Koecher-Maaß Dirichlet series for Eisenstein series of Klingen type

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1 Introduction

Let $f(Z)$ be a Siegel modular form of weight $k$ belonging to the symplectic group $\Gamma_n = Sp_n(\mathbb{Z})$. Then $f(Z)$ has a Fourier expansion of the form:

$$f(Z) = \sum_A a_f(A) \exp(2\pi i \text{tr}(AZ)),\nonumber$$

where $A$ runs over all semi-positive definite half-integral matrices over $\mathbb{Z}$ of degree $n$ and $\text{tr}(X)$ denotes the trace of a matrix $X$. We then define the Koecher-Maaß Dirichlet series $L(f, s)$ by

$$L(f, s) = \sum_A \frac{a_f(A)}{e(A)(\det A)^s},\nonumber$$

where $A$ runs over a complete set of representatives of $GL_n(\mathbb{Z})$-equivalence classes of positive definite half-integral matrices of degree $n$, and $e(A)$ denotes the order of the orthogonal group of $A$. The Koecher-Maaß Dirichlet series can also be obtained as the Mellin transform of $F$, and therefore its analytic properties are relatively known. As for this, we refer to Maaß [M], and Arakawa [Ar1],[Ar2]. However we had little knowledge about its arithmetic properties. Thus we present the following problem:

**Problem 1:** Investigate the arithmetic properties of $L(f, s)$.

To this problem, Böcherer and Shulze-Pillot have made a large contribution. As for this, we refer to [B-R1],[B-R2], and [B-R3]. In those papers, they mainly
treat the case of Yoshida lifting. In this note, we take another approach to this problem. Namely we consider the Koecher-Maaß Dirichlet series for Eisenstein series of Klingen type; let $f$ be a cusp form of weight $k$ belonging to $\Gamma_r$ ($0 \leq r \leq n$) and define $[f]^n_r(Z)$ as

$$[f]^n_r(Z) = \sum_{M \in \Delta_{n,r} \setminus \Gamma_n} f(M < Z >^*) j(M, Z)^{-k},$$

where $\Delta_{n,r} = \{ \begin{pmatrix} * & \ast \\ O_{n-r,n+r} & * \end{pmatrix} \in \Gamma_n \}$, and for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ let $M < Z >^*$ denote the upper left $r \times r$-block of the matrix $(AZ + B)(CZ + D)^{-1}$ and $j(M, Z) = \det(CZ + D)$. We note that $[1]^0_n(Z)$ is nothing but the Siegel Eisenstein series $E_{n,k}(Z)$ of weight $k$. We then propose the following problem:

**Problem 2.** Let $0 \leq r < n$. Then give an explicit form of $L(\{f\}^n_r, s)$ in terms of $f$.

In [B2] among others Böcherer gave an explicit form of $L(\{f\}^2_r, s)$ for $r = 0, 1$. In [I-K1] we gave an explicit form of $L(E_{n,k}(Z), s)$ for an arbitrary $n$. We note that $L(E_{n,k}(Z), s)$ is also regarded as the zeta function of prehomogeneous vector space. From this point of view, Saito gave a generalization of our result (cf. [Sa]). In relation to Problem 2 we should remark that a certain Dirichlet series attached to $f$ appears in the explicit formula for $L(\{f\}^2_1, s)$ by [B2]. This Dirichlet series is a modification of the Dirichlet series originally defined by Kohnen and Zagier [K-Z], and is of importance in its own right. Böcherer obtained a functionnal equation for the Dirichlet series from a general theory of the Koecher-Maaß Dirichlet series. Hence the following problem seems very interesting.

**Problem 3.** Investigate the analytic and arithmetic properties of the Dirichlet series related to $f$ appearing in an explicit formula for $L(\{f\}^n_r, s)$.

In this note, we give an answer to Problems 2 and 3 for the case $\{f\}^1_1$ with $f$ a cusp form belonging to $\Gamma_1$ and $n$ even. This also gives a certain generalization of Böcherer’s result in [B2].

