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NASH FUNCTIONS ON NONCOMPACT NASH MANIFOLDS

MICHEL COSTE and MASASHIRO SHIOTA

§1. INTRODUCTION

A Nash manifold is a semialgebraic $C^\infty$ submanifold of a Euclidean space. A Nash function on a Nash manifold is a $C^\infty$ function with semialgebraic graph. Let $M$ be a Nash manifold. Let $\mathcal{N}$ denote the sheaf of Nash function germs on $M$. (We write $\mathcal{N}_M$ if we need to emphasize $M$.) Let $\mathcal{O}$ (or $\mathcal{O}_M$) denote the sheaf of $C^\omega$ function germs on $M$. We call a sheaf of ideals $\mathcal{I}$ of $\mathcal{N}$ finite if there exists a finite open semialgebraic covering $\{U_i\}$ of $M$ such that for each $i$, $\mathcal{I}|_{U_i}$ is generated by Nash functions on $U_i$. (See [S] and [C-R-S$_2$] for elementary properties of sheaves of $\mathcal{N}$-ideals and $\mathcal{N}$-modules.) Let $\mathcal{N}(M)$ denote the ring of Nash functions on $M$ and let $\mathcal{O}(M)$ denote the ring of $C^\omega$ functions on $M$.

[C-R-S$_2$] showed that the following three elementary conjectures are equivalent, and [C-R-S$_1$] gave a positive answer to the conjectures in the case where the manifold of domain $M$ is compact.

Separation conjecture. Let $M$ be a Nash manifold. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{N}(M)$. Then $\mathfrak{p}\mathcal{O}(M)$ is a prime ideal of $\mathcal{O}(M)$.

Global equation conjecture. For the same $M$ as above, every finite sheaf $\mathcal{I}$ of $\mathcal{N}_M$-ideals is generated by global Nash functions on $M$.

Extension conjecture. For the same $M$ and $\mathcal{I}$ as above, the following natural homomorphism is surjective:

$$H^0(M,\mathcal{N}) \longrightarrow H^0(M,\mathcal{N}/\mathcal{I}).$$

If these conjectures hold true, then the following conjecture also holds [C-R-S$_2$].

Factorization conjecture. Given a Nash function $f$ on a Nash manifold $M$ and a $C^\omega$ factorization $f = f_1f_2$, there exist Nash functions $g_1$ and $g_2$ on $M$ and positive $C^\omega$ functions $\varphi_1$ and $\varphi_2$ such that $\varphi_1\varphi_2 = 1$, $f_1 = \varphi_1g_1$ and $f_2 = \varphi_2g_2$.

In the present paper, we prove the conjectures in the noncompact case. It suffices to show the following theorem.

Theorem. Let $M \subset \mathbb{R}^n$ be a noncompact Nash manifold. Let $U$ and $V$ be open semialgebraic subset of $M$ such that $M = U \cup V$. Let $\mathcal{I}$ be a sheaf of $\mathcal{N}_M$-ideals such that $\mathcal{I}|_U$ and $\mathcal{I}|_V$ are generated by global cross-sections on $U$ and $V$ respectively. Then $\mathcal{I}$ itself is generated by global cross-sections on $M$.

The following proof of this theorem is completely different to the proof in [C-R-S$_1$] in the compact case. The proof in [C-R-S$_1$] is algebraic and based on the Néron
desingularization. On the other hand, the present proof is geometric, and the key
is Lemma 1 (Proposition VI.2.8 in [S]) of the next section on extension of Nash
functions to a compact domain.

We refer meanings and a history of the conjectures to [C-R-S1,2].

§2. Proof of the theorem

A manifold stands for a manifold without boundary unless otherwise specified.
A manifold with corners is, by definition, not a manifold but locally diffeomorphic
to an open subset of $\mathbb{R}^n_+$, where $\mathbb{R}_+ = \{x \in \mathbb{R}: x \geq 0\}$. Let $M$ be a manifold
with corners. Int $M$—the interior of $M$—is the subset of $M$ where $M$ is locally
diffeomorphic to $\mathbb{R}^n$. $\partial M$—the boundary of $M$—is the complement. A manifold
with boundary is a manifold with corners such that the boundary is a manifold.

An abstract Nash manifold of dimension $m$ is a $C^\omega$ manifold with a finite system
of coordinate neighborhoods $\{\psi_i: U_i \to \mathbb{R}^m\}$ such that for each pair $i$ and $j$,
$\psi_i(U_i \cap U_j)$ is an open semialgebraic subset of $\mathbb{R}^m$ and the map

$$
\psi_j \circ \psi_i^{-1}: \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)
$$

is a Nash diffeomorphism. A $C^1$ Nash manifold is a $C^1$ semialgebraic submanifold
of a Euclidean space. An abstract $C^1$ Nash manifold is a $C^1$ manifold with a
finite system of coordinate neighborhoods of $C^1$ semialgebraic class. Note that a
Nash manifold is an abstract Nash manifold, but an abstract Nash manifold is not
necessarily affine, i.e., an abstract Nash manifold cannot be always Nash imbedded
in a Euclidean space (Mazur). On the other hand, a $C^1$ Nash manifold is an
abstract $C^1$ Nash manifold and, conversely, an abstract $C^1$ Nash manifold is affine
(Theorem III.1.1 in [S]).

For a Nash manifold with corners $M$, we say that the boundary of $M$ is shrunk if
we replace $M$ with $M$ – (a small closed semialgebraic neighborhood of $\partial M - \partial M \subset \partial M$). We call the replaced manifold with corners a Nash submanifold with shrunk corners of $M$.

The index $x$ denotes the stalk of a sheaf at $x$ or the germ of a set or a map at $x$.

Note. The theorem holds true if the closure $\overline{M}$ of $M$ in $\mathbb{R}^n$ is compact and
contained in a Nash manifold $M'$ of the same dimension as $M$ and if $\mathcal{I}$ can be
extended to a coherent sheaf $\mathcal{T}'$ of $\mathcal{N}_{M''}$-ideals on $M'$ for the following reason.

