THOM'S CONJECTURE ON TRIANGULATIONS OF MAPS

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§1. INTRODUCTION

Let $f_i: X_i \to Y_i$, $i = 1, 2$, be proper $C^0$ maps between closed sets in Euclidean spaces. We call $f_1$ and $f_2$ $\mathcal{R}$-$\mathcal{L}$ equivalent if there exist homeomorphisms $\eta: Y_1 \to Y_2$ and $\tau: X_1 \to X_2$ such that $\eta \circ f_1 = f_2 \circ \tau$. We call $f_1$ triangulable if it is $\mathcal{R}$-$\mathcal{L}$ equivalent to a PL map between closed polyhedra in Euclidean spaces.

Thom [T] conjectured that a so-called "Thom map", which Thom called an application stratifiée sans éclatement, is triangulable. In the present paper we solve the conjecture in a more general form. Partial solutions were given by Teissier [Te] and Proposition IV.1.10 in [S].

A tube system $\{T_j\} = \{(|T_j|, \pi_j, \rho_j)\}_{j=1, \ldots, k}$ for a $C^\infty$ stratification $\{Y_j\}_{j=1, \ldots, k}$ with $Y = \cup_j Y_j \subset \mathbb{R}^n$ and $\dim Y_j < \dim Y_{j+1}$ consists of one tube $T_j$ at each $Y_j$, where $\pi_j: |T_j| \to Y_j$ is a $C^\infty$ open tubular neighborhood of $Y_j$ in $\mathbb{R}^n$ and $\rho_j$ is a non-negative $C^\infty$ function on $|T_j|$ such that $\rho_j^{-1}(0) = Y_j$ and each point $y$ of $Y_j$ is a unique and non-degenerate critical point of $\rho_j|_{\pi_j^{-1}(y)}$. We call a tube system $\{T_j\}$ strongly controlled if for each pair $j$ and $j'$ with $j < j'$, the following property holds true:

$$\operatorname{ct}(T_j, T_{j'}) = \pi_j \circ \pi_{j'} = \pi_j \quad \text{and} \quad \rho_j \circ \pi_{j'} = \rho_j$$

on $|T_j| \cap |T_{j'}|$, and (sc) the map $(\pi_j, \rho_j)|_{Y_j \cap |T_j|}$ is a $C^\infty$ submersion into $Y_j \times \mathbb{R}$. Note that any Whitney stratification admits a strongly controlled tube system. An example of a $C^\infty$ stratification which admits a strongly controlled tube system but is not a Whitney stratification is $\{(x, y) \in \mathbb{R}^3: y = z^2 \sin x/z, z \neq 0\}$.

Let $\{X_{i,j}\}_{i=1, \ldots, k_j}$ and $\{Y_{j}\}_{j=1, \ldots, k}$ be $C^\infty$ stratifications of sets $X$ and $Y$ in $\mathbb{R}^n$, respectively, such that $\dim X_{i,j} < \dim X_{i+1,j}$ and $\dim Y_j < \dim Y_{j+1}$, and let $f: X \to Y$ be a $C^\infty$ map (i.e., the restriction to $X$ of a $C^\infty$ map $\tilde{f}$ between restrictions of $X$ and $Y$) such that each restriction $f|_{X_{i,j}}$ is a submersion into $Y_j$. Let $\{T_j = (|T_j|, \pi_j, \rho_j)\}$ be a strongly controlled tube system for $\{Y_j\}$, and let $\{T_{i,j} = (|T_{i,j}|, \pi_{i,j}, \rho_{i,j})\}$ be a tube system for $\{X_{i,j}\}$. We call $\{T_{i,j}\}$ strongly controlled over $\{T_j\}$ if the following conditions are satisfied. (sc1) For each $(i, j)$, it holds that $f \circ \pi_{i,j} = \pi_j \circ \tilde{f}$ on $|T_{i,j}|$. (sc2) For each $j$, $\{T_{i,j}\}_{i=1, \ldots, k_j}$ is a strongly controlled tube system for $\{X_{i,j}\}$. (sc3) For any pair $(i, j)$ and $(i', j')$ with $j < j'$, it holds that $\pi_{i,j} \circ \pi_{i', j'} = \pi_{i,j}$ on $|T_{i,j}| \cap |T_{i', j'}|$, and $(\pi_{i,j}, f)|_{X_{i,j} \cap |T_{i,j}|}$ is a $C^\infty$ submersion into the $C^\infty$ manifold $\{(x, y) \in X_{i,j} \times (Y_j \cap |T_j|): f(x) = \pi_j(y)\}$. (An example of $f: X \to Y$ where there do not exist such tube systems $\{T_j\}$ and $\{T_{i,j}\}$ is the blow-up of $S^n$, $n > 1$, at a point of $S^n$.)
Theorem. Let \( \{X_{i,j}\} \) and \( \{Y_{j}\} \) be \( C^\infty \) stratifications of closed sets \( X \subset \mathbb{R}^n \) and \( Y \subset \mathbb{R}^n \) respectively, and let \( f: X \to Y \) be a \( C^\infty \) proper map such that each restriction \( f|_{X_{i,j}} \) is a submersion into \( Y_{j} \). Assume there exist a strongly controlled tube system \( \{T_{j}\} \) for \( \{Y_{j}\} \) and a tube system \( \{T_{i,j}\} \) for \( \{X_{i,j}\} \) strongly controlled over \( \{T_{j}\} \). Then \( f \) is triangulable.

The theorem is proved by a theory developed in [S] and hence can be proved also in the semialgebraic, subanalytic and \( \mathcal{X} \) categories. (See [S] for the definition of \( \mathcal{X} \).) (In the subanalytic and \( \mathcal{X} \) cases, we argue in the \( C^\infty \) category for a positive integer \( r \).) In the following proof we use integrations of vector fields. But we can avoid this in the above important special cases as shown in [S]. Note also that we can construct effectively a triangulation, i.e., polyhedra \( X' \) and \( Y' \) and homeomorphisms \( \tau: X' \to X \) and \( \eta: Y' \to Y \) such that \( \eta^{-1} \circ f \circ \tau \) is PL in the cases. Hence the following assertion seems true.

Let \( k, l, m \in \mathbb{N} \). The cardinal number of the \( \mathcal{R}-\mathcal{L} \) equivalence classes of all proper semialgebraic Thom maps between closed semialgebraic sets in \( \mathbb{R}^k \) whose graphs are defined by equalities or inequalities of \( l \)-polynomials of degree \( \leq m \) is bounded by some recursive function in variables \( (k, l, m) \).

For the proof it suffices to find an effective method of choosing a Thom stratification \( f: \{X_{i,j}\} \to \{Y_{j}\} \) of a Thom map \( f: X \to Y \), because we can effectively construct strongly controlled tube systems \( \{T_{i,j}\} \) and \( \{T_{j}\} \) of a Thom stratification \( f: \{X_{i,j}\} \to \{Y_{j}\} \) [S]. (See [G-al] for the definitions of a Thom map and a Thom stratification.) Therefore, we can prove the above assertion if we replace the phrase "Thom maps" with the one "Thom stratifications \( f: \{X_{i,j}\} \to \{Y_{j}\} \)" and add the condition that \( \{X_{i,j}\} \) and \( \{Y_{j}\} \) are defined by \( l \)-polynomials as graph \( f \).

\section{\( C^\infty \) triangulations}

In this paper, \( K \) and \( L \) always denote simplicial complexes in some Euclidean space. Let \( |K| \) denote the underlying polyhedron of \( K \). For a point \( x \) in \( |K| \), let \( \text{st}(x, K) \) denote the subcomplex of \( K \) generated by the simplexes containing \( x \). We denote by \( K^k \) the \( k \)th skeleton of \( K \) for a non-negative integer \( k \). For a simplex or a manifold \( \sigma \), \( \text{Int} \sigma \) and \( \partial \sigma \) denote the interior and the boundary of \( \sigma \) respectively. If \( K \subset L \), the simplicial neighborhood \( N(K, L) \) of \( K \) in \( L \) is the smallest subcomplex of \( K \) whose underlying polyhedron is a neighborhood of \( |K| \) in \( |L| \). If a subset \( W \) of \( |L| \) is the underlying polyhedron of a subcomplex of \( L \), we call the subcomplex \( L|W \). For each simplex \( \sigma \) of \( K \), let \( v_{\sigma} \) denote the barycenter of \( \sigma \). The barycentric subdivision \( K' \) of \( K \) consists of all the simplexes spanned by \( v_{\sigma_1}, \ldots, v_{\sigma_k} \) for \( \sigma_1 \subset \cdots \subset \sigma_k \in K \).

A \( C^\infty \) map \( h: K \to \mathbb{R}^n \) is a continuous map \( h: |K| \to \mathbb{R}^n \) such that all the restrictions \( h|_{\sigma}, \sigma \in K, \) are of class \( C^\infty \). Let \( b \in |K| \). We define \( dh_b: |\text{st}(b, K)| \to \mathbb{R}^n \) by

\[
dh_b(x) = d(h|_{\sigma})_b(x - b) \quad \text{for} \quad \sigma \in \text{st}(b, K), \quad x \in \sigma.
\]

We call \( h \) a \( C^\infty \) imbedding if \( h \) and \( dh_b \) for all \( b \in |K| \) are homeomorphisms onto the images. Let \( Z \subset \mathbb{R}^n \). A \( C^\infty \) triangulation of \( Z \) is a pair of \( K \) and a \( C^\infty \) imbedding \( h: K \to \mathbb{R}^n \) such that \( h(|K|) = Z \). (A triangulation of \( Z \) consists of \( K \) and a homeomorphism from \(|K| \) to \( Z \).) An approximation of \( h \) is a \( C^\infty \) map.
\( \hat{h}: \hat{K} \to \mathbb{R}^n \) such that \( \hat{K} \) is a subdivision of \( K \),

\[
|h(x) - \hat{h}(x)| \leq c \quad \text{for} \quad x \in |K|,
\]

and

\[
|dh_b(x) - \hat{h}(x)| \leq c|x - b| \quad \text{for} \quad b \in |K|, \quad x \in |st(b, K')|
\]

for a small positive number \( c \).

Let \( \alpha: K_1 \to K_2 \) be a simplicial map between finite simplicial complexes in \( \mathbb{R}^n \). By induction on \( \dim K_1 \) we define the mapping cylinder \( C_\alpha(K_1, K_2) \) of \( \alpha \) which is a simplicial complex in \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) and whose underlying polyhedron can be regarded as the mapping cylinder \( C_\alpha(|K_1|, |K_2|) \) of the topological map \( \alpha: |K_1| \to |K_2| \). Let \( K_1 \) and \( K_2 \) be given in \( \mathbb{R}^n \times 0 \times 0 \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) and \( 0 \times \mathbb{R}^n \times 1 \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) respectively, and let \( K_1' \) and \( K_2' \) denote the barycentric subdivision of \( K_1 \) and \( K_2 \) respectively. If \( \dim K_1 = -1 \), i.e., \( K_1 = \emptyset \), then set \( C_\alpha(K_1, K_2) = K_2' \). Let \( \dim K_1 = k \) and assume we have already defined the mapping cylinder \( C_\alpha(K_1^{k-1}, K_2) \). For \( \sigma \in K_1 - K_1^{k-1} \), let \( a_\sigma \) denote the middle point of the barycenters of \( \sigma \) and of \( \alpha(\sigma) \) in \( \mathbb{R}^n \times \mathbb{R}^n \times 1/2 \). We set

\[
C_\alpha(K_1, K_2) = C_\alpha(K_1^{k-1}, K_2) \\
\cup \bigcup_{\sigma \in K_1 - K_1^{k-1}, \sigma_1 \in K_1' \cup K_2' \cup C_\alpha(\sigma, K_1, K_2)} \langle a_\sigma, \sigma_1, a_\sigma \star \sigma_1: \sigma_1 \in K_1' \cup K_2' \cup C_\alpha(\sigma, K_1, K_2) \rangle,
\]

where \( a_\sigma \star \sigma_1 \) denotes the cone with vertex \( a_\sigma \) and base \( \sigma_1 \).