Now to state our main result, for a non-zero integer $m$ such that $m \equiv 1$ mod 4 or $\equiv 0$ mod 4, let $\psi_m$ denote the character of the quadratic field $K$ whose discriminant is $m$. Here we understand that $\psi_1 = 1$. Put

$$\mathcal{F}_n = \{ d_0 \in \mathbb{Z}_+; d_0 \text{ is the fundamental discriminant of a quadratic field or } 1 \}.$$
For a positive integer $D = D_0m^2$ with $D_0 \in \mathcal{F}_n$ and $m > 0$, put

$$L_D(s) = L(s, \psi_{(-1)^{n/2}D_0}) \sum_{d|m} \mu(d)\psi_{(-1)^{n/2}D_0}(d)d^{-s} \sum_{c|m} c^{1-2s},$$

where $L(s, \psi_{(-1)^{n/2}D_0})$ is Dirichlet L-function attached to $\psi_{(-1)^{n/2}D_0}$, and $\mu$ is the Möbius function. Write $L_D(s)$ as

$$L_D(s) = \sum_{m=1}^{\infty} \epsilon_D(m)m^{-s},$$

and for a modular form $f(z) = \sum_{m=1}^{\infty} a(m)e^{2\pi imz}$ of weight $k$ with respect to $\Gamma_1$ put

$$L(f, s, D) = \sum_{m=1}^{\infty} a(m)\epsilon_D(m)m^{-s},$$

and

$$\mathcal{L}(f, \lambda, s) = \sum_{D} L(f, \lambda, S)D^{-s},$$

where $D$ runs over all positive integers such that $(-1)^{n/2}D \equiv 1, 0 \text{ mod } 4$. This type of Dirichlet series was originally introduced by Kohnen and Zagier [K-Z]. Further let $\zeta^{+}(f, s)$ denote the standard zeta function of $f$. Note that we have

$$\mathcal{L}(f, \lambda, s) = \zeta^{+}(f; 2s + 2\lambda - 1)\zeta(2s) \sum_{D_0 \in \mathcal{F}_n} D_0^{-s}\zeta(f; \psi_{(-1)^{n/2}D_0}; \lambda)$$

$$\times \prod_{p} \{(1 + p^{-2s+k-1-2\lambda})\psi_{(-1)^{n/2}D_0}(p)^2)(1 + p^{-2s+k-2\lambda})$$

$$-a(p)\psi_{(-1)^{n/2}D_0}(p)p^{-2s-\lambda}(1 + p^{k-2\lambda})\},$$

where $\zeta(f; \psi_{(-1)^{n/2}D_0}; s)$ denotes the twisted zeta function of $f$ by $\psi_{(-1)^{n/2}D_0}$.

**Theorem 1** Let $n$ be even. Then we have

$$L([f]_1^n, s) = 2^{ns}\gamma_n[k] \frac{\zeta(f; k - n/2)}{\zeta^{+}(f; k - 1)} \frac{\zeta(f; k - n/2)}{\zeta^{+}(f; k - 1)} \prod_{i=1}^{n/2} \zeta(2s - 2i + 1) \prod_{i=1}^{n/2-1} \zeta(2s - 2k + 2i + 2)$$

$$\times \mathcal{L}(f, k - 1, s - k + 3/2)$$

$$+ (-1)^{n(n-2)/8} \frac{\zeta(f; k - 1)}{\zeta^{+}(f; k - 1)} \prod_{i=1}^{n/2-1} \zeta(2s - 2i) \prod_{i=1}^{n/2-1} \zeta(2s - 2k + 2i + 1)$$

$$\times \mathcal{L}(f, k - n/2, s - k + (n + 1)/2)],$$
where $\gamma_{n,k}$ is a constant depending only on $n$ and $k$.

By the above theorem combined with a general theory of $L([f]_1^n, s)$ by Maaß [M], we obtain

**Corollary.** Put

$$L(f, n, s) = \pi^{-2s} \zeta(2s + 2k - 2n) \Gamma(s + k - (n + 1)/2) \Gamma(s + k - (n + 2)/2) L(f; k - n/2, s).$$

Then $L(f, n, s)$ can be continued analytically to a meromorphic function of $s$ in the whole complex plane, and has the following functional equation:

$$L(f, n, n + 1 - s - k) = L(f, n, s).$$

**Remark 1.** If $n = 2$, the two terms inside the brackets in Theorem 1 coincide with each other, and unify in one term. This is nothing but Böcherer’s result in [B2].