It is easy to find a compact Nash manifold with boundary $M''$ with $M \subset \text{Int } M''$ and $M'' \subset M'$. Using the double of $M''$, we easily construct a compact Nash manifold $M^{(3)}$ and a Nash map $\rho: M^{(3)} \to M''$ such that $\rho|_{\rho^{-1}(\text{Int } M'')} : \rho^{-1}(\text{Int } M'') \to \text{Int } M''$ is a trivial double covering. Let $\Omega$ be a union of connected components of $\rho^{-1}(\text{Int } M'')$ such that $\rho|_{\Omega}$ is a diffeomorphism onto $\text{Int } M''$. Let $\mathcal{T}^{(3)}$ denote the pull back of $\mathcal{T}'$ by $\rho$. Then $\mathcal{T}^{(3)}$ is finite and hence generated by global cross-sections. Hence $\mathcal{T}'|_{\text{Int } M''}$ and then $\mathcal{T}$ are generated by global cross-sections, because we can identify $\mathcal{T}'|_{\text{Int } M''}$ with $\mathcal{T}^{(3)}|_{\Omega}$.

Hence we will imbed $M$ in a Euclidean space so that the image has such properties. The following lemma assures it.
Lemma 1 (Proposition VI.2.8 in [S]). Let $M$ be a noncompact Nash manifold, and let $f: M \rightarrow \mathbb{R}^n$ be a bounded Nash map. Then there exists a compact Nash manifold with corners $M'$ and a Nash diffeomorphism $\pi: \text{Int } M' \rightarrow M$ such that $f \circ \pi$ can be extended to a Nash map $M' \rightarrow \mathbb{R}^n$.

Using this lemma, we shall reduce the theorem to the following lemma.

Lemma 2. Let $M'$ and $M''$ be (not necessarily compact) Nash submanifolds of $\mathbb{R}^n$ without boundary and with corners, respectively, such that $\overline{M}^\prime$ is compact and contained in a Nash manifold of the same dimension, $\overline{M}''$ is a compact Nash manifold with corners and $\text{Int } M'' = \text{Int } \overline{M}''$ (i.e., $M''$ is a compact Nash manifold with corners) — (a closed semialgebraic subset of the boundary). Let $p: M' \rightarrow \mathbb{R}^n$ be a Nash map such that $p|_{\text{Int } M''}$ is a Nash imbedding into $M'$ and $p(\partial M'')$ is contained in $\overline{M}^\prime - M'$. Shrink the boundary of $M''$. Then the abstract Nash manifold $M' \cup p|_{\text{Int } M''} M''$, defined to be the union of $M'$ and $M''$ pasted by the Nash diffeomorphism $p|_{\text{Int } M''}: \text{Int } M'' \rightarrow p(\text{Int } M'')$, is affine.

Proof of the theorem. We can assume that $M$ is bounded in $\mathbb{R}^n$ because $\mathbb{R}^n$ is Nash diffeomorphic to $S^n - \text{a point}$. Let the dimension of $M$ be $m$. By the separation theorem of Mostowski [M], we have a Nash function $\psi$ on $M$ such that $-2 \leq \psi \leq 2$, $\psi > 1$ on $M - V (= U - V)$, and $\psi < -1$ on $M - U (= V - U)$. Replace $M$ with graph $\psi$. Then we can assume that $M - V$ and $M - U$ have distance. Apply Lemma 1 to the inclusion map $M \rightarrow \mathbb{R}^n$. Then we assume, moreover, that $M$ is a Nash manifold with corners. Let $\varphi$ be a positive Nash function on $M$ such that $\varphi(x) \rightarrow 0$ as $M \ni x \rightarrow \text{a point of } \partial M$.

Let $f_1, \ldots, f_k \in H^0(U, \mathcal{I}|_U)$ and $g_1, \ldots, g_k \in H^0(V, \mathcal{I}|_V)$ be generators of $\mathcal{I}|_U$ and $\mathcal{I}|_V$ respectively. Multiplying small positive Nash functions, we can assume the generators are all bounded. Note that the restrictions of the both generators to $U \cap V$ are generators of $\mathcal{I}|_{U \cap V}$. Hence by I.6.5 in [S] there exist Nash functions $\alpha_{i,j}$ and $\beta_{i,j}$ on $U \cap V$, $i, j = 1, \ldots, k$, such that for each $i$,

\[
(*) \quad f_i = \sum_{j=1}^{k} \alpha_{i,j} g_j \quad \text{and} \quad g_i = \sum_{j=1}^{k} \beta_{i,j} f_j \quad \text{on} \quad U \cap V.
\]

Shrink $U$ and $V$ keeping the property that $M - V$ and $M - U$ have distance. Then by Lojasiewicz Inequality, all $\varphi^l \alpha_{i,j}$ and $\varphi^l \beta_{i,j}$ are bounded for a positive integer $l$. Apply Lemma 1 to $f_i|_{U \cap V}, g_i|_{U \cap V}, \varphi^l \alpha_{i,j}, \varphi^l \beta_{i,j}, \varphi|_{U \cap V}$ and the inclusion map $U \cap V \rightarrow \mathbb{R}^n$. Then there exists a compact Nash manifold with corners $X$ and a Nash diffeomorphism $\pi: \text{Int } X \rightarrow U \cap V$ such that all the Nash functions on $\text{Int } X$: $f_i \circ \pi, g_i \circ \pi, (\varphi^l \alpha_{i,j}) \circ \pi, (\varphi^l \beta_{i,j}) \circ \pi$ and $\varphi \circ \pi$ can be extended to $X$, and $\overline{\pi^{-1}(M - U)}$ and $\overline{\pi^{-1}(M - V)}$ have distance, where $\overline{\pi}$ denotes the extension of (the inclusion map) $\pi: \text{Int } X \rightarrow \mathbb{R}^n$ to $X \rightarrow \mathbb{R}^n$.