We show some good properties of \( C_\alpha(K_1, K_2) \). Clearly it is a simplicial complex in \( \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \), \( K_1' \) and \( K_2' \) are subcomplexes of \( C_\alpha(K_1, K_2) \), and there is a natural simplicial map \( C_\alpha(K_1, K_2) \to K_2' \), which is a retraction and carries the barycenter of a simplex \( \sigma \) of \( K_1 \) and the above-mentioned \( a_\sigma \) to the barycenter of \( \alpha(\sigma) \). Given a commutative diagram of simplicial maps

\[
\begin{array}{ccc}
L_1 & \stackrel{\beta}{\longrightarrow} & L_2 \\
\downarrow & & \downarrow \\
K_1 & \stackrel{\alpha}{\longrightarrow} & K_2,
\end{array}
\]

there exists a natural simplicial map \( C_\beta(L_1, L_2) \to C_\alpha(K_1, K_2) \). On the other hand, \( C_{\text{id}}(K_1, K_1) \) is naturally and simplicially isomorphic to the barycentric subdivision \( L \) of the cell complex \( K_1 \times \{0, 1, [0, 1]\} \). Hence we have a natural simplicial map \( L \to C_\alpha(K_1, K_2) \), which equals the identity map on \( |K_1| \times 0 \) and \( \alpha \) on \( |K_1| \times 1 \). Through this map we identify \( |C_\alpha(K_1, K_2)| \) with the mapping cylinder of the topological map \( \alpha \).

Let \( M \) be a subset of \( \mathbb{R}^n \). We call \( M \) a \( C^\infty \) manifold possibly with corners of dimension \( m \) if it is locally \( C^\infty \) diffeomorphic to an open subset of \( \mathbb{R}_+^m \), where \( \mathbb{R}_+ = [0, \infty] \). Note that such an \( M \) admits the canonical \( C^\infty \) stratification \( \{Z_i\}_{i=0, \ldots, m} \) such that each \( Z_i \) is the subset of \( \bigcup_{j=0}^i Z_j \) where \( \bigcup_{j=0}^i Z_j \) is locally \( C^\infty \) diffeomorphic to \( \mathbb{R}^i \). Faces of \( M \) are the closures of the connected components of \( Z_i \). For a face \( M' \) of \( M \) of dimension \( m' \), set \( \text{Sing} M' = M' \cap \bigcup_{i=0}^{m'-1} Z_i \).
For continuous maps $\psi_i: A_i \to B$, $i = 1, 2$, let $A_1 \times (\psi_1, \psi_2) A_2$ denote the fibre product — $\{(a_1, a_2) \in A_1 \times A_2 : \psi_1(a_1) = \psi_2(a_2)\}$.

The key of proof of the theorem is the following lemma, which is similar to Proposition I.3.20 in [S].

**Lemma 1.** Let $M$ and $M'_1$ be compact $C^\infty$ manifolds possibly with corners. Let $\psi: M \to M'_1$ be a surjective $C^\infty$ submersion which carries surjectively and submersively any face of $M$ to some face of $M'_1$. Let $M'$ be a face of $M$. Let $(L, g)$ and $(K, h)$ be $C^\infty$ triangulations of $M'_1$ and a neighborhood of a union of subfaces of $M'$ in $M$, respectively, such that $g^{-1} \circ \varphi \circ h$ is a PL map from $|K|$ to $|L|$. Shrink the neighborhood of the union and subdivide $K$. Then keeping the property that $g^{-1} \circ \varphi \circ h$ is PL, we can extend $h$ to a $C^\infty$ triangulation of a neighborhood of $M'$ in $M$.

**Proof of Lemma 1.** We can assume that the given neighborhood is a neighborhood of $\text{Sing} M'$ in $M$. Recall the following assertion in the proof of Proposition I.3.20 in [S].

**Assertion.** Let $n > n_1$ be non-negative integers, let $p: \mathbb{R}^n_{+} \to \mathbb{R}^n_{+}$ be the projection onto the first $n_1$-factors, let $\alpha: A \to \mathbb{R}^n_{+}$ be a $C^\infty$ imbedding of a finite simplicial complex $A$, let $(B, \beta)$ be a $C^\infty$ triangulation of $\mathbb{R}^n_{+}$ such that $\beta^{-1} \circ p \circ \alpha$ is PL, and let $C$ be a compact subset of $\mathbb{R}^n_{+}$. Then there exist a simplicial complex $A_0$ and a $C^\infty$ imbedding $\alpha_0: A_0 \to \mathbb{R}^n_{+}$ such that some subdivision of $A$ is a subcomplex of $A_0$, the restriction $\alpha_0 |_{|A|}: A_0 |_{|A|} \to \mathbb{R}^n_{+}$ is a strong approximation of $\alpha$,

$$A_1 \subset A_0, \quad \alpha_0 |_{|A_1|} = \alpha |_{|A_1|}, \quad \alpha_0 (|A_0|) \supset C, \quad (|A_0| - |A|) \cap |A_1| = \emptyset,$$

and $\beta^{-1} \circ p \circ \alpha_0$ is PL, where $A_1 = \{\sigma \in A : \alpha(\sigma) \cap C = \emptyset\}$.

It is easy to see that $h^{-1}(M')$ and $h^{-1}(\text{Sing} M')$ are the underlying polyhedra of some subcomplexes of $K$. Set $U = \text{h}(\text{N}(K|h^{-1}(\text{Sing} M'), K))$. Then $U$ is a compact neighborhood of $\text{Sing} M'$ in $M$, and we can assume $U \cap \overline{M - h(|K|)} = \emptyset$. (Here replace $K$ with its barycentric subdivision if necessary.) Let $\{C_i\}_{i=1, \ldots, k}$ be a covering of $M' - h(|K|)$ by compact sets such that for each $i$, there exist an open neighborhood $V_i$ of $C_i$ in $M$ and $C^\infty$ imbeddings $\tau_i: V_i \to \mathbb{R}^m_+$ and $\theta_i: \varphi(V_i) \to \mathbb{R}^m_+$, where $m = \dim M$ and $m_1 = \dim M'_1$, such that $V_i \cap U = \emptyset$, and the composite $\theta_i \circ \varphi \circ \tau_i^{-1}: \tau_i(V_i) \to \mathbb{R}^m_{+}$ is the restriction of the projection of $\mathbb{R}^m_+$ onto the first $m_1$-factors.

Let $0 < l < k$ be an integer. Assume we have already constructed a $C^\infty$ triangulation $(K_{l-1}, h_{l-1})$ of a neighborhood of $U \cup \bigcup_{i=1}^{l-1} C_i$ in $M$ such that $g^{-1} \circ \varphi \circ h_{l-1}$ is PL, some subdivision of $K$ is a subcomplex of $K_{l-1}$, $h_{l-1}|_{|K_l|}$ is a strong approximation of $h$, and $h = h_{l-1}$ on $h^{-1}(U)$. Then it suffices to obtain $(K_l, h_l)$ with the corresponding properties.

Subdividing finely $L$ and then $K_{l-1}$, we can assume that (i) for $\sigma \in K_{l-1}$, if $h_{l-1}(\sigma) \cap C_i \neq \emptyset$ then $h_{l-1}(\sigma) \subset V_i$, (ii) for $\sigma_1, \sigma_2 \in L$, if $\sigma_1 \cap C_2 \neq \emptyset$ and $g(\sigma_1) \cap \varphi(C_i) \neq \emptyset$ then $g(\sigma_2) \subset \varphi(V_i)$, and (iii) for $\sigma \in K_{l-1}$ and $\sigma_1 \in L$, if $h_{l-1}(\sigma) \cap C_i \neq \emptyset$ and $\varphi \circ h_{l-1}(\sigma) \cap g(\sigma_1) \neq \emptyset$ then $g(\sigma_1) \cap \varphi(C_i) \neq \emptyset$. Let $D$ denote the complex generated by $\sigma \in L$ with $g(\sigma) \cap \varphi(C_i) \neq \emptyset$. 


Apply the assertion to
\[ n = m_1, \quad n_1 = 0, \]
\[ (A, \alpha) = (\{\sigma \in L : g(\sigma) \subset \varphi(V_i)\}, \theta_l \circ (g|_{A_l})), \]
\[ (B, \beta) = (\{0\}, \text{id}), \quad \text{and} \quad C = [0, c]^n - \alpha(|A|) \]
for a large number \( c \). Then by (ii) we have a \( C^\infty \) triangulation \((A_0, \alpha_0)\) of a neighborhood of \( [0, c]^n \) in \( \mathbb{R}^m_+ \) such that \( A_0 \supset D \) and \( \alpha_0 = \alpha \) on \( |D| \). Repeat a similar argument for \( c_1 = c, c_2, \ldots \rightarrow \infty \). Then we obtain a \( C^\infty \) triangulation \((B, \beta)\) of \( \mathbb{R}^{m_1}_+ \) such that \( \beta = \theta_l \circ g \) on \( |D| \).