**Remark 2.** A similar formula holds for any $1 \leq r < n$. In particular we obtain an explicit formula for $r = 1$ and $n$ odd.

Theorem 1 cannot be derived directly from the commutativity of Siegel operator and Hecke operators. The main idea of the proof is to relate the Koecher-Maaß Dirichlet series for a modular form $F$ to the standard zeta function for $F$. To be more precise, in Section 2 on the set of half-integral matrices we introduce a certain arithmetic function, which we call the squared Möbius function, and give a certain induction formula for the number of representations of half-integral matrices (cf. Theorem 2). In Section 3, we express the Koecher-Maaß Dirichlet series $L(F, s)$ in terms of the squared Möbius function, the standard zeta function, and the ”primitive coefficients” of $F$. (cf. Theorem 3.1). The primitive coefficients of Eisenstein series of Klingen type is well-known (cf. Proposition 4.1). Thus, in Section 4, applying Theorem 3.1 to $F = [f]_1^n$ with $f$ a cusp form belonging to $\Gamma_1$, we express $L([f]_1^n, s)$ as a sum of Euler products (cf. Theorem 4.2), and complete the proof in the final section. For the detail, see [I-K2] and [I-K3].
2 Squared Möbius function for half-integral matrices

For an integral domain $R$ of characteristic 0 let $\mathcal{H}_n(R)$ denote the set of half-integral matrices over $R$. Further let $\mathcal{H}_n(\mathbb{Z})_{>0}$ (resp. $\mathcal{H}_n(\mathbb{Z})_{\geq 0}$) denote the set of positive definite (resp. semi-positive definite) half-integral matrices over $\mathbb{Z}$. Throughout this note, for two half-integral matrices $A$ and $B$ over $\mathbb{Z}_p$ of degree $n$ we write $A \sim B$ if there is a unimodular matrix $X$ of degree $n$ with entries in $\mathbb{Z}_p$ such that $^tXAX = B$. Further for two square matrices $U$ and $V$ we write $U \perp V = \left( \begin{array}{cc} U & 0 \\ 0 & V \end{array} \right)$.

A half-integral matrix $A$ over $\mathbb{Z}_p$ is called non-degenerate modulo $p$ if the quadratic form $\overline{A}[x]$ over $\mathbb{Z}_p/p\mathbb{Z}_p$ associated with $A$ is non-degenerate. We should remark that $A$ is non-degenerate modulo $p$ if and only if $A$ is unimodular in the case of $p \neq 2$, where as it is non-degenerate modulo 2 if and only if $A = \frac{1}{2}U$ or $A \sim \frac{1}{2}U \perp c$ with $U$ an even-integral unimodular matrix and $c \in \mathbb{Z}_2^*$ in the case of $p = 2$. To define the arithmetic function in the introduction, first we define

$$K'_n(\mathbb{Z}_p) = \{ A \in \mathcal{H}_n(\mathbb{Z}_p); A \sim V_0 \perp V_1 \text{ with } V_0, V_1 \text{ non-degenerate modulo } p \}.$$ 

Next let $p = 2$. We then define a subset $K''_n(\mathbb{Z}_2)$ of $\mathcal{H}_n(\mathbb{Z}_2)$ by

$$K''_n(\mathbb{Z}_2) = \{ A \in \mathcal{H}_n(\mathbb{Z}_2); A \sim \frac{1}{2}V_0 \perp V \perp V_1 \text{ with } V_0, V_1 \text{ even integral matrices} \}.$$ 

and $V$ a diagonal unimodular matrix of degree 2 such that $\det V \equiv 1 \mod 4$, and

$$K_n(\mathbb{Z}_p) = K'_n(\mathbb{Z}_2) \cup K''_n(\mathbb{Z}_2)$$ or $K'_n(\mathbb{Z}_p)$

according as $p = 2$ or not. For a $p$-adic number $c$ put

$$\chi_p(c) = 1, -1 \text{ or } 0$$

according as $\mathbb{Q}_p(\sqrt{c}) = \mathbb{Q}_p, \mathbb{Q}_p(\sqrt{c})/\mathbb{Q}_p$ is quadratic unramified, or $\mathbb{Q}_p(\sqrt{c})/\mathbb{Q}_p$ is quadratic ramified. Further for a symmetric matrix $A$ of even degree $n$ with entries in $\mathbb{Q}_p$ we put