First we modify the inclusion map of $M$ into $\mathbb{R}^n$ so that $\mathcal{I}$ can be extended to $\overline{M} - (\overline{M - U}) - (\overline{M - V})$. We can assume that the abstract Nash manifold with corners $M \cup \pi (X - \partial X \cap \overline{\pi^{-1}(M)})$ is affine for the following reason. Set

\[
M' = M, \quad M'' = X - \partial X \cap \overline{\pi^{-1}(M)} \quad \text{and} \quad p = \overline{\pi}|_{M''}.
\]
Then the assumptions in Lemma 2 are satisfied. Hence if we let \( \tilde{M}'' \) be a Nash submanifold with shrunk corners of \( M'' \), then \( M' \cup_{\text{pl}_{\text{int}}M''} \tilde{M}'' \) is affine. Shrink \( U \) and \( V \) a little so that (the shrunk \( U \)) \( \cap \) (the shrunk \( V \)) and \( (M - \text{the original} \) \( U \)) \( \cap \) (\( M - \) the shrunk \( U \)) and \( (M - \) the shrunk \( V \)) have distance. Then the new \( M \cup_{\pi} (X - \partial X \cap \overline{\partial -1(M)} \) coincides with \( M' \cup_{\text{pl}_{\text{int}}M''} \tilde{M}'' \) for some \( \tilde{M}'' \), and hence it is affine.

Set \( M_1 = M \cup_{\pi} (X - \partial X \cap \overline{\partial -1(M)} \), let \( M_1 \) be contained in \( \mathbb{R}^n \), and regard \( M \) as a submanifold of \( M_1 \). Then \( M_1 = \tilde{M} - (\tilde{M} - U) - (\tilde{M} - V) \), and it is a Nash manifold with corners. Set

\[
U_1 = U \cup \partial M_1 \quad \text{and} \quad V_1 = V \cup \partial M_1.
\]

Then we have \( M_1 = U_1 \cup V_1 \), and \( U_1 \) and \( V_1 \) are open in \( M_1 \). Since \( f_i \circ \pi, g_i \circ \pi \) and \( \varphi \circ \pi \) can be extended to \( X \), we have Nash function extensions \( f_{1,i} \) of \( f_i \) to \( U_1 \), \( g_{1,i} \) of \( g_i \) to \( V_1 \), \( i = 1, \ldots, k \) and \( \varphi_i \) of \( \varphi \) to \( M_1 \). Hence we have sheaf extensions \( \mathcal{I}_1^U \) and \( \mathcal{I}_1^V \) of \( \mathcal{I} \) to \( M_1 \) such that \( \mathcal{I}_1^U \mid_{U_1} \) is generated by \( f_{1,1}, \ldots, f_{1,k} \), and \( \mathcal{I}_1^V \mid_{V_1} \) is generated by \( g_{1,1}, \ldots, g_{1,k} \). By (*) and by the fact that \( (\varphi_i \alpha_{1,j}) \circ \pi \) and \( (\varphi_i \beta_{i,j}) \circ \pi \) can be extended to \( X \), say, \( \alpha_{1,i,j} \) and \( \beta_{1,i,j} \) respectively, we have

\[
\varphi_{1}^{l} \mathcal{I}_{1}^{U} \subset \mathcal{I}_{1}^{V} \quad \text{and} \quad \varphi_{1}^{l} \mathcal{I}_{1}^{V} \subset \mathcal{I}_{1}^{U}.
\]

Let \( M_2 \subset \mathbb{R}^n \) be a Nash manifold of dimension \( m \) such that \( M_1 \subset M_2 \) and any connected component of \( M_2 \) touches \( M_1 \). Here also we can assume \( M_2 \) is a compact Nash manifold with corners. Set

\[
U_2 = U_1 \cup (M_2 - M_1) \quad \text{and} \quad V_2 = V_1 \cup (M_2 - M_1).
\]

Then we have \( M_2 = U_2 \cup V_2 \), and \( U_2 \) and \( V_2 \) are open in \( M_2 \). Choose \( M_2 \) so small that \( f_{2,i} \), \( g_{2,i} \), \( \alpha_{2,i,j} \), \( \beta_{2,i,j} \) and \( \varphi_i \) can be extended to \( U_2 \), \( V_2 \), \( U_2 \cap V_2 \), \( U_2 \cap V_2 \) and \( M_2 \) respectively. Let \( f_{2,i} \), \( g_{2,i} \), \( \alpha_{2,2,i,j} \), \( \beta_{2,2,i,j} \) and \( \varphi_i \) denote the respective extensions. Then there exist sheaves \( \mathcal{I}_2^U \) and \( \mathcal{I}_2^V \) of \( \mathcal{N}_{M_2} \)-ideals such that \( \mathcal{I}_2^U \mid_{M_1} = \mathcal{I}_1^U \), \( \mathcal{I}_2^V \mid_{M_1} = \mathcal{I}_1^V \), \( \mathcal{I}_2^U \mid_{U_2} \) is generated by \( f_{2,1}, \ldots, f_{2,k} \), \( \mathcal{I}_2^V \mid_{V_2} \) is generated by \( g_{2,1}, \ldots, g_{2,k} \) and

\[
\varphi_{2}^{l} \mathcal{I}_{2}^{U} \subset \mathcal{I}_{2}^{V} \quad \text{and} \quad \varphi_{2}^{l} \mathcal{I}_{2}^{V} \subset \mathcal{I}_{2}^{U}.
\]

Second, we want to extend \( \mathcal{I}_2^U \) to \( M_2 - V_2 \). Apply Lemma 1 to \( f_{2,1}, \ldots, f_{2,k} \), \( \varphi_i \mid_{U_2} \) and the inclusion map \( U_2 \to \mathbb{R}^n \). Then we have a compact Nash manifold with corners \( Y \) and a Nash diffeomorphism \( \tau: \text{Int} Y \to U_2 \) such that each \( f_{2,i} \circ \tau \) and \( \varphi_i \circ \tau \) can be extended to \( Y \), and \( \overline{\tau^{-1}(M_2 - U_2)} \) and \( \overline{\tau^{-1}(M_2 - V_2)} \) have distance, where \( \overline{\tau} \) is defined by \( \tau \) as \( \overline{\tau} \). Let \( M_3 \) denote the following abstract Nash manifold with corners:

\[
M_2 \cup_{\tau \mid_{\text{Int} Y}} \left( \text{Int} Y \cup \right.
\]

(a small open semialgebraic neighborhood of \( \partial Y \cap \overline{\tau^{-1}(U_2 - V_2)} \) in \( \partial Y \)).

