Since $L_1$ is $f$-ample, there exists an integer $n \in \mathbb{N}$ such that $\mathcal{O}_F(L_n)$ is a very ample invertible sheaf and the following sequence is exact for any $m \in \mathbb{N}$:
\[
0 \to f_* (\mathcal{I}_F \mathcal{O}_Y(L_{mn} + F)) \to f_* \mathcal{O}_Y(L_{mn} + F) \to H^0(\mathcal{O}_F(L_{mn} + F)) \to 0.
\]
By [1, III, Ex. 5.2], there exists a polynomial $q'$ of degree 2 such that 
\[
\dim \mathbb{C} f_* \mathcal{O}_Y(L_{mn} + F)/f_* (\mathcal{I}_F \mathcal{O}_Y(L_{mn} + F)) = q'(m)
\]
for $m >> 0$. Since $\mathcal{I}_F \mathcal{O}_Y(L_{mn} + F)$ is isomorphic to $\mathcal{O}_Y(L_{mn})$ outside a one-dimensional subvariety in $F$, there exists a polynomial $q$ of degree 2 such that $\dim \mathbb{C} f_* \mathcal{O}_Y(L_{mn} + F)/\mathcal{I}_{mn} \geq q(m)$ for $m >> 0$. Since any section of the sheaf $f_* \mathcal{O}_Y(L_{mn} + F)/\mathcal{I}_{mn}$ is a torsion element of $\mathcal{Q}_{mn}$, we obtain the inequality (cf. (4.2))
\[
\dim \mathbb{C} \mathcal{Q}_{mn} \otimes \mathbb{C}(t) \leq \dim \mathbb{C} \mathcal{Q}_{mn} \otimes \mathbb{C}(0) - q(m).
\]
Since $\dim \mathbb{C} \mathcal{Q}_{mn} \otimes \mathbb{C}(0) - \dim \mathbb{C} \mathcal{Q}_{mn} \otimes \mathbb{C}(t)$ is bounded by a linear function, we are led to a contradiction.

**Remark 5.6.** From the proof above, we see that $Y_0$ is irreducible. Thus any irreducible component of $E$ dominates $T$. Since $Y_0$ is a principal divisor, for any irreducible component $F$ of $E$, the intersection $F \cap Y_0$ is a one-dimensional variety.

**Lemma 5.7.** $\mathcal{I}_{m,0} = \mathcal{A}_m$ for any $m \in \mathbb{N}$, if $\mathcal{P}$ is constant.

**Proof.** The inclusion $\mathcal{I}_{m,0} \subset \mathcal{A}_m$ follows from Corollary 4.6. Let $\omega$ be a section of $\mathcal{A}_m$ and $\omega'$ a section of $\mathcal{O}_X(mK_X)$ of which the image in $\mathcal{O}_{X_0}(mK_{X_0})$ is $\omega$. If $v_F(\omega') < -m$ for an irreducible component $F$ of $E$, then there exists an irreducible component $A_i$ of $A$ lying over the variety $F \cap Y_0$ such that $v_{A_i}(\psi^* \omega) < -m$. It contradicts the definition of $\omega$. Hence $\omega'$ belongs to $\mathcal{I}_m$, and $\omega$ also belongs to $\mathcal{I}_{m,0}$.

**Theorem 5.8.** The following conditions are equivalent.
(1) \( \pi: X \rightarrow T \) admits the SLC model.
(2) The map \( \Lambda_m: T \rightarrow \mathbb{Z} \) is constant for any \( m \in \mathbb{N} \).
(3) The map \( \mathcal{P}: T \rightarrow \mathbb{Q} \) is constant.

Proof. We consider the following condition: (2)' The map \( \Lambda_m: T \rightarrow \mathbb{Z} \) is constant for \( m \gg 0 \). By Proposition 5.4 (1) implies (2)'. It follows from Theorem 3.4 that (2)' implies (3). We assume that \( \mathcal{P} \) is constant. Then, from Lemma 4.1 and lemmas above, we obtain the following equalities for any \( m \in \mathbb{N} \):

\[ \mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0} = \mathcal{A}_m, \quad \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0). \]

Now it is clear that (2) holds, and that \( Y_0 = \text{Proj}(\bigoplus_{m \in \mathbb{N}} \mathcal{I}_m \otimes \mathbb{C}(0)) \) is the log-canonical model of \( X_0 \). Since \( \mathcal{A}_m = \mathcal{I}_m \otimes \mathbb{C}(0) = f_* \mathcal{O}_{Y_0}(m(I_{i}Y_{0}+E_{0})) \) for \( m \gg 0 \) (cf. proof of Proposition 5.4) and \( K_{Y_0} + E_0 \) is ample, \( K_{Y_0} + E_0 \) is log-canonical. On the other hand, \( f_t: Y_t \rightarrow X_t \) is the log-canonical model for \( t \in T^* \) by Lemma 4.2. Hence we obtain the condition (1). \( \square \)

References