$$\xi_p(A) = \chi_p((-1)^{n/2} \det A).$$

For a non-degenerate half-integral matrix $A$ we define $\sigma_p(A)$ as follows; first assume that $A$ belongs to $K'_n(\mathbb{Z}_p)$. Then we have $A \sim \frac{1}{2}V_0 \perp \frac{1}{2}V_1$ with $V_0, V_1$ non-degenerate
matrices modulo $p$ of degree $n_0$ and $n_1$, respectively. Then we put

$$
\sigma_p(A) = \begin{cases} 
(-1)^{n_1/2} \xi_p(V_1)p^{(n_1^2-2n_1)/4} & \text{if } n_1 \text{ is even} \\
(-1)^{(n_1-1)/2}p^{(n_1-1)/4} & \text{if } n_1 \text{ is odd.}
\end{cases}
$$

Next let $p = 2$ and assume that $A$ belongs to $\mathcal{K}_n'(\mathbb{Z}_2)$. Then we have $A \sim \frac{1}{2} V_0 \perp V \perp V_1$ with $V_0, V_1$ even-integral unimodular matrices of degree $n_0$ and $n_1$, respectively, and $V$ a unimodular diagonal matrix of degree 2 such that $\det V \equiv 1 \mod 4$. Then we put

$$
\sigma_p(A) = (-1)^{n_1/2}p^{n_1^2/4}.
$$

Finally if $A$ does not belong to $\mathcal{K}_n(\mathbb{Z}_p)$ we put $\sigma_p(A) = 0$. For a non-degenerate half-integral matrix $A$ over $\mathbb{Z}$ put

$$
\sigma(A) = \prod_p \sigma_p(A).
$$

By definition $\sigma(A)$ depends only on the genus of $A$. Put $\mathcal{K}_n(\mathbb{Z}) = \mathcal{H}_n(\mathbb{Z}) \cap \prod_p \mathcal{K}_n(\mathbb{Z}_p)$. Then by definition we have $\sigma(A) = 0$ for $A \not\in \mathcal{K}_n(\mathbb{Z})$. We remark that $\mathcal{H}_1(\mathbb{Z}) \cap GL_1(\mathbb{Q})$ can be identified with the set of all non-zero integers. Further by definition we have $\sigma(a) = 1$ or 0 according as $a$ is square free or not, and therefore, $\sigma$ is nothing but the square of the usual Möbius function in case $n = 1$. Thus we call $\sigma$ the squared Möbius function over $\mathcal{H}_n(\mathbb{Z})$. Now for a non-degenerate positive definite half-integral matrices $A$ and $B$ of degree $n$ over $\mathbb{Z}$ put

$$
G(A, B) = \sum_{A' \in \mathcal{G}(A)} \frac{a(A', B)}{a(A', A')},
$$

where $\mathcal{G}(A)$ denotes the set of equivalence classes belonging to the genus of $A$, and $a(A, B)$ the representation number of $B$ by $A$. As is well-known $G(A, B)$ is determined by $\mathcal{G}(A)$ and $\mathcal{G}(B)$. Then we have

**Theorem 1.** Let $A$ be a positive definite half-integral matrix of degree $n$ over $\mathbb{Z}$. Then we have

$$
\sum_{A_0} \sigma(A_0) G(A_0, A) = 1,
$$

where $A_0$ runs over all genera of positive definite half-integral matrices of degree $n$.

For a proof, see [I-K2].
3 Koecher-Maaß Dirichlet series and the standard zeta function

From now on for a $p$-adic number $c$ let $\nu(c) = \nu_p(c)$ denote the normalized additive valuation on $\mathbb{Q}_p$. Now for a Siegel modular form $f$ of weight $k$ belonging to $\Gamma_n$ we define the Koecher-Maaß Dirichlet series $L(f, s)$ for $f$ as in Introduction. For a non-degenerate half-integral matrix $A$ over $\mathbb{Z}_p$ let $r = r_p(A)$ denote the rank of a maximal totally singular subspace of the quadratic space over $\mathbb{Z}_p/p\mathbb{Z}_p$ associated with $A$. If $n - r$ is even, we have $A \sim \frac{1}{2}U_0 \perp \frac{1}{2}U_1$ with $U_0$ an even integral unimodular matrix of degree $n - r$ and $U_1$ an even integral matrix. We then put $\eta_p(A) = \xi_p(\frac{1}{2}U_0)$.