Then by the same reason as above, \( M_3 \) is affine, and we can assume \( M_2 \subset M_3 \subset \mathbb{R}^n \). Set

\[
U_3 = U_2 \cup (M_3 - M_2) \quad \text{and} \quad V_3 = V_2.
\]
Then we have
\[
\overline{M} - \overline{V} \subset U_3, \quad M_3 = U_3 \cup V_3 \quad \text{and} \quad U_3 \cap V_3 = U_2 \cap V_2,
\]
and \(U_3\) and \(V_3\) are open in \(M_3\). Since \(f_{2,i} \circ \tau\) and \(\varphi_2 \circ \tau\) are extended to \(Y\), \(f_{2,i}\) and \(\varphi_2\) can be extended to Nash functions \(f_{3,i}\) on \(U_3\) and \(\varphi_3\) on \(M_3\). Let \(\mathcal{I}^{U}_3\) denote the sheaf of \(N_{U_3}\)-ideals on \(U_3\) (not on \(M_3\)) generated by \(f_{3,1}, \ldots, f_{3,k}\).

Third, as the above extension of \(M_1\) to \(M_2\) and then to \(M_3\), we obtain a Nash manifold \(M_4\) of dimension \(m\), open semialgebraic subsets \(U_4\) and \(V_4\) of \(M_4\), Nash functions \(f_{4,i}\) on \(U_4\), \(g_{4,i}\) on \(V_4\), \(i = 1, \ldots, k\) and \(\varphi_4\) on \(M_4\), and sheaves \(\mathcal{I}^{U}_4\) of \(N_{U_4}\)-ideals and \(\mathcal{I}^{V}_4\) of \(N_{V_4}\)-ideals such that

\[
\overline{M} \subset M_4, \quad M_4 = U_4 \cup V_4, \\
U_4 \cap M = U, \quad V_4 \cap M = V, \\
f_{4,i}|U = f_i, \quad g_{4,i}|V = g_i, \quad \varphi_4|M = \varphi, \\
(\ast\ast) \quad \varphi_4^l \mathcal{I}^{U}_4 \subset \mathcal{I}^{V}_4, \quad \varphi_4^l \mathcal{I}^{V}_4 \subset \mathcal{I}^{U}_4 \quad \text{on} \quad U_4 \cap V_4,
\]

\(\mathcal{I}^{U}_4\) is generated by \(f_{4,1}, \ldots, f_{4,k}\), and \(\mathcal{I}^{V}_4\) is generated by \(g_{4,1}, \ldots, g_{4,k}\).

Finally, we define a sheaf \(\mathcal{I}_4\) of \(N_{M_4}\)-ideals so that for each \(x \in M_4\),

\[
\mathcal{I}_4 = \left\{ \begin{array}{ll}
\{ h \in N_x : \varphi_4^l h \in \mathcal{I}^{U}_4 \text{ for some } l' \} & \text{if } x \in U_4 \\
\{ h \in N_x : \varphi_4^l h \in \mathcal{I}^{V}_4 \text{ for some } l' \} & \text{if } x \in V_4.
\end{array} \right.
\]

By \((\ast\ast)\), \(\mathcal{I}_4\) is a well-defined coherent sheaf, and by the fact that \(\varphi\) is positive, it is an extension of \(\mathcal{I}\). Hence the theorem follows from the note. \(\square\)

**Proof of Lemma 2.** Let \(\dim M' = m\). Regard \(M' \cup_{\text{pl} M''} M''\) as an abstract \(C^1\) Nash manifold with corners which is of class \(C^\omega\) around its boundary. By Theorem III.1.1 in [S], there exists its \(C^1\) Nash imbedding into a Euclidean space, say, \(\mathbb{R}^{n'}\). By the proof of Theorem III.1.1, the imbedding map can be of class \(C^\omega\) around the boundary. Hence the image can be of class \(C^\omega\) around the boundary. By Theorem III.1.3, ibid., and its proof, the image is modified to be a Nash manifold with corners through a \(C^1\) Nash diffeomorphism of class \(C^\omega\) around the boundary. Consequently, we have a Nash manifold with corners \(M_1 \subset \mathbb{R}^{n'}\) and a \(C^1\) Nash diffeomorphism \(\rho: M_1 \rightarrow M' \cup_{\text{pl} M''} M''\) of class \(C^\omega\) around \(\partial M_1\). Here by the same arguments as before, we can assume \(M_1\) is compact and contained in a Nash manifold \(M_2\) of dimension \(m\). It suffices to approximate \(\rho\) by a Nash map in the \(C^1\) topology, because a strong \(C^1\) Nash approximation of a \(C^1\) Nash diffeomorphism in the \(C^1\) topology is a diffeomorphism by Lemma II.1.7, ibid. (See Chapter II, ibid., for the topology.) Define a \(C^1\) Nash map \(\xi: M_1 \rightarrow \mathbb{R}^n\) by

\[
\xi = \left\{ \begin{array}{ll}
\rho & \text{on } \rho^{-1}(M') \\
\rho \circ \rho & \text{on } \rho^{-1}(M'').
\end{array} \right.
\]

Then \(\xi(M_1) \subset \overline{M'}\), \(\xi\) is of class \(C^\omega\) around \(\partial M_1\), and \(\xi|_{\text{Int} M_1}\) is a \(C^1\) diffeomorphism onto \(M'\).
Shrink $\partial M_1$. Then there exists a strong Nash approximation $\xi'$ of $\xi$ in the $C^1$ topology such that $\xi' = \xi$ on $\partial M_1$ and $\xi'(\text{Int} M_1) = M'$ for the following reason.