In consideration of application of the assertion, set newly
\[ n = m, \quad n_1 = m_1, \]
\[ (A, \alpha) = (\text{the complex generated by } \sigma \in K_{l-1} \text{ with } h_{l-1}(\sigma) \cap C_l \neq \emptyset, \tau_l \circ (h_{l-1}|_{A_l})), \]
\[ (B, \beta) = (\tilde{B}, \tilde{\beta}), \quad \text{and} \quad C = \tau_l(C_l). \]

By (i), \( \alpha \) is well-defined. By (iii), \( \alpha(|A|) \subset \beta(|D|) \). Hence \( \beta^{-1} \circ p \circ \alpha = g^{-1} \circ \theta_l^{-1} \circ p \circ \tau_l \circ h_{l-1} = g^{-1} \circ \varphi \circ h_{l-1} \) is PL. Thus the conditions in the assertion are satisfied. Let \( \alpha_0 : A_0 \rightarrow \mathbb{R}^m_+ \) be a resulting \( C^\infty \) imbedding. Set \( \tilde{K}_{l-1} = \{\sigma \in K_{l-1} : h_{l-1}(\sigma) \cap C_l = \emptyset\} \). Remember that
\[ (A_0, \alpha_0) = (A, \alpha) \quad \text{on} \quad |\{ \sigma \in A : h_{l-1}(\sigma) \cap C_l = \emptyset \}|, \]
and regard
\[ |A_0| \cap |\tilde{K}_{l-1}| = |\{ \sigma \in A : h_{l-1}(\sigma) \cap C_l = \emptyset \}|. \]

Let \( E' \) denote the barycentric subdivision of a simplicial complex \( E \) as always. Then the family \( A_0 \cup \tilde{K}_{l-1} \) is a simplicial complex. Let \( K_l \) denote the complex. We can assume that \( \alpha_0(|A_0|) \subset \tau_l(V_i) \). Set
\[ h_l = \begin{cases} \tau_l^{-1} \circ \alpha_0 & \text{on} \quad |A_0| \\ h_{l-1} & \text{on} \quad |\tilde{K}_{l-1}| - |A_0|. \end{cases} \]

Then this map is well-defined and a \( C^\infty \) imbedding by 8.8 in [M], and \((K_l, h_l)\) fulfills the requirements. \( \square \)

§3. VECTOR FIELDS AND REMOVAL DATA

Let \( X, Y, \{X_{i,j}\}, \{Y_j\}, f : X \rightarrow Y, \{T_{i,j}\} \) and \( \{T_j\} \) be the same as in the theorem except for the assumption that \( f \) is proper. Assume \( \dim Y_j < \dim Y_{j+1} \) and \( \dim X_{i,j} < X_{i+1,j} \). Let the set of indexes of \( \{X_{i,j}\} \) be \( \overline{H} = \{(i, j) \in \mathbb{N}^2 : 1 \leq j \leq k, 1 \leq i \leq k_j\} \). Set \( H = \overline{H} - \{(k_k, k)\} \). Give a lexicographic order to \( H \) and \( \overline{H} \) so that \( (i, j) < (i', j') \) if \( j < j' \) or \( j = j' \) and \( i < i' \).
A vector field $v^Y$ on $\{Y_j\}$ consists of one $C^\infty$ vector field $v_j$ on each $Y_j$. We call $v^Y$ controlled if for each pair $j$ and $j'$, 
\[ \begin{align*}
\text{cv}(T_j, T_{j'}) &= \frac{d\pi_j v_{j'} y}{v_j \pi_j(y)} = \frac{d\rho_j v_{j'} y}{v_j \rho_j(y)} = 0 \\
\text{for } y &\in Y_j' \cap U_j, \\
\end{align*} \]
where $U_j$ is some neighborhood of $Y_j$ in $|T_j|$. If only the former equality is assumed, we call $v^Y$ weakly controlled. We call a vector field $v^X = \{v_{i,j}\}$ on $\{X_{i,j}\}$ controlled over $v^Y$ if the former equality of $\text{cv}(T_{i,j}, T_{i',j'})$ for each pair $(i, j)$ and $(i', j')$, the latter for each pair $(i, j)$ and $(i', j)$, and the following equality for each $(i, j)$ hold:
\[ df v_{i,j} x = v_{j'i}(x) \text{ for } x \in X_{i,j}. \]

Let $v^Y = \{v_j\}$ be a vector field on $\{Y_j\}$. For each $j$, let $\omega_j : \Omega_j \to Y_j$, $\Omega_j \subset Y_j \times \mathbb{R}$, be the maximal $C^\infty$ flow defined by $v_j$. Set $\Omega = \cup \Omega_j$ and define a map $\omega : \Omega \to Y$ by $\omega|_{\Omega_j} = \omega_j$ for each $j$. We call $\omega$ the flow of $v^Y$. We call $v^Y$ locally integrable if $\Omega$ is open in $Y \times \mathbb{R}$ and the flow is continuous.

Assume $X$ and $Y$ are compact. Let $0 < \epsilon_{k-1} \ll \cdots \ll \epsilon_1 \ll \infty$ be numbers. Then for $j \leq l$, (1) the following set is a $C^\infty$ submanifold possibly with corners of $Y_j$:
\[ Y_{j,l} = Y_l \cap |T_j| - \rho^{-1}_j(0, \epsilon_1/2) - \cdots - \rho^{-1}_{j-1}(0, \epsilon_{j-1}/2), \]
(2) if $j < l$, the restriction of $(\pi_j, \rho_j)$ to $Y_{j,l} \cap \rho_j^{-1}([0, 2\epsilon_j])$ is a $C^\infty$ submersion into $Y_{j,l} \times [0, 2\epsilon_j]$, and (3) the sets $Y_{j,l}$ and $\cup_{j' \geq j} Y_{j,j'} \cap \rho_{j'}^{-1}([0, 2\epsilon_j])$ are compact. We call $\epsilon = \{\epsilon_j\}_{1 \leq j \leq k}$ with such properties a removal data of $\{T_j\}_{j=1}^k$.

A removal data $\epsilon = \{\epsilon_{i,j}\}_{(i,j) \in \overline{H}}$ of $\{T_j, T_{i,j}\}_{(i,j) \in \overline{H}}$ is such that the following eight conditions are satisfied. Let $(i_1, j_1) \leq (i_2, j_2) \in \overline{H}$. (1) Each $\epsilon_{i,j}$ is a small positive number. Set $\epsilon_{k,j} = \epsilon_j$. (2) $\{\epsilon_{i,j}\}_{1 \leq i \leq k, j}$ is a removal data of $\{T_j\}_{j=1}^k$.

(3) The following set is a $C^\infty$ manifold possibly with corners:
\[ X_{i_1,j_1,i_2,j_2} = X_{i_2,j_2} \cap |T_{i_1,j_1}| \cap (\rho_{j_1} \circ f)^{-1}([0, 2\epsilon_{j_1}]) \]
\[ - \bigcup_{j < j_1} (\rho_j \circ f)^{-1}([0, \epsilon_{j,j}/2]) - \bigcup_{i < i_1} \rho_{i,j_1}^{-1}([0, \epsilon_{i,j_1}/2]). \]
(Here we ignore $(\rho_{j_1} \circ f)^{-1}([0, 2\epsilon_{j_1}])$ if $j_1 = k$.) (4) If $j_1 = j_2$ and if $i_1 < i_2$, the restriction of $(\pi_{i_1,j_1}, \rho_{i_1,j_1})$ to $X_{i_1,j_1,i_2,j_2} \cap \rho_{i_1,j_1}^{-1}([0, 2\epsilon_{i_1,j_1}])$ is a $C^\infty$ submersion into $X_{i_1,j_1,i_1,j_1} \times [0, 2\epsilon_{i_1,j_1}]$. (5) If $j_1 < j_2$, the restriction of $(\pi_{k,j_1,j_1}, \rho_{j_1} \circ f)$ to $X_{k,j_1,i_2,j_2}$ is a $C^\infty$ submersion into $X_{k,j_1,i_2,j_1} \times [0, 2\epsilon_{i_2,j_1}]$. (6) If $j_1 < j_2$ and if $i_1 < k$, the restriction of $(\pi_{i_1,j_1}, f, \rho_{i_1,j_1})$ to $X_{i_1,j_1,i_2,j_2} \cap \rho_{i_1,j_1}^{-1}([\varepsilon_{i_1,j_1}/2, 2\varepsilon_{i_1,j_1}])$ is a $C^\infty$ submersion into $(X_{i_1,j_1,i_2,j_2} \times \{\varepsilon_{i_1,j_1}/2, 2\varepsilon_{i_1,j_1}\}) \times [\varepsilon_{i_1,j_1}/2, 2\varepsilon_{i_1,j_1}]$. (7) The set $\cup_{(i,j) \geq (i_1,j_1)} X_{i,j}$ is compact. (8) If $i_1 < k_1$, the set $\cup_{(i,j) \geq (i_1,j_1)} X_{i_1,j_1,i,j} \cap \rho_{i_1,j_1}^{-1}([0, 2\epsilon_{i_1,j_1}])$ is compact.

It is easy to see existence of a removal data of $\{T_j, T_{i,j}\}_{(i,j) \in \overline{H}}$. Indeed, it suffices to choose $\{\epsilon_{i,j}\}$ so that $0 < \epsilon_{1,1} \ll \infty$ and $\epsilon_{i,j} \gg \epsilon_{i',j'}$ if $(i, j) < (i', j')$. (Only condition (6) is nontrivial. For each $(i_3, j_1) > (i_1, j_1)$, the restriction of $(\pi_{i_3,j_1}, f)$
to $X_{i_2,j_2} \cap |T_{i_3,j_1}|$ and $(\pi_{i_1,j_1}, \rho_{i_1,j_1})$ to $X_{i_3,j_1} \cap |T_{i_1,j_1}|$ are $C^\infty$ submersion into $X_{i_3,j_1} \times (f, \pi_{j_1}) \ Y_{j_2} \cap |T_{j_1}|$ and $X_{i_1,j_1} \times \mathbb{R}$, respectively, by conditions (sc2) and (sc3). Hence (6) holds.

In the case where $f$ is proper and the connected components of $Y_j$ are bounded in $\mathbb{R}^n$, we need to and can easily generalize the above definition of a removal data. For each $j$, let $\{Y_{j}^{(i)}\}_{i \in \Gamma_{j}}$ denote the family of the connected components of $Y_j$. Replace the above $\{\epsilon_{i,j}\}$, $X_{i_1,j_1,i_2,j_2}, \ldots$ with $\{\epsilon_{i,j,l}\}_{(i,j) \in H, l \in \Gamma_{j}}$,

$$X_{i_2,j_2} \cap f^{-1}(Y_{j_2}^{(i)}) \cap \pi_{i_1,j_1}^{-1}(f^{-1}(Y_{j_1}^{(i)})) \cap (\rho_{j_1} \circ f)^{-1}([0, 2\epsilon_{j_1,l_1}])$$

$$- \bigcup_{j < j_1, l \in \Gamma_{j}} (\rho_{j} \circ f)^{-1}([0, \epsilon_{j,l}/2]) - \bigcup_{i < i_1} \rho_{i,j_1}^{-1}([0, \epsilon_{i,j_1,l_1}/2])$$

for $l_1 \in \Gamma_{j_1}$ and $l_2 \in \Gamma_{j_2}$.

... Then the generalization is clear. We omit the details.

If we undo the assumption that the connected components of $Y_j$ are bounded, the generalization becomes complicated. See [S] for it. We need not consider this case in the present paper by the following lemma.

**Lemma 2.** In the theorem, we can assume that each connected component of $Y_j$ is bounded in $\mathbb{R}^n$.

**Proof of Lemma 2.** In this paper we shall frequently shrink $|T_{i,j}|$ and $|T_{j}|$ without telling. Considering the unions of strata of same dimensions, we assume $\dim Y_j = j$, $j = 0, \ldots, k$, only now. It is easy to construct a $C^\infty$ proper function $\xi$ on $\mathbb{R}^n$ such that for each $y \in Y_j$, $\xi$ is constant on $\pi_{j}^{-1}(y)$, $\mathbb{Z} + [-1/3, 1/3] = \cup_{z \in \mathbb{Z}} [z-1/3, z+1/3]$ is common $C^\infty$ regular values of all $\xi$ and $\xi|_{Y_j}$, $j \neq 0$, and $\xi(Y_0) \cap (\mathbb{Z} + [-1/3, 1/3]) = \emptyset$. Set

$$Y_{j}' = Y_j - \xi^{-1}(\mathbb{Z}) \quad \text{and} \quad Y_{j}'' = Y_{j+1} \cap \xi^{-1}(\mathbb{Z}).$$

Clearly $\{Y_{j}', Y_{j}''\}$ is a $C^\infty$ stratification of $Y$ such that the connected components of the strata are bounded in $\mathbb{R}^n$ and each $Y_j$ is the union of $Y_{j}'$ and $Y_{j-1}''$. We want to construct a strongly controlled tube system $\{T_{j}' = ([|T_{j}'|], \pi_{j}', \rho_{j}'), T_{j}'' = ([|T_{j}''|], \pi_{j}'', \rho_{j}'')\}$ for $\{Y_{j}', Y_{j}''\}$.

Set

$$|T_{j}'| = |T_{j}| - \xi^{-1}(\mathbb{Z}), \quad \rho_{j}' = \rho_j \quad \text{on} \quad |T_{j}'| \quad \text{and}$$

$$|T_{j}''| = |T_{j+1}| \cap \xi^{-1}(\mathbb{Z} + [-1/3, 1/3]).$$

Let $\xi'$ be a $C^\infty$ function on $\mathbb{R}$ such that

$$\xi'(x) = (x - z)^2 \quad \text{on} \quad [z-1/3, z+1/3] \quad \text{for each} \quad z \in \mathbb{Z}.$$  

Set

$$\rho_{j}'' = \rho_{j+1} + \xi' \circ \xi \quad \text{on} \quad |T_{j''}|.$$  

For the moment, set $\pi_j' = \pi_j$, which we need to modify.
We want to define $\pi''_j$ first on $Y_{j+1} \cap |T''_j|$ so that for $j < j'$,
\[
\begin{align*}
\pi''_j \circ \pi_{j+1} &= \pi_{j+1} \circ \pi''_j, \\
\rho''_j \circ \pi''_j &= \rho''_j
\end{align*}
\] on $Y_{j'+1} \cap |T''_j|$. 

Shrink $|T''_j|$ sufficiently. Assume that there exist a vector field $\{v_{j+1}\}$ on $\{Y_{j+1} \cap |T''_j|\}$ such that $v_{j+1} \xi = 1$, and for $j < j'$,

\[
\begin{align*}
d\pi_{j+1}v_{j'+1} &= v_{j+1} + \pi_{j+1}(y) \\
\rho''_jv_{j'+1} &= 0
\end{align*}
\]

for $y \in Y_{j'+1} \cap |T''_j|$. 

Define $\pi''_j$ on $Y_{j+1} \cap |T''_j|$ so that $\{\pi''_{j-1}(y)\}_{y \in Y_j''}$ is the integral curves of $v_{j+1}$. Then $\pi''_j$ satisfies the required properties. Extend $\pi''_j$ to $|T''_j|$ by setting $\pi''_j = \pi''_j \circ \pi_{j+1}$. Then it is easy to see that $\{T''_j\}$ is a strongly controlled tube system for $\{Y_j''\}$, and for $j < j'$, the former equality of ct$(T''_j, T''_{j'})$ and (sc) for $(\pi''_j, \rho''_j)|Y_{j'} \cap |T''_j|$ hold.

We now construct $v_j$. Since $\xi|_{Y_1}$ is $C^\infty$ regular at $Y_1 \cap \xi^{-1}(Z)$, there clearly exists $v_1$. Assume that we have already constructed $v_j$ for all $j < k$. It suffices to construct $v_k$. Moreover, consider the following downward induction. Let $l < k$ be a nonnegative integer. Assume we have defined $v_k$ on $Y_k \cap |T''_{k-1}| \cap (\cup_{l < j < k} |T''_j|)$ so that $cv'(l + 1, k)$ hold on $Y_k \cap |T''_{k-1}| \cap |T''_j|$ for all $j$ with $l < j < k-1$. Then it suffices to extend $v_k$ to $Y_k \cap |T''_{k-1}| \cap |T''_j|$ so that $cv'(l + 1, k)$ holds on $Y_k \cap |T''_{k-1}| \cap |T''_j|$, because we easily extend $v_k$ defined on $Y_k \cap |T''_{k-1}| \cap (\cup_{l < j < k-1} |T''_j|)$ to $Y_k \cap |T''_{k-1}|$ by using a $C^\infty$ partition of unity.

Note that $cv'(l + 1, k)$ for $v_k$ holds on $Y_k \cap |T''_{k-1}| \cap |T''_j| \cap (\cup_{l < j < k-1} |T''_j|)$. Indeed, the former equality follows from ct$(T_{l+1}, T_{j+1})$, cv'(j + 1, k) and cv'(l + 1, j + 1), and we have

\[
\begin{align*}
d\rho''_jv_{ky} &= d\rho_{l+1}v_{ky} + d(\xi' \circ \xi)v_{ky} \\
&= d\rho_{l+1} \circ d\pi_{j+1}v_{ky} + d(\xi' \circ \xi) \circ d\pi_{j+1}v_{ky} \\
&= d\rho''_j \circ d\pi_{j+1}v_{ky} = d\rho''_j v_{j+1} + \pi_{j+1}(y) = 0
\end{align*}
\]

for $y \in Y_k \cap |T''_{k-1}| \cap |T''_j|$, $l < j < k - 1$.

Forget $T''_j$, $l < j < k - 1$, and consider only $T''_j$. For sufficiently small $|T''_{k-1}|$, the map $(\pi''_{l+1}, \rho''_{l+1})|Y_k \cap |T''_{k-1}| \cap |T''_j|$ is a $C^\infty$ submersion into $Y_{l+1} \times \mathbb{R}$. Hence we have a $C^\infty$ vector field $v_k|_{Y_k \cap |T''_{k-1}| \cap |T''_j|}$ such that $v_k \xi = 1$ and $cv'(l + 1, k)$ holds. Consequently, pasting $v_k$ and $v_k|_{Y_k}$ by a partition of unity, we can extend $v_k$ to $Y_k \cap |T''_{k-1}| \cap |T''_j|$. To be precise, let $\theta$ be a $C^\infty$ function on $Y_k$ such that $0 \leq \theta \leq 1$, $\theta = 1$ outside $Y_k \cap (a$ sufficiently small neighborhood of $\cup_{l < j < k-1} Y''_j$ in $\mathbb{R}^n$) and $\theta = 0$ on $Y_k \cap (a$ smaller one). Shrink $|T''_j|, l \leq j < k - 1$. Define $v_k$ to be $\theta v_k + (1 - \theta) v_k$ on $Y_k \cap |T''_{k-1}| \cap |T''_j| \cap (\cup_{l < j < k-1} |T''_j|)$, $v_k|_{Y_k \cap |T''_{k-1}| \cap |T''_j|}$ $- (\cup_{l < j < k-1} |T''_j|)$ and $v_k|_{Y_k \cap |T''_{k-1}| \cap (\cup_{l < j < k-1} |T''_j|) - |T''_j|}$. Then $v_k$ satisfies the required conditions. Thus we obtain $\{T''_j\}$. 

It is easy to see that for \( j < j' \), \( \text{ct}(T'_j, T'_j') \), the former equality of \( \text{ct}(T''_j, T'_j') \), (sc) and the conditions of a tube system hold. If \( j + 1 < j' \), then the latter of \( \text{ct}(T'_j, T'_j') \) also holds because

\[
\rho''_j \circ \pi_{j'} = \rho''_j \circ \pi_{j'} = \rho_{j+1} \circ \pi_{j'} + \xi' \circ \xi \circ \pi_{j'},
\]

\[
= \rho_{j+1} + \xi' \circ \xi = \rho''_j \text{ on } |T''_j| \cap |T'_j'|.
\]

But the latter of \( \text{ct}(T''_j, T''_j') \) is not correct. (We need not consider \( \text{ct}(T''_j, T''_j') \) because we can choose \( |T'_j| \) and \( |T'_j'| \) so that they do not intersect.) We modify \( \pi'_{j+1} \) so that this holds as follows.

Shrinking \( |T_{j+1}| \) we assume \( \rho_{j+1} \leq 1 \). Let \( \overline{V_1} \subset V_2 \) be small open neighborhoods of \( Y''_j \times \mathbb{Z} \times 0 \) in \( Y''_j \times \mathbb{R} \times [0,1] \) such that \( \overline{V_1} \subset V_2 \), and the image of \( \overline{V_2} \) under the projection \( Y''_j \times \mathbb{R} \to Y''_j \times \mathbb{R} \) is contained in \( (\pi'', \xi)(Y_{j+1} \cap |T''_j|) \). Let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) be a \( C^\infty \) diffeomorphism of \( Y''_j \times (\mathbb{R} - \mathbb{Z}) \times [0,1] \) such that

\[
\alpha = \text{id} \text{ on } Y''_j \times (\mathbb{R} - \mathbb{Z}) \times 0,
\]

\[
\alpha_1(y, s, t) = y, \quad \alpha_3(y, s, t) = t, \quad \text{and}
\]

\[
\alpha_2(y, s, t) = \begin{cases} 
\pm((s-z)^2 + t)^{1/2} + z & \text{on } V_1 \cap (Y''_j \times ([z-1/3, z+1/3] - z) \times [0,1]), \ z \in \mathbb{Z} \\
\text{outside } V_2, 
\end{cases}
\]

whose existence is easily shown if \( V_1 \) is sufficiently small.

Modify \( \pi'_{j+1} \) to be

\[
((\pi'', \xi)|Y_{j+1} \cap |T''_j|)^{-1} \circ (\alpha_1, \alpha_2) \circ (\pi'', \xi, \rho_{j+1}) \text{ on } |T''_j| \cap |T''_j'|,
\]

and do not change \( \pi'_{j+1} \) on \( |T''_{j+1}| - |T''_j'| \). Then it is clear that \( \{T''_j\} \) is a tube system and (sc) is satisfied. Note that \( \pi'_{j+1} \) does not change outside \( (\pi'', \xi, \rho_{j+1})^{-1}(V_2) \).

Hence \( \text{ct}(T''_{j+1}, T''_{j+1}) \) can hold for any \( j' < j \) because we can choose small \( V_2 \) and shrink \( |T''_{j+1}| \) so that \( (\pi'', \xi, \rho_{j+1})^{-1}(V_2) \) and \( |T''_{j+1}| \) do not intersect.