Let $\tilde{M}' \subset \mathbb{R}^n$ be a Nash manifold that contains $\tilde{M}$ and is of dimension $m$. Shrink $\partial M_1$. Then by Lemma 3 below, there exists a Nash function $\varphi$ on $M_1$ with zero set $= \partial M_1$. Let $U$ be a small open semialgebraic neighborhood of $\partial M_1$ in $M_2$ where $\varphi|_{U \cap M_1}$ and $\xi'|_{U \cap M_1}$ can be extended as a Nash function and a Nash map to $\tilde{M}'$ respectively. Set $M_3 = M_1 \cup U$, and let $\tilde{\varphi}: M_3 \rightarrow \mathbb{R}$ and $\tilde{\xi}: M_3 \rightarrow \tilde{M}'$ denote the respective extensions. Apply Theorem II.5.2 in [S] to $\tilde{\varphi}$, $\tilde{\xi}$, $M_3$ and $\tilde{M}'$. Then there exists a Nash approximation $\tilde{\xi}' : M_3 \rightarrow \tilde{M}'$ of $\tilde{\xi}$ in the $C^1$ topology such that $\tilde{\xi}' = \tilde{\xi}$ on $\tilde{\varphi}^{-1}(0)$ and $\tilde{\xi}'(M_3) = \tilde{\xi}(M_3)$. If we set $\xi' = \tilde{\xi}'|_{M_1}$ then $\xi'$ is a Nash approximation of $\xi$ in the $C^1$ topology and satisfies the required conditions.

Moreover, $\xi'|_{\text{Int} M_1}$ can be a Nash diffeomorphism onto $M'$ for the following reason.

First we prove that $\xi'|_{\text{Int} M_1}$ can be an immersion. For each $i = 1, \ldots, n$, let $v_i$ denote the Nash vector field on $M_1$ such that for each $x \in M_1$,

$$
\left( \frac{\partial}{\partial x_i} \right)_x = v_i + (\text{a vector normal to the tangent space of } M_1 \text{ at } x).
$$

For a $C^1$ map $\chi = (\chi_1, \ldots, \chi_n): M_1 \rightarrow \mathbb{R}^n$, let $\alpha(\chi)$ denote the sum of the squares of the minors of degree $m$ of the $n \times n$ matrix whose $(i, j)$-element is $v_i \chi_j$. Then $\chi|_{\text{Int} M_1}$ is an immersion if and only if $\alpha(\chi)$ is positive on $\text{Int} M_1$. It follows from Lojasiewicz Inequality and the property $\alpha(\xi) > 0$ on $\text{Int} M_1$ that $\alpha(\xi') > 0$ on $\text{Int} M_1$ if we choose $\xi'$ so that $\xi' - \xi$ is the product of $\varphi'$ and a $C^1$ Nash map close to the zero map in the $C^1$ topology for a large integer $l$ and for the above $\varphi$. Hence $\xi'|_{\text{Int} M_1}$ can be an immersion.

Second, we see that $\xi'|_{\text{Int} M_1}$ can be injective. For a map $\chi: M_1 \rightarrow \mathbb{R}^n$, let $\beta(\chi): M_1 \times M_1 \rightarrow \mathbb{R}^n$ be defined by

$$
\beta(\chi)(x_1, x_2) = \chi(x_1) - \chi(x_2) \quad \text{for} \quad (x_1, x_2) \in M_1 \times M_1.
$$

Let $\Delta$ denote the diagonal of $M_1 \times M_1$. Then $\chi|_{\text{Int} M_1}$ is injective if and only if

\[(*) \quad \beta(\chi)^{-1}(0) = \Delta \quad \text{in} \quad \text{Int} M_1 \times \text{Int} M_1,
\]

the zero set of $\beta(\xi)$ contains $\Delta$ and is contained in $\partial M_1 \times \partial M_1 \cup \Delta$, and the rank of the Jacobian matrix of $\beta(\xi)$ at each point of $\text{Int} \Delta$ equals $m$. Note that $\dim \Delta = m$. Let $l$ be a large integer and let $\gamma: M_1 \times M_1 \rightarrow \mathbb{R}^n$ be a $C^1$ Nash map which vanishes on $\Delta$ and is close to the zero map in the $C^1$ topology. Then by Lojasiewicz Inequality, it is easy to see that the zero set of the map

$$
M_1 \times M_1 \ni (x_1, x_2) \mapsto \beta(\xi)(x_1, x_2) + (\varphi^{2l}(x_1) + \varphi^{2l}(x_2))\gamma(x_1, x_2) \in \mathbb{R}^n
$$

coincides with the zero set of $\beta(\xi)$. Choose $\xi'$ so that $\xi' - \xi$ is the product of $\varphi''$ and a $C^1$ Nash map close to the zero map in the $C^1$ topology for a much larger integer $l'$. Then $\beta(\xi')$ is of the above form. Hence $\xi'$ has the property $(*)$. Thus $\xi'|_{\text{Int} M_1}$ can be injective.
By the above two facts, \( \xi'_{\text{Int} M} \) can be a diffeomorphism onto \( M' \) because \((\xi - \xi')(x)\) converges to \( 0 \in \mathbb{R}^n \) as a point \( x \) in \( \text{Int} M_1 \) converges to a point of \( \overline{M}_1 - \text{Int} M_1 \).