Moreover, we have

\[
\pi''_j \circ \pi'_{j+1} = \pi''_j \circ ((\pi'', \xi)|Y_{j+1} \cap |T''_j|)^{-1} \circ (\alpha_1, \alpha_2) \circ (\pi'', \xi, \rho_{j+1})
\]

\[
= \alpha_1 \circ (\pi'', \xi, \rho_{j+1}) = \pi''_j
\]

on \( (\pi'', \xi, \rho_{j+1})^{-1}(V_1) \cap \xi^{-1}([z-1/3, z+1/3] - z), \ z \in \mathbb{Z}, \)

and

\[
\rho''_j \circ \pi'_{j+1} = \rho_j \circ ((\pi'', \xi)|Y_{j+1} \cap |T''_j|)^{-1} \circ (\alpha_1, \alpha_2) \circ (\pi'', \xi, \rho_{j+1})
\]

\[
+ \xi' \circ \xi \circ ((\pi'', \xi)|Y_{j+1} \cap |T''_j|)^{-1} \circ (\alpha_1, \alpha_2) \circ (\pi'', \xi, \rho_{j+1})
\]

\[
= 0 + \xi' \circ \alpha_2 \circ (\pi'', \xi, \rho_{j+1}) = (\xi - z)^2 + \rho_{j+1} = \rho''_j
\]

on the same domain.

Therefore, if we shrink \( |T''_j| \), \( \text{ct}(T''_j, T''_{j+1}) \) holds.

If \( j' < j \), \( \text{ct}(T''_j, T''_{j+1}) \) continues to hold. Indeed, this is clear on \( |T''_j| \cap |T''_{j+1}| - (\pi'', \xi, \rho_{j+1})^{-1}(V_2) \). Shrink \( |T_{j+1}| \) and \( |T''_j| \) so that \( |T''_{j+1}| \cap |T''_j| \subset (\pi'', \xi, \rho_{j+1})^{-1}(V_1) \). Then, on \( |T''_j| \cap |T''_{j+1}| \cap (\pi'', \xi, \rho_{j+1})^{-1}(V_2) \), we have

\[
((\pi'', \rho'_{j'}) \circ \pi'_{j+1} = ((\pi'', \rho'_{j'}) \circ \pi''_j) \circ \pi'_{j+1}
\]

\[
= (\pi'', \rho'_{j'}) \circ (\pi''_j \circ \pi'_{j+1}) = (\pi'', \rho'_{j'}) \circ \pi''_j = (\pi'', \rho''_j).
\]
Thus a strongly controlled tube system \( \{T'_j, T''_j\} \) is constructed.

From now on we remove the assumption \( \dim Y_j = j \), and we change the definition of \( Y_j'' 
\)
 of \( \{\tau_{i,j}'\} \) for \( \{X_{i,j}'\} \) over \( \{Y_j''\} \) for \( \{X_{i,j}''\} \) strongly controlled over \( \{T_j', T''_j\} \). Let \( f \) denote the extension of \( f \)
in condition (scl1) of strong controlledness. Set

\[
|T_{i,j}'| = |T_{i,j}| - (\xi \circ f)^{-1}(Z), \quad |T_{i,j}''| = |T_{i,j}| \cap (\xi \circ f)^{-1}(Z + [-1/3, 1/3]),
\]

\[
\begin{align*}
\pi_{i,j}' &= \pi_{i,j} \quad \text{on } |T_{i,j}'|, \\
\rho_{i,j}' &= \rho_{i,j} \quad \text{on } |T_{i,j}'|,
\end{align*}
\]

\[
\rho_{i,j}'' = \rho_{i,j} + \xi \circ f \quad \text{on } |T_{i,j}''|.
\]

The definition of \( \pi_{i,j}'' \) is similar to that of \( \pi_{i,j}' \) as follows. Shrink \( |T_{i,j}''| \) sufficiently.

Then there exist \( C^\infty \) imbeddings

\[
\theta_{i,j} : X_{i,j} \cap |T_{i,j}''| \longrightarrow X_{i,j}'' \times \mathbb{R}
\]

of the form \( (\theta_{i,j}', \xi \circ f) \) such that

\[
\theta_{i,j}' = \text{id} \quad \text{on } X_{i,j}'', \quad \theta_{i,j} = \text{id} \quad \text{on } X_{i,j}'' \cap \tau_{i,j}'', \quad \theta_{i,j}'' = \text{id} \quad \text{on } X_{i,j}' \cap \tau_{i,j}'', \\
\theta_{i,j}'' = \text{id} \quad \text{on } X_{i,j}'' \cap \tau_{i,j}'',
\]

\[
\rho_{i,j}'' = \rho_{i,j} + \xi \circ f \quad \text{on } |T_{i,j}''|.
\]

Set \( \pi_{i,j}'' = \theta_{i,j}' \) on \( X_{i,j} \cap \tau_{i,j}'' \), and extend it to \( |T_{i,j}''| \) by setting \( \pi_{i,j}'' = \pi_{i,j}' \circ \pi_{i,j} \).

The tube system \( \{T_{i,j}', T_{i,j}''\} \) satisfies the required conditions except that

\[
f \circ \pi_{i,j}' = \pi_j' \circ f \quad \text{on } |T_{i,j}'|.
\]

But we can modify \( \pi_{i,j}' \), so that this equality holds in the same way that we did \( \pi_{i,j}' \).

We omit the details. Thus we prove the lemma. \( \square \)

**Lemma 3** (I.3.2 in [G-al] and its proof). Let \( X, Y, \{X_{i,j}\}, \{Y_j\}, f : X \rightarrow Y \), \( \{T_{i,j}\} \) and \( \{T_j\} \) be the same as in the theorem except for the assumption that \( f \)
is proper. Assume \( \dim Y_1 < \dim Y_j \) for \( j \neq 1 \).

Given a \( C^\infty \) vector field \( v_1 \) on \( Y_1 \), there exists a controlled vector field on \( \{Y_j\} \)
which is an extension of \( v_1 \).

Given a weakly controlled vector field \( v^Y = \{v_j\} \) on \( \{Y_j\} \) and a vector field \( \{v_{i,1}\}_i \) on \( \{X_{i,1}\}_i \) controlled over \( \{v_1\} \), there exists a vector field on \( \{X_{i,j}\}_i,j \) which is an extension of \( \{v_{i,1}\}_i \) and controlled over \( v^Y \).

[G-al] treats only Thom maps. But the proof works in our situation. See [S].

**Lemma 4** (I.4.6 in [G-al]). In the same situation as in Lemma 3, a controlled vector field on \( \{Y_j\} \) and a vector field on \( \{X_{i,j}\} \) controlled over a locally integrable vector field on \( \{Y_j\} \) are locally integrable.
§4. Proof of the theorem

Proof of the theorem. Assume \( \dim Y_j < \dim Y_{j+1} \) and \( \dim X_{i,j} < X_{i+1,j} \). Let the sets of indexes \( H \) and \( \overline{H} \) and an order in \( H \) and \( \overline{H} \) be given as in §3. By Lemma 2 we can assume that each connected component of \( Y_j \) is bounded in \( \mathbb{R}^n \). But, only for simplicity of notations, we assume, moreover, that \( Y \) is compact. The following arguments work in the noncompact case. (See a generalization of the definition of a removal data in §3.) Let a removal data \( \varepsilon = \{ \varepsilon_{i,j} \}_{(i,j) \in H} \) of \( \{ T_i, T_{i,j} \}_{(i,j) \in \overline{H}} \) be fixed. Set \( \varepsilon_{k,j} = \varepsilon_j \).

Set

\[
Y_j^\varepsilon = Y_j - \bigcup_{l < j} \rho_{l_l}^{-1}([0, \varepsilon_l]), \quad j = 1, \ldots, k,
\]

which are compact \( C^\infty \) manifolds possibly with corners. We want \( C^\infty \) triangulations \( (L_j, g_j) \) of \( Y_j^\varepsilon \) such that for \( j < j' \), the restriction of \( g_j^{-1} \circ \pi_j \circ g_{j'} \) to a neighborhood of \( g_j^{-1}(\rho_j^{-1}(\varepsilon_j)) \) in \( |L_{j'}| \) is a PL map to \( |L_j| \). We call the property PL\((j, j')\). (Proposition I.3.20 in [S] shows the existence. But we repeat the proof because we shall use the idea.)

We construct the triangulations by induction. If we apply Lemma 1 to the constant map \( Y_1^\varepsilon \to 0 \), existence of \((L_1, g_1)\) follows. Let \( 1 \leq l_1 < l_2 \leq k \) be integers. Assume we have constructed \((L_j, g_j)\) for all \( j \) with \( j < l_2 \) and a \( C^\infty \) triangulation \((L_{l_2}, g_{l_2})\) of a neighborhood of \( Y_{l_2}^\varepsilon \cap (\bigcup_{l_1 < j < l_2} \rho_{j-1}^{-1}(\varepsilon_j)) \) in \( Y_{l_2}^\varepsilon \) with PL\((j, l_2)\) for all \( j \) with \( l_1 < j < l_2 \). Then shrinking the neighborhood we need to extend \((L_{l_2}, g_{l_2})\) to a \( C^\infty \) triangulation of a neighborhood of \( Y_{l_2}^\varepsilon \cap (\bigcup_{l_1 < j < l_2} \rho_{j-1}^{-1}(\varepsilon_j)) \) with PL\((l_1, l_2)\). Let \( l_1 < j < l_2 \). By PL\((l_1, l_1), PL(j, l_2)\) and \( \mathrm{ct}(T_{l_1}, T_j) \), the restriction of \( g_{l_1}^{-1} \circ \pi_{l_1} \circ g_{l_2} \) to a neighborhood of \( g_{l_1}^{-1}(\rho_{l_1}^{-1}(\varepsilon_{l_1})) \cap \rho_{j-1}^{-1}(\varepsilon_j) \) in \( |L_{l_1}| \) is a PL map to \( |L_{l_1}| \). Note that \( Y_{l_2}^\varepsilon \cap \rho_{l_1}^{-1}(\varepsilon_{l_1}) \) is a disjoint union of faces of \( Y_{l_2}^\varepsilon \), and \( Y_{l_2}^\varepsilon \cap \rho_{l_1}^{-1}(\varepsilon_{l_1}) \cap (\bigcup_{l_1 < j < l_2} \rho_{j-1}^{-1}(\varepsilon_j)) \) is a union of subfaces of \( Y_{l_2}^\varepsilon \cap \rho_{l_1}^{-1}(\varepsilon_{l_1}) \). Hence by Lemma 1 we can extend \((L_{l_2}, g_{l_2})\) as required. Thus we have a \( C^\infty \) triangulation \((L_{l_2}, g_{l_2})\) of a neighborhood of \( \partial Y_{l_2}^\varepsilon \) in \( Y_{l_2}^\varepsilon \) with PL\((j, l_2)\) for all \( j < l_2 \). A further extension to whole \( Y_k^\varepsilon \) follows from Lemma 1 applied to the map \( Y_k^\varepsilon \to 0 \). Therefore, there exist \((L_j, g_j)\), \( j = 1, \ldots, k \).

Note that for \( 1 \leq j < j' \leq k \), \( g_{j'}^{-1}(\rho_{j'}^{-1}(\varepsilon_{j'})) \) is the underlying polyhedron of a subcomplex of \( L_j' \). For a simplicial complex \( K \), let \( K' \) and \( \hat{K} \) denote the barycentric and some subdivisions of \( K \) respectively.

Set

\[
Y_j^+ = Y - \bigcup_{l < j} \rho_l^{-1}([0, \varepsilon_l]), \quad j = 1, \ldots, k.
\]

Note that

\[
Y_1^+ = Y, \quad Y_k^+ = Y_k^\varepsilon \quad \text{and} \quad Y_j^+ = Y_{j+1}^+ \cup (Y_j^+ \cap \rho_j^{-1}([0, \varepsilon_j])), \quad j = 1, \ldots, k - 1.
\]

We want to construct (not necessarily \( C^\infty \)) triangulations \((L_j^+, g_j^+)\) of \( Y_j^+ \) such that for \( 1 \leq j < j' \leq k \), \( g_{j'}^{-1}(\rho_{j'}^{-1}(\varepsilon_{j'})) \) is the underlying polyhedron of some subcomplex
$L^+_j(j)$ of $L^+_j$, the map $\alpha^+_j(j): |L^+_j(j)| \to |L_j|$ is PL,

$$L^+_j = (\hat{L}^+_j) \cup C_{\alpha^+_j(j)}(\hat{L}^+_j(j), \hat{L}_j),$$

$$\hat{L}^+_j((j')) = (\hat{L}^+_j) \cap C_{\alpha^+_j(j)}(\hat{L}^+_j((j)), \hat{L}_j),$$

$$g^+_j|_{L^+_j(j)} = g^+_j$$

where

$$\alpha^+_j(j) = g^+_j \circ \pi_j \circ (g^+_j|_{L^+_j(j)}).$$

(This is shown in the proof of Corollary I.3.21 in [S]. We shall need the same procedure.)

We define $(L^+_j, g^+_j)$ by downward induction on $j$. Clearly we set $L^+_k = L_k$ and $g^+_k = g_k$. Let $1 \leq j < k$ be an integer, and assume $(L^+_j, g^+_j)$. Set

$$g^+_j = \begin{cases} g^+_j+1 & \text{on } |L^+_j+1| \\ g_j & \text{on } |L_j|. \end{cases}$$

We need to subdivide $L^+_j+1$ and $L_j$ so that $\alpha^+_j(j): \hat{L}^+_j(j) \to \hat{L}^+_j$ is a simplicial map and then to extend $g^+_j$ to $C_{\alpha^+_j(j)}(|L^+_j(j)|, |L_j|)$. The former requirement is clearly fulfilled since $\alpha^+_j(j)$ is PL. For the latter it suffices to find a homeomorphism $\theta^+_j: Y^+_j \cap \rho^+_j(0, \epsilon_j) \to (Y^+_j \cap \rho^+_j(\epsilon_j)) \times [0, \epsilon_j]$ of the form $(\theta^+_j, \rho^+_j)$ such that $\pi_j \circ \theta^+_j = \pi^+_j$ and $\theta^+_j = \text{id}$ on $Y^+_j \cap \rho^+_j(\epsilon_j)$. Indeed, by such $\theta^+_j$ we can identify $Y^+_j \cap \rho^+_j(0, \epsilon_j)$ with $C_{\pi_j}(Y^+_j \cap \rho^+_j(\epsilon_j), Y^+_j)$, and we can naturally extend $g^+_j$ to $C_{\alpha^+_j(j)}(|L^+_j(j)|, |L_j|)$. It is clear by ct($T', T$), PL($j', j$) for $j' < j$ and by the properties of a mapping cylinder that $(L^+_j, \text{the extension})$ satisfies all the requirements.

Existence of $\theta^+_j$ immediately follows if we apply Thom’s Second Isotopy Lemma to the sequence of maps $Y^+_j \cap \rho^+_j(0, \epsilon_j) \quad (\pi_j, \rho_j^+) \quad \pi_j(Y^+_j \cap \rho^+_j(\epsilon_j)) \times [0, \epsilon_j] \quad \text{proj} \quad [0, \epsilon_j]$.

(Note that $\pi_j(Y^+_j \cap \rho^+_j(\epsilon_j))$ does not necessarily coincide with $Y^+_j$. We will show a more precise construction of $\theta^+_j$ later because we need another additional property.) Thus we have the required $(L^+_j, g^+_j)$.

Set

$$X^e_{i,j} = X_{i,j} - \bigcup_{j' < j} (\rho_{j'} \circ f)^{-1}([0, \epsilon_{j'}]) - \bigcup_{\nu < i} \rho^{-1}_{\nu,j}([0, \epsilon_{\nu,j}])$$

for $(i, j) \in \overline{H}$,

which also are compact $C^\infty$ manifolds possibly with corners. We will construct $C^\infty$ triangulations $(K_{i,j}, h_{i,j})$ of $X^e_{i,j}$ with the following three properties. (1) For $(i, j) \in \overline{H}$, the map $g^{-1}_j \circ f \circ h_{i,j}: [K_{i,j}] \to |L_j|$ is PL. Let $(i_1, j_1) < (i_2, j_2) \in \overline{H}$. (2) If $j_1 < j_2$, the restriction of $h^{-1}_{i_1,j_1} \circ \pi_{i_1,j_1} \circ h_{i_2,j_2}$ to a neighborhood of $h_{i_2,j_2}((\rho_{j_1} \circ f)^{-1}(\epsilon_{j_1}) \cap \rho^{-1}_{i_1,j_1}(0, \epsilon_{i_1,j_1}) \cap \pi_{i_1,j_1}(X^e_{i_1,j_1}))$ in $h^{-1}_{i_2,j_2}(\rho^{-1}_{i_1,j_1}(0, \epsilon_{i_1,j_1}) \cap \pi_{i_1,j_1}(X^e_{i_1,j_1}))$ is a PL map to $|K_{i_1,j_1}|$. (Here we ignore $\rho^{-1}_{i_1,j_1}(0, \epsilon_{i_1,j_1})$ if $i_1 = k_{j_1}$.) (3) If $j_1 = j_2$,
the restriction of $h_{i_2,j_2}^{-1} \circ \pi_{i_1,j_1} \circ h_{i_2,j_2}$ to a neighborhood of $h_{i_2,j_2}^{-1}(\rho_{i_1,j_1}(e_{i_1,j_1}))$ in $|K_{i_2,j_2}|$ is a PL map to $|K_{i_1,j_1}|$.

As in the case of $Y^\varepsilon$, we construct them by induction. Existence of $(K_{1,1}, h_{1,1})$ with (1) is clear by Lemma 1. Let $(i_1,j_1) < (i_2,j_2) \in \overline{H}$. Assume we have $(K_{i,j}, h_{i,j})$ for all $(i,j) < (i_2,j_2)$ and a $C^\infty$ triangulation $(K_{i_2,j_2}, h_{i_2,j_2})$ of the following set with property (1) for $(i_2,j_2)$, (2) for any pair $(i',j') < (i_2,j_2)$ with $(i',j') > (i_1,j_1)$ and (3) for any pair $(i',j_2) < (i_2,j_2)$ with $(i',j_2) > (i_1,j_1)$:

$$\bigcup_{(i,j) > (i_1,j_1), j < j_2} \left( \text{a neighborhood of } X_{i_2,j_2}^\varepsilon \cap (\rho_j \circ f)^{-1}(e_j) \cap \pi_{i,j}^{-1}(X_{i,j}^\varepsilon) \right)$$

in $X_{i_2,j_2}^\varepsilon \cap \pi_{i,j}^{-1}(X_{i,j}^\varepsilon)$

$$\bigcup_{(i_1,j_1) < (i',j') < (i_2,j_2)} \left( \text{a neighborhood of } X_{i_2,j_2}^\varepsilon \cap \rho_{i',j_2}^{-1}(e_{i',j_2}) \in X_{i_2,j_2}^\varepsilon \right).$$

We call such $(K_{i_2,j_2}, h_{i_2,j_2})$ a $C^\infty$ triangulation of $R(i_2,j_2,i_1',j_1')$, where $(i_1',j_1')$ denotes the minimum of the elements of $\overline{H}$ greater than $(i_1,j_1)$. We extend $(K_{i_2,j_2}, h_{i_2,j_2})$ to a $C^\infty$ triangulation of $R(i_2,j_2,i_1,j_1)$. Let $e_{j_1} > e_j$ be a number sufficiently close to $e_{j_1}$.

There are four possible cases: (i) $j_1 = j_2$, (ii) $j_1 < j_2$ and $i_1 = k_{j_1}$, (iii) $j_1 < j_2$, $i_1 < k_{j_1}$ and $i_2 = 1$ or (iv) $j_1 < j_2$, $i_1 < k_{j_2}$ and $i_2 > 1$. In case (i), the arguments on the extension are the same as in the case of $Y^\varepsilon$, because we do not need consider (2) and because (1) follows from (1) for $(i_1,j_1)$ and (3).

Assume (ii). We easily see the following three facts. First the fibre product $|K_{i_1,j_1} \times_{(f,o_{h_{1,j_1}, \pi_{j_1}, e_{j_1}})} g_{j_2}^{-1}(\rho_{i_2,j_2}(e_{j_1}, e'_{j_1}))|$ is a polyhedron. (We treat not $g_{j_2}^{-1}(\rho_{i_2,j_2}^{-1}([e_{j_1}, e'_{j_1}]))$, because $g_{j_2}^{-1}(\rho_{i_2,j_2}^{-1}([e_{j_1}, e'_{j_1}]))$ is not always a polyhedron. But $g^{-1}_{j_2}(\rho_{j_2}^{-1}([e_{j_1}, e'_{j_1}]))$ is non-compact and hence does not admit a finite simplicial decomposition.) Second, the restriction of the map $(h_{i_1,j_1}, g_{j_2})$ to some simplicial complex whose underlying polyhedron is this polyhedron is a $C^\infty$ triangulation of the fibre product $X_{i_1,j_1}^\varepsilon \times_{(f,o_{\pi_{j_1}})} Y_{j_2}^\varepsilon \cap \rho_{j_1}^{-1}([e_{j_1}, e'_{j_1}])$, which is a $C^\infty$ manifold possibly with corners. Third, the restriction of $(\pi_{i_1,j_1}, f)$ to $X_{i_1,j_1}^\varepsilon \cap (\rho_{j_1} \circ f)^{-1}([e_{j_1}, e'_{j_1}]) \cap \pi_{i_1,j_1}^{-1}(X_{i_1,j_1}^\varepsilon)$ is a $C^\infty$ submersion onto a union of some connected components of the preceding manifold possibly with corners and, moreover, satisfies the conditions in Lemma 1. (Lemma 1 treats only compact sets, and the present sets are not compact. But the problem is only around the compact set $X_{i_1,j_1}^\varepsilon \cap (\rho_{j_1} \circ f)^{-1}([e_{j_1}, e'_{j_1}]) \cap \pi_{i_1,j_1}^{-1}(X_{i_1,j_1}^\varepsilon)$. Hence Lemma 1 is applicable.) Therefore, an extension of $(K_{i_2,j_2}, h_{i_2,j_2})$ to a $C^\infty$ triangulation of $R(i_2,j_2,i_1,j_1)$ is possible.