Define a Nash map \( \rho' : \text{Int} M_1 \to M' \cup_{\partial \text{Int} M''} M'' \) to be \( \xi' \). Choose \( \xi' \) so that the map \( \xi' - \xi : M_1 \to \mathbb{R}^n \) is the product of \( \phi^l \) and a \( C^1 \) Nash map \( M_1 \to \mathbb{R}^n \) for a sufficiently large integer \( l \) (Theorem II.5.2, ibid.). Then by Lojasiewicz Inequality we can extend \( \rho' \) to a semialgebraic homeomorphism \( \rho' : M_1 \to M' \cup_{\partial \text{Int} M''} M'' \) which equals \( \rho \) on \( \partial M_1 \). Clearly \( \rho'_{\text{Int} M_1} \) is a Nash diffeomorphism onto \( M' \). Hence Lemma 2 follows if we can choose \( \xi' \) so that \( \rho' \) is a Nash diffeomorphism around \( \partial M_1 \). For that it suffices to prove the following assertion.

Let \( \tau \) be the semialgebraic homeomorphism of \( M_1 \) such that \( \rho \circ \tau = \rho' \). Then we can choose \( \xi' \) so that \( \tau \) is a Nash diffeomorphism around \( \partial M_1 \).

It follows from \( \rho \circ \tau = \rho' \) that \( \xi \circ \tau = \xi' \). Since \( \tau \) is unique and since \( \tau = \text{id} \) on \( \partial M_1 \), the problem is local at \( \partial M_1 \). Hence we can reduce the above assertion to the next one.

We can choose \( \xi' \) so that for each \( x \in \partial M_1 \) there exists a Nash diffeomorphism germ \( \tau \) of \( M_{1x} \) such that \( \xi \circ \tau = \xi'_{x} \).

We can assume \( M_1 \subset \mathbb{R}^m \) and \( M' \subset \mathbb{R}^m \) since the problem is local. Let \( J \) denote the Jacobian of \( \xi \). Then we precisely state the above assertion as follows, which is due to [T].

There exists such \( \tau \) if for each \( x \in \partial M_1 \), \( \xi_{x}' = \xi_{x} - \xi_{x} = J_{x}^{2} \varphi_{x} \) and a Nash map germ.

Such \( \xi' \) exists by the above construction of \( \xi' \) if we have a Nash function \( J' \) on \( M_1 \) such that \( J'^{-1}(0) \subset \partial M_1 \), and for each \( x \in \partial M_1 \), \( J_{x}' \) is the product of \( J_{x} \) and a Nash function germ. Let \( J \) denote the finite sheaf of \( N_{M_{1}} \)-ideals defined to be \( JN_{M_{1}} \) around \( \partial M_1 \) and \( N_{M_{1}} \) outside of \( \partial M_1 \). Then by Lemma 3, \( J \) has finite generators if we shrink \( \partial M_1 \). The sum of the squares of the generators fulfills the requirements for \( J' \).

It remains to show the last assertion. We assume \( M_1 = M' = \mathbb{R}^m \) for simplicity of notation. Let \( g : \mathbb{R}^m \to \mathbb{R}^m \) be the Nash map germ such that \( \xi_{0}' = \xi_{0} = J_{0}^{2} \varphi_{0}g \).

By the Taylor expansion formula we have
\[
\xi_{0}(x + y) = \xi_{0}(x) + y \cdot \frac{\partial \xi_{0}}{\partial x} + \sum_{i,j=1}^{m} y_{i}y_{j} f_{i,j}(x, y), \quad x, \ y = (y_{1}, \ldots, y_{m}) \in \mathbb{R}^m,
\]
for some Nash map germs \( f_{i,j} : \mathbb{R}^{2m} \to \mathbb{R}^m \), where \( \frac{\partial}{\partial x} \) denotes the Jacobian matrix. Substitute \( y \) with \( J_{0}(x)y \). Then
\[
\xi_{0}(x + J_{0}(x)y) - \xi_{0}(x) = J_{0}(x)y \cdot \frac{\partial \xi_{0}}{\partial x}(x) + J_{0}^{2}(x) \sum_{i,j=1}^{m} y_{i}y_{j} f'_{i,j}(x, y)
\]
for some Nash map germs \( f'_{i,j} \). Hence we need only find a Nash map germ \( y = y(x) : \mathbb{R}^m \to \mathbb{R}^m \) such that \( y(0) = 0 \) and
\[
J_{0}(x)y(x) \cdot \frac{\partial \xi_{0}}{\partial x}(x) + J_{0}^{2}(x) \sum_{i,j=1}^{m} y_{i}(x)y_{j}(x) f'_{i,j}(x, y(x)) = J_{0}^{2}(x) \varphi_{0}(x)g(x).
\]
Multiply this equality by the cofactor matrix of $\frac{\partial f}{\partial x}(x)$. Then it is equivalent to

$$y(x) + \sum_{i,j=1}^{m} y_i(x)y_j(x)f''_{i,j}(x, y(x)) = \varphi_0(x)g'(x),$$

where $f''_{i,j}$ and $g'$ are some Nash map germs. By the implicit function theorem, the last equality is solved. \(\Box\)

**Lemma 3.** Let $M \subset \mathbb{R}^n$ be a Nash manifold with corners. Let $\mathcal{I}$ be a finite sheaf of $N_M$-ideals on $M$ such that $\mathcal{I}_x = N_x$ for $x \in \text{Int} M$. Shrink $\partial M$. Then Global equation conjecture and Extension conjecture for this $\mathcal{I}$ hold true.

**Proof.** We can assume $M - M$ is a point. Let $\varphi$ be the function on $M$ which measures distance from $M - M$, and let $\varepsilon$ be a small positive number. Then $\varphi$ is of class Nash on $\varphi^{-1}([0, \varepsilon])$ and $C^1$ regular on $(\text{Int} M) \cap \varphi^{-1}([0, \varepsilon])$ and on (each face of $\partial M) \cap \varphi^{-1}([0, \varepsilon])$. Hence $M_1 = \varphi^{-1}([\varepsilon, \infty[)$ is a compact Nash manifold with corners. Set

$$M_2 = M - \{x \in \partial M : \varphi(x) \leq \varepsilon\} \quad \text{and} \quad M_3 = \varphi^{-1}([\varepsilon, \infty[),$$

which are Nash manifolds with corners. By the semialgebraic version of Thom's First Isotopy Lemma [C-S2], we have a semialgebraic map $\tau : \varphi^{-1}([0, \varepsilon]) \to \varphi^{-1}(\varepsilon)$ such that $\tau = \text{id}$ on $\varphi^{-1}(\varepsilon)$ and $(\tau, \varphi)|_{M_2 \cap \varphi^{-1}([0, \varepsilon])}$ is a Nash diffeomorphism onto $(M_2 \cap \varphi^{-1}(\varepsilon)) \times [0, \varepsilon]$. Using $\tau$ we easily construct a $C^1$ Nash diffeomorphism $\pi : M_3 \to M_2$ which is the identity on a small semialgebraic neighborhood of $\partial M_3$ in $M_3$.