Assume (iii) or (iv). In these cases, the preceding arguments do not work. Indeed, the given $(K_{i_2,j_2}, h_{i_2,j_2})$ defines only a $C^\infty$ triangulation of a neighborhood of $X_{i_2,j_2}^\varepsilon \cap (\rho_{j_1} \circ f)^{-1}([e_{j_1}, e'_{j_1}]) \cap \pi_{i_1,j_1}^{-1}(X_{i_1,j_1}^\varepsilon)$ in $X_{i_2,j_2}^\varepsilon \cap \pi_{i_1,j_1}^{-1}(X_{i_1,j_1}^\varepsilon)$ and $\rho_{i_1,j_1}^{-1}(e_{i_1,j_1})$, but for application of Lemma 1 in the preceding way, what is necessary is a $C^\infty$ triangulation of a neighborhood of the same set in $X_{i_2,j_2}^\varepsilon \cap \pi_{i_1,j_1}^{-1}(X_{i_1,j_1}^\varepsilon)$ and $\rho_{i_1,j_1}^{-1}([0, e_{i_1,j_1}])$. Hence we need such an extension of the $C^\infty$ triangulation.

To be precise, set

$$M = X_{i_2,j_2}^\varepsilon \cap (\rho_{j_1} \circ f)^{-1}([e_{j_1}, e'_{j_1}]) \cap \pi_{i_1,j_1}^{-1}(X_{i_1,j_1}^\varepsilon) \cap \rho_{i_1,j_1}^{-1}([0, e_{i_1,j_1}]),$$
which is a $C^\infty$ manifold possibly with corners. Then we have

$$\partial M = A \cup B \cup C \cup D,$$

where

$$A = M \cap (\rho_{j_1} \circ f)^{-1}(\varepsilon_{j_1}), \quad B = M \cap \left( \bigcup_{i<i_2} \rho_{i_j}^{-1}(\varepsilon_{i,j_2}) \right),$$

$$C = M \cap \rho_{i_1,j_1}^{-1}(\varepsilon_{i_1,j_1}),$$

and

$$D = M \cap \left( \bigcup_{i<i_1} \rho_{i_1,j_1}^{-1}(\varepsilon_{i_1,j_1}) \right).$$

$h_{i_2,j_2}(M)$ is the intersection of the open neighborhood $h_{i_2,j_2}^{-1}((\rho_{j_1} \circ f)^{-1}(\varepsilon_{j_1}))$ of $h_{i_2,j_2}^{-1}(A)$ in $|K_{i_2,j_2}|$ and the closed polyhedron $h_{i_2,j_2}^{-1}(X_{i_2}^{\varepsilon_{i_2,j_2}} \cap \rho_{i_1,j_1}^{-1}([0,\varepsilon_{i_1,j_1}]))$, and $M \cap \text{Im} h_{i_2,j_2}$ is the union of $C$ and a closed neighborhood $U$ of $B$ in $M$. Hence $(K_{i_2,j_2}, h_{i_2,j_2})$ induces a $C^\infty$ triangulation, say, $(K, h)$ for simplicity of notation, of $U \cup C$, which equals $(K_{i_2,j_2}, h_{i_2,j_2})$ around $h_{i_2,j_2}^{-1}(A)$. Shrinking $U$, we need to extend $(K, h)$ to a $C^\infty$ triangulation of $U \cup \{ \text{a neighborhood of } A \cap C \text{ in } M \}$.

Assume (iii). Then $B = \emptyset$. Hence the extension follows from the following note, which is clear by condition (6) of a removal data of $\{T_{i,j}\}$.

Note: There exists a $C^\infty$ diffeomorphism $\theta: M \cap \rho_{i_1,j_1}^{-1}([\varepsilon_{i_1,j_1}/2, \varepsilon_{i_1,j_1}]) \to C \times [\varepsilon_{i_1,j_1}/2, \varepsilon_{i_1,j_1}]$ of the form $(\theta^*, \rho_{i_1,j_1})$ with $\pi_{i_1,j_1} \circ \theta^* = \pi_{i_1,j_1}$ and $f \circ \theta^* = f$.

Case (iv) remains. The situation is more complicated. The note is not sufficient. Indeed, $(K, h)$ would change if we used only the note, since $B \neq \emptyset$. Given a subset $E$ of $M$ such that $h^{-1}(E)$ is the underlying polyhedron of some subcomplex of $K$, let $K_E$ denote the subcomplex by abuse of notation. We can assume that the closure of the interior $U^0$ of $U$ as a subset of $M$ coincides with $U$, and $|N(K_B, K)|$ does not intersect with the boundary of $|K_U|$ as a subset of $|K|$. Let $a > 1$ be a number close to 1. Let $\beta$ be the simplicial function on $K$ defined by $\beta = a$ at the vertices $|K_A| \cap |K_B| = h^{-1}(U^0)$ and $\beta = 1$ at any other vertex. Clearly $\beta = 1$ on $|N(K_B, K)|$, and the polyhedron $\cup_{u \in |K_C|} [u \times [1, \beta(u)]]$ has a natural cell complex structure. Paste the barycentric subdivision of this cell complex with $K'$ by the identification of $|K_C| \times 1$ with $|K_C|$ in $|K|$. Let $\tilde{K}$ denote this simplicial complex.

We want to define a $C^\infty$ imbedding $\tilde{h}: \tilde{K} \to M$ so that $(\tilde{K}, \tilde{h})$ is the required $C^\infty$ triangulation. By $\theta$ in the note in case (iii), we can regard $(M, C)$ as $(C \times [\varepsilon_{i_1,j_1}/2, \varepsilon_{i_1,j_1}], C \times [\varepsilon_{i_1,j_1}])$, because the problem is only local around $C$. We call the latter pair $(C \times [0, 1], C \times 1)$ for simplicity of notation. Let $h$ be of the form $(h_1, h_2)$, where $h_1: |K| \to C$ and $h_2: |K| \to [0, 1]$. Set

$$\tilde{h} = \begin{cases} 
(h_1, (2 - \beta)h_2) & \text{on } |K| \\
(h_1(u), t + 1 - \beta(u)) & \text{for } u \in |K_C| \text{ and } t \in [1, \beta(u)].
\end{cases}$$

Note that $\tilde{h} = h$ on $|N(K_B, K)|$. Let $a$ be sufficiently close to 1. Then $\tilde{h}|_{K'}$ is a strong approximation of $h$. Hence by 8.8 in [M], $\tilde{h}|_{K'}$ is a $C^\infty$ imbedding. On the other hand, by the above definition of $\tilde{h}$, $\tilde{h}$ outside $K'$ also is a $C^\infty$ imbedding. Moreover, it is clear that $\tilde{h}$ is a $C^\infty$ triangulation of a neighborhood of $B \cup (A \cap C)$ in $M$.

In both cases of (iii) and (iv), we can extend $(K_{i_2,j_2}, h_{i_2,j_2})$ to a $C^\infty$ triangulation of $R(i_2, j_2, i_1, j_1)$ in the same way as in case of (ii). That completes the
induction step. Thus by induction we have a $C^\infty$ triangulation $(K_{i_2,j_2}, h_{i_2,j_2})$ of a neighborhood of $\partial X^\epsilon_{i_2,j_2}$ in $X^\epsilon_{i_2,j_2}$. Its further extension to a $C^\infty$ triangulation of $X^\epsilon_{i_2,j_2}$ with (1) follows if we apply Lemma 1 to the map $f|X^\epsilon_{i_2,j_2}: X^\epsilon_{i_2,j_2} \rightarrow Y^\epsilon_{j_2}$.

As in the case of $Y_j$, note the following property. Let $(i_1, j_1) < (i_2, j_2) \in \overline{H}$. The following set is the underlying polyhedron of some subcomplex of $K_{i_2,j_2}$:

$$h_{i_2,j_2}^{-1}(\rho_{i_1,j_1}^{-1}(\epsilon_{i_1,j_1})) \quad \text{if } j_1 = j_2,$$

$$h_{i_2,j_2}^{-1}((\rho_{j_1} \circ f)^{-1}(\epsilon_{j_1}) \cap \pi_{i_1,j_1}^{-1}(X^\epsilon_{i_1,j_1})) \quad \text{if } i_1 = k_{j_1} \text{ and } j_1 < j_2,$$

$$h_{i_2,j_2}^{-1}(((\rho_{j_1} \circ f)^{-1}(\epsilon_{j_1}) \cap \pi_{i_1,j_1}^{-1}(X^\epsilon_{i_1,j_1}) \cap \rho_{i_1,j_1}^{-1}(\epsilon_{i_1,j_1}))) \quad \text{otherwise}.$$  

For each $(i, j) \in \overline{H}$, set

$$N_{i,j} = \begin{cases} X \cap \pi_{i,j}^{-1}(X^\epsilon_{i,j}) \cap \rho_{i,j}^{-1}([0, \epsilon_{i,j}]) & \text{if } j = k, \\ X \cap (\rho_j \circ f)^{-1}([0, \epsilon_{j}]) \cap \pi_{i,j}^{-1}(X^\epsilon_{i,j}) & \text{if } i = k_j, j < k, \\ X \cap (\rho_j \circ f)^{-1}([0, \epsilon_{j}]) \cap \pi_{i,j}^{-1}(X^\epsilon_{i,j}) \cap \rho_{i,j}^{-1}([0, \epsilon_{i,j}]) & \text{otherwise,} \end{cases}$$

$$N'_{i,j} = N_{i,j} \cap \bigcup_{(i',j') > (i,j)} N_{i',j'} \quad \text{and} \quad X^+_i = \bigcup_{(i',j') \geq (i,j)} N_{i',j'}.$$  

Since $X^+_{1,1} = X$, the theorem follows if we can construct triangulations $(K^+_{i,j}, h^+_{i,j})$ of $X^+_{i,j}$ such that the following three conditions are satisfied. For $(i, j) \in \overline{H}$, $g^+_j \circ f \circ h^+_{i,j} : |K^+_{i,j}| \rightarrow |L^+_j|$ is PL. For $(i_1, j_1) < (i_2, j_2) \in \overline{H}$, $h^+_{i_2,j_2}(N_{i_1,j_1})$ is the underlying polyhedron of some subcomplex $K^+_{i_2,j_2}(i_1, j_1)$ of $K^+_{i_2,j_2}$, and the map $\alpha^+_{i_2,j_2}(i_1, j_1) : |K^+_{i_2,j_2}(i_1, j_1)| \rightarrow |K_{i_1,j_1}|$ is PL, where

$$\alpha^+_{i_2,j_2}(i_1, j_1) = h^+_{i_1,j_1} \circ \pi_{i_1,j_1} \circ (h^+_{i_2,j_2}|_{K^+_{i_2,j_2}(i_1, j_1)}).$$  

For $(i, j) \in H$, let $(i', j')$ denote the minimum of the elements of $\overline{H}$ greater than $(i, j)$. Then

$$K^+_{i,j} = (K^+_{i',j'}') \cup C^+_{\alpha^+_{i',j'}(i,j)}(K^+_{i',j'}(i,j), \hat{K}_{i,j}),$$

$$\hat{K}^+_{i',j'}(i,j)' = (K^+_{i',j'}') \cap C^+_{\alpha^+_{i',j'}(i,j)}(K^+_{i',j'}(i,j), \hat{K}_{i,j}),$$

$$h^+_{i,j}|_{K^+_{i',j'}(i,j)} = h^+_{i',j'} \quad \text{and} \quad h^+_{i,j}|_{K_{i,j}} = h_{i,j}.$$  

Here ' and $^*$ denote the barycentric and some subdivisions respectively.