From the note it follows that there exists a Nash function on $M_3$ with zero set $= \partial M_3$, and $\mathcal{I}|_{M_3}$ is generated by global cross-sections. We show that $\mathcal{I}|_{M_2}$ also is generated by global cross-sections. For that it suffices to find a Nash approximation $\pi' : M_3 \to M_2$ of $\pi$ in the $C^1$ topology such that $\pi' = \text{id}$ on $\partial M_3$ and the pull back of $\mathcal{I}|_{M_2}$ by $\pi'$ equals $\mathcal{I}|_{M_3}$.

Let $\psi$ be a global cross-section of $\mathcal{I}|_{M_3}$ with zero set $= \partial M_3$. By Theorem II.5.2 in [S] there exists a Nash approximation $\pi'$ of $\pi$ such that the map $\pi' - \pi : M_3 \to \mathbb{R}^n$ is the product of $\psi$ and a $C^1$ Nash map $\alpha : M_3 \to \mathbb{R}^n$ of class $C^\omega$ around $\partial M_3$. We need only prove that for each $a \in \partial M_3$ and for each $f \in N_a$, $f$ is contained in $\mathcal{I}_a$ if and only if $f \circ \pi'_a$ is in $\mathcal{I}_a$. (Note that $\pi'(a) = a$.) As the problem is local, we can assume $M \subset \mathbb{R}^m$ and $a = 0$, where $m = \dim M$. In general, for a Nash function germ $g$ at $0$ in $\mathbb{R}^m$ there exists a Nash function germ $h$ at $0$ in $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ such that

$$g(x + zy) = g(x) + zh(x, y, z) \quad \text{for} \ (x, y, z) \ \text{around} \ 0 \ \text{in} \ \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}.$$

Hence we have

$$f \circ \pi'_0(x) = f(x + \psi_0(x)\alpha_0(x)) = f(x) + \psi_0(x)f_1(x, \alpha_0(x), \psi_0(x))$$

for some Nash function germ $f_1$ at $0$ in $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$. Therefore, $f \in \mathcal{I}_0$ if and only if $f \circ \pi'_0 \in \mathcal{I}_0$. \(\Box\)
Remark. Global equation and Extension conjectures hold true for any real closed field $R$ which contains $R$.

We prove this in the same way as in the proof of the implication (i)$\Rightarrow$(ii) of Theorem 2.4 in [C-S1]. For semialgebraic subsets $X$ and $Y$ of $R^n$ and for a semialgebraic map $f: X \to Y$, let $X_R$, $Y_R$ and $f_R: X_R \to Y_R$ denote the extensions to $R$ of $X$, $Y$ and $f$ respectively.

**Proof of Global equation conjecture.** It suffices to prove the theorem for a (not necessarily noncompact) Nash manifold $M$ in $R^n$. Let $\dim M = m$. By Theorem 2.4 in [C-S1], we can assume there exists a Nash manifold $M^R \subset R^n$ such that $M$ is diffeomorphic to $M^R_R$. Hence let $M = M^R_R$. Moreover, by its proof we can assume $U = U^R_R$ and $V = V^R_R$ for some open semialgebraic sets $U^R$ and $V^R$ of $M^R$. Let $f_1, \ldots, f_k \in H^0(U, I_U)$ and $g_1, \ldots, g_k \in H^0(V, I_V)$ be generators of $I_U$ and $I_V$ respectively. Let $\gamma_{i,j}: U \cap V \to R$ and $\delta_{i,j}: U \cap V \to R$, $i, j = 1, \ldots, k$, be Nash functions such that for each $i$,

\[(*) \quad f_i = \sum_{j=1}^{k} \gamma_{i,j} g_j \quad \text{and} \quad g_i = \sum_{j=1}^{k} \delta_{i,j} f_j \quad \text{on} \quad U \cap V.\]

Let $f: M \to R$ be a Nash function. Then we have a presentation

\[
\text{graph } f = \bigcup_{\text{finite}} \{x \in R^{n+1}: \varphi(x, a) = 0, \varphi_1(x, a) > 0, \ldots, \varphi_l(x, a) > 0\},
\]

where $\varphi$ and $\varphi_i$ are polynomials with coefficients in $Z$ and $a$ is a $p$-uple of elements of $R$. For $b \in R^p$, set

\[
X_b = \bigcup_{\text{finite}} \{x \in R^{n+1}: \varphi(x, b) = 0, \varphi_1(x, b) > 0, \ldots, \varphi_l(x, b) > 0\}.
\]

Then, as noted in the proof of Theorem 2.4 in [C-S1], the set of $b$ such that $X_b$ is a Nash manifold of dimension $m$ is semialgebraic in $R^p$. Moreover, by the same reason as in the proof, the set $B \subset R^p$ of $b$ such that $X_b$ is the graph of a Nash function on $M^R$ is semialgebraic. Note that $X_b \subset M^R_R \times R$. Set $X = \cup_{b \in B} X_b \times b$.

By Theorem 2.4 there exists a finite semialgebraic stratification $B = \cup B_i^i$ of $B$ into Nash manifolds such that for each $i$, $X^i = X \cap R^{n+1} \times B_i^i$ is a Nash manifold and that there is a Nash diffeomorphism $\xi^i: M^R \times B_i \to X^i$ compatible with the projection onto $B_i^i$. For $(x, b) \in M^R \times B_i^i$, $\xi^i(x, b)$ is of the form $(\xi^i_1(x, b), \xi^i_2(x, b), b) \in M^R \times R \times B_i^i$. Then it is easy to see that the map $M^R \times B_i^i \ni (x, b) \to (\xi^i_1(x, b), b) \in M^R \times B_i^i$ is a diffeomorphism. Hence we can assume $\xi^i_1$ is the identity map of $M^R$ and we have a Nash function $h^i: M^R \times B_i^i \to R$ such that for each $b \in B_i^i$, the graph of the function $h^i(\cdot, b): M^R \to R$ coincides with $X_b$.

Note that there exists $i$ such that $a \in B_i^i_R$, i.e., $f = h_i^i(\cdot, a)$.

Consequently, there exist Nash manifolds $A$ and $C$ over $R$, Nash maps $F = (F_1, \ldots, F_k): U^R \times A \to R^k$, $G = (G_1, \ldots, G_k): V^R \times A \to R^k$, Nash functions
\( \Gamma_{i,j}: (U^R \cap V^R) \times C \to \mathbb{R} \) and \( \Delta_{i,j}: (U^R \cap V^R) \times C \to \mathbb{R} \), \( i, j = 1, \ldots, k \), and points \( a \in A_R \) and \( c \in C_R \) such that

\[
F_R(\cdot, a) = (f_1, \ldots, f_k), \quad G_R(\cdot, a) = (g_1, \ldots, g_k),
\]

\[
\Gamma_{i,j} R(\cdot, c) = \gamma_{i,j} \quad \text{and} \quad \Delta_{i,j} R(\cdot, c) = \delta_{i,j}.
\]

Replace \( A, C, a \) and \( c \) with \( A \times C, A \times C, (a, c) \) and \( (a, c) \) respectively. Then we can assume \( A = C \) and \( a = c \). Moreover, we can choose \( A, F, G, \Gamma_{i,j} \) and \( \Delta_{i,j} \) so that for each \( i \),

\[
(**) \quad F_i = \sum_{j=1}^{k} \Gamma_{i,j} G_j \quad \text{and} \quad G_i = \sum_{j=1}^{k} \Delta_{i,j} F_j \quad \text{on} \quad (U^R \cap V^R) \times A
\]

by the same reason as above, because it is possible to express by a formula of the first order theory of real closed field the fact that the equality \((**)\) holds.

By \((**)\) there exists a sheaf of \( \mathcal{N}_{M^R \times A}\)-ideals \( \mathcal{J} \) on \( M^R \times A \) such that \( \mathcal{J}|_{U^R \times A} \) and \( \mathcal{J}|_{V^R \times A} \) are generated by \( F_1, \ldots, F_k \) and \( G_1, \ldots, G_k \) respectively. By the theorem we have a finite number of generators \( H_i \) of \( \mathcal{J} \). Then it is easy to see that \( H_i R(\cdot, a) \) generate \( \mathcal{I} \).

**Proof of Extension conjecture.** It is sufficient to prove the following assertion.

Let \( M \subset \mathbb{R}^n \) be a Nash manifold. Let \( U \) and \( V \) be open semialgebraic subsets of \( M \) such that \( M = U \cup V \). Let \( \mathcal{I} \) be a sheaf of \( \mathcal{N}_M \)-ideals generated by a finite number of global Nash functions. Let \( f: U \to \mathbb{R} \) and \( g: V \to \mathbb{R} \) be Nash functions such that \( f - g \) is a cross-section of \( \mathcal{I}|_{U \cap V} \). Then there exists a Nash function \( h: M \to \mathbb{R} \) such that \( h|_U - f \) and \( h|_V - g \) are cross-sections of \( \mathcal{I}|_U \) and \( \mathcal{I}|_V \) respectively.

Let \( \varphi_i, i = 1, \ldots, l \), be generators of \( \mathcal{I} \). We have

\[
f - g = \sum_{i=1}^{l} \gamma_i \varphi_i \quad \text{on} \quad U \cap V
\]

for some Nash functions \( \gamma_i: U \cap V \to \mathbb{R} \). Then, as in the preceding proof of Global equation conjecture, we can assume \( M = M^R_R, U = U^R_R \) and \( V = V^R_R \) for some Nash manifold \( M^R \) over \( \mathbb{R} \) and open semialgebraic subsets \( U^R \) and \( V^R \) of \( M^R \), and we obtain a Nash manifold \( A \) over \( \mathbb{R} \), a point \( a \) of \( A_R \) and Nash functions \( F: U^R \times A \to \mathbb{R}, G: V^R \times A \to \mathbb{R}, \Phi_i: M^R \times A \to \mathbb{R}, i = 1, \ldots, l \), and \( \Gamma_i: (U^R \cap V^R) \times A \to \mathbb{R}, i = 1, \ldots, l \), such that

\[
F - G = \sum_{i=1}^{l} \Gamma_i \Phi_i \quad \text{on} \quad (U^R \cap V^R) \times A,
\]

\[
F_R(\cdot, a) = f, \quad G_R(\cdot, a) = g, \quad \Phi_i R(\cdot, a) = \varphi_i \quad \text{and} \quad \Gamma_i R(\cdot, a) = \gamma_i.
\]

Let \( \mathcal{J} \) be the sheaf of \( \mathcal{N}_{M^R \times A}\)-ideals on \( M^R \times A \) generated by \( \Phi_i \). Then, since Extension conjecture holds true for \( \mathbb{R} \), there exists a Nash function \( H: M^R \times A \to \mathbb{R} \) such that \( H|_{U^R \times A} - F \) and \( H|_{V^R \times A} - G \) are cross-sections of \( \mathcal{J}|_{U^R \times A} \) and \( \mathcal{I}|_{V^R \times A} \) respectively. Clearly \( h = H R(\cdot, a) \) fulfills the requirements.

**Problem.** Open problems are Global extension and Extension conjectures for a general real closed field.
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