We construct $(K^+_{i,j}, h^+_{i,j})$ by downward induction as $(L^+_j, g^+_j)$. Then by the same reason, it suffices to find a homeomorphism $\theta_{i,j}: N_{i,j} - X^\epsilon_{i,j} \rightarrow N'_{i,j} \times [0, 1]$ of the form $(\theta^*_{i,j}, \theta^*_{i,j})$ for each $(i, j) \in H$ such that

$$(a) \quad \theta^*_{i,j} = \text{id \ on } N'_{i,j}, \quad \pi_{i,j} = \pi_{i,j} \circ \theta^*_{i,j},$$

$$(b) \quad \rho_j \circ f = \theta^*_j \circ \rho_j \circ f \circ \theta^*_{i,j} \quad \text{if } j < k,$$

$$(c) \quad \theta^*_j \circ f = \theta^*_j \circ f \circ \theta^*_{i,j} \quad \text{on } N_{i,j} - (\rho_j \circ f)^{-1}(0) \quad \text{if } j < k.$$
If \( j = k \), \( \theta_{i,j} \) is constructed as \( \theta_{j} \). So assume \( j < k \). To distinguish elements of \( \overline{H} \), we call \((i, j)\) \((i, j_{0})\) and use the notation \((i, j)\) for a general element. Since the problem is local around \( N_{i_{0},j_{0}} \), we assume

\[
|T_{i,j}| \subset |T_{i_{0},j_{0}}| \quad \text{and} \quad |T_{j}| \subset |T_{j_{0}}| \quad \text{for all} \quad (i, j) > (i_{0}, j_{0}).
\]

Set

\[
X_{(i,j)} = \bigcup_{(i', j') \neq (i,j)} X_{i',j'} \quad \text{and} \quad Y_{j} = \bigcup_{j' \geq j} Y_{j'} \quad \text{for} \quad (i, j) \in \overline{H} \quad \text{and} \quad ? \in \{\geq, >\},
\]

and let \( \otimes Z \) or \( \otimes (Z) \) in \( \mathbb{R}^{\alpha} \times \mathbb{R}^{\alpha} \) denote the fibre product \( X_{i_{0},j_{0}} \times_{(f_{0},\pi_{j_{0}})} Z \) for a subset \( Z \) of \( Y_{\geq j_{0}} \). Define naturally a \( C^{\infty} \) map \( \otimes f : X_{\geq (i_{0},j_{0})} \rightarrow \otimes Y_{\geq j_{0}} \). Then we can easily construct a strongly controlled tube system \( \{\otimes T_{j} = (|\otimes T_{j}|, \otimes \pi_{j}, \otimes \rho_{j})\}_{j \geq j_{0}} \) for \( \{\otimes Y_{j}\}_{j \geq j_{0}} \) such that for each \( j \geq j_{0}, \)

\[
\otimes|T_{j}| \subset |\otimes T_{j}|,
\]

\[
\begin{align*}
\otimes \pi_{j}(x, y) &= (x, \pi_{j}(y)) \\
\otimes \rho_{j}(x, y) &= \rho_{j}(y)
\end{align*}
\]

for \( (x, y) \in \otimes|T_{j}|, \)

and \( \{T_{i,j}\}_{(i,j) \geq (i_{0},j_{0})} \) is strongly controlled over \( \{\otimes T_{j}\}_{j \geq j_{0}} \). Let \( p_{X} : \otimes Y_{\geq j_{0}} \rightarrow X_{i_{0},j_{0}} \) and \( p_{Y} : \otimes Y_{\geq j_{0}} \rightarrow Y_{\geq j_{0}} \) denote the projections.

Let us specify the construction of \( \theta_{j}^{*} \) as in the proof of I.5.8 (Thom’s Second Isotopy Lemma) in [G-al]. There exists a controlled vector field \( \{v_{j}\}_{j > j_{0}} \) on \( \{Y_{j} \cap \rho_{j_{0}}^{-1}(0, 2\varepsilon_{j_{0}}]\}_{j \geq j_{0}} \) such that

\[
(*) \quad d\pi_{j_{0}}v_{j} = 0 \quad \text{and} \quad \nu_{j}\rho_{j_{0}} = 1, \quad j > j_{0}.
\]

(The existence follows if we apply Lemma 3 to the map \( (\pi_{j_{0}}, \rho_{j_{0}}) : Y \cap \rho_{j_{0}}^{-1}(0, 2\varepsilon_{j_{0}}] \rightarrow Y_{j_{0}} \times (0, 2\varepsilon_{j_{0}}[, \text{Then by Lemma 4,} \{v_{j}\} \text{is locally integrable. Hence if we define} \)

\[
\theta_{j_{0}} = (\theta_{j_{0}}^{*}, \rho_{j_{0}}) \text{so that for each} \quad y \in Y_{j_{0}} \cap \rho_{j_{0}}^{-1}(\varepsilon_{j_{0}}), \]

\[
\theta_{j_{0}}^{*}\rho_{j_{0}}^{-1}(y) = \rho_{j_{0}}^{-1}(0, \varepsilon_{j_{0}}] \cap \text{(the integral curve of} \{v_{j}\} \text{passing through} y),
\]

which is possible by condition (3) of a removal data of \( \{T_{j}\} \), then \( \theta_{j_{0}} \) fulfills the requirements.

Multiplying \( v_{j} \) by \( \rho_{j_{0}} \), we replace the latter equality of (*) with \( v_{j}\rho_{j_{0}} = \rho_{j_{0}} \). Let \( (*)' \) denote the new equalities. Define a \( C^{\infty} \) vector field \( v_{j_{0}} \) on \( Y_{j_{0}} \) to be 0. Then \( v^{Y} = \{v_{j}\}_{j \geq j_{0}} \) is a locally integrable and weakly controlled vector field on \( \{Y_{j}\}_{j \geq j_{0}} \).

(Local integrability around \( Y_{j_{0}} \) follows from \( (*)' \).)

We want to lift \( v^{Y} \) to a vector field \( v^{X} \) on \( \{X_{i,j}\}_{(i,j) \geq (i_{0},j_{0})} \) which induces \( \theta_{i,j}^{*} \) as \( v^{Y} \) does \( \theta_{j}^{*} \). First we lift \( v^{Y} \) to \( \{\otimes Y_{j}\} \). Since \( d\pi_{j_{0}}v_{j} = 0 \), there exists uniquely a vector field \( v^{\otimes Y} = \{v^{\otimes v_{j}}\}_{j \geq j_{0}} \) on \( \{\otimes Y_{j}\}_{j \geq j_{0}} \) such that

\[
dp_{X} \otimes v_{jx,y} = 0 \quad \text{and} \quad dp_{Y} \otimes v_{jx,y} = v_{jy} \quad \text{for} \quad (x, y) \in \otimes Y_{j}, \quad j \geq j_{0}.
\]
Clearly \( v^{\otimes Y} \) is locally integrable and weakly controlled, and it induces the homeomorphism

\[ \otimes(Y_{j0}^{+} \cap \rho_{j0}^{-1}(0, \varepsilon_{j0})) \ni (x, y) \mapsto (x, \theta_{j0}(y)) \in \otimes(Y_{j0}^{+} \cap \rho_{j0}^{-1}(\varepsilon_{j0})) \times [0, \varepsilon_{j0}]. \]

Second, by the same reason as above we obtain a controlled vector field \( \{v_{i,j0}\}_{i > i_0} \) on \( \{X_{i,j0}\}_{i > i_0} \) such that

\[ (**') \quad d\pi_{i_0,j_0}v_{i,j0} = 0 \quad \text{and} \quad v_{i,j0}\rho_{i_0,j0} = \rho_{i_0,j0}, \quad i > i_0. \]

Set \( v_{i_0,j0} = 0 \) on \( X_{i_0,j0} \). Then \( \{v_{i,j0}\}_{i \geq i_0} \) is a locally integrable vector field on \( \{X_{i,j0}\}_{i \geq i_0} \).

Third, by Lemma 3 there exists a vector field \( v^{X} = \{v_{i,j}\}_{(i,j) \geq (i_0,j_0)} \) on \( \{X_{i,j}\}_{(i,j) \geq (i_0,j_0)} \) which is an extension of \( \{v_{i,j0}\}_{i \geq i_0} \) and such that \( \{v_{i,j}\}_{(i,j) > (i_0,j_0)} \) is controlled over \( v^{\otimes Y} \). Lemma 4 claims that \( \{v_{i,j}\}_{(i,j) > (i_0,j_0)} \) is locally integrable. Moreover, it follows from \((*)'\), \((**)\) and controlledness over \( v^{\otimes Y} \) that \( v^{X} \) is locally integrable around \( X_{i_0,j0} \).

In the same way as we defined \( \theta^{*}_{i,j0} \), we do \( \theta^{*}_{i_0,j0} \) so that for each \( x \in N'_{i_0,j0} \),

\[ \theta^{*}_{i_0,j0}^{-1}(x) = N_{i_0,j0} \cap \{ \text{the integral curve of } v^{X} \text{ passing through } x \}, \]

which is possible by conditions (7) and (8) of a removal data of \( \{T_{i,j}\} \), if \( v^{X} \) points outside of \( N'_{i_0,j0} \) at each point of \( N'_{i_0,j0} \). The last condition is satisfied at \( N'_{i_0,j0} \cap (\rho_{j0} \circ f)^{-1}\{0, \varepsilon_{j0}\} \), and hence, by weak controlledness of \( v^{X} \), at a neighborhood of \( N'_{i_0,j0} \cap (\rho_{j0} \circ f)^{-1}(0) \) in \( N'_{i_0,j0} \). Therefore, it suffices to choose sufficiently small \( \varepsilon_{j0} \). This means that when we fix \( \{\varepsilon_{i,j}\} \) at the beginning of the proof, we construct also \( \theta_{i,j0}^{*} \).

By (b), \( \theta_{i_0,j0}^{**} \) is automatically defined on \( N_{i_0,j0} - (\rho_{j0} \circ f)^{-1}(0) \). It is extendible to \( N_{i_0,j0} \cap (\rho_{j0} \circ f)^{-1}(0) - X_{i_0,j0}^{\varepsilon} \) for the following reason. Let \( \omega: \Omega \rightarrow X_{(i_0,j_0)} \), \( \Omega \subset X_{(i_0,j_0)} \times R \), denote the flow of \( v^{X} \). Then by \((*)\) we have

\[ \omega(x, \log t) = \theta_{i_0,j0}^{* -1}(x, t) \quad \text{for} \quad (x, t) \in (N'_{i_0,j0} - (\rho_{j0} \circ f)^{-1}(0)) \times [0, 1]. \]

Conditions (a), (b) and (c) are satisfied. Indeed, the former equality of (a) is trivial. The latter follows from controlledness of \( \{v_{i,j}\}_{(i,j) > (i_0,j_0)} \) over \( v^{\otimes Y} \). (c) is clear by the definition of \( \theta_{j0}^{*} \) and \( \theta_{i_0,j0}^{*} \) and the same controlledness.

\[ \square \]

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