Now we define a polynomial $B_p(v; A)$ by

$$B_p(v, A) = \begin{cases} 
(1 + v)(1 - \eta_p(A)p^{-(n-r)/2}v) \prod_{i=1}^{(n-r)/2-1}(1 - p^{-2i}v^2) & \text{if } n - r \text{ is even} \\
(1 + v)\prod_{i=1}^{(n-r-1)/2}(1 - p^{-2i}v^2) & \text{if } n - r \text{ is odd}. 
\end{cases}$$

Here we make the convention that $B_p(v, A) = 1$ if $r = n$. For a non-degenerate half-integral matrix $A$ over $\mathbb{Z}$ put

$$B(s; A) = \prod_p B(p-s; A).$$

For a positive definite half-integral matrix $A$ of degree $n$ over $\mathbb{Z}$, put

$$M(A) = \sum_{A' \in \Theta(A)} \frac{1}{a(A', A')}.$$

Now let $A$ and $B$ be non-degenerate half-integral matrices of degree $n$ over $\mathbb{Z}$. We say $A$ dominates $B$ over $\mathbb{Z}$ if there is a square matrix $D$ with entries in $R$ such that $B = ^tDAD$, and define a finite Euler product $T(s; A, B)$ by

$$T(s; A, B) = \prod_p \prod_{i=1}^{m_p}(1 - p^{-s-n+i}).$$

where $m_p = 1/2(\nu_p(\text{det } B) - \nu_p(\text{det } A))$. We also put $T(s; A, B) = 1$ if $A$ does not dominate $B$ over $\mathbb{Z}$.

Now following [B-R], we define a "primitive" Fourier coefficient $a_f^*(A)$ by means of the relation:

$$a_f(A) = \sum_D a_f^*(^tD^{-1}AD^{-1}),$$
where $D$ runs over a complete set of representatives of left $GL_n(\mathbb{Z})$-equivalence classes of non-degenerate square matrices of degree $n$, and put

$$G_J^*(C_0) = \sum_{C \in \mathcal{G}(C_0)} a_J^*(C).$$

Put

$$K(f, s) = \sum_{A_0} \frac{\sigma(A_0)B(2s + 1 - k, A_0)}{(\det A_0)^s}$$

$$\times M(A_0) \sum_{C_0} \frac{G(C_0, A_0)G_J^*(C_0)}{M(C_0)} T(2s + 2 - 2k; C_0, A),$$

where $A_0$ and $C_0$ run over all genera of positive definite matrices of degree $n$.

Now let $L_{np} = L(GSp_n(\mathbb{Q}_p), Sp_n(\mathbb{Z}_p))$ be the Hecke algebra associated with the pair $(GSp_n(\mathbb{Q}_p), Sp_n(\mathbb{Z}_p))$ for each prime $p$. Assume that $f$ is an eigen function for all the Hecke operators, and for each prime $p$ let $\alpha_{0,p}, \alpha_{1,p}, ..., \alpha_{n,p}$ denote the Satake parameters of $L_{np}$ determined by $f$. We then define the standard zeta function $\zeta^+(f, s)$ of $f$ by

$$\zeta^+(f, s) = \prod_p \left\{ \prod_{i=1}^n (1 - \alpha_{i,p}p^{-s})(1 - \alpha_{i,p}^{-1}p^{-s}) \right\}^{-1}.$$ 

We note that the analytic and arithmetic properties of $\zeta^+(f, s)$ are fairly well known (cf. [An2],[B1],[Sh]). Then by [An1, Theorem 1],[An2, Theorem 4.3.19] and Theorem 1 we obtain

**Theorem. 3.1** Let $A \in \mathcal{K}_n(\mathbb{Z})$. We have

$$L(f, s) = \zeta^+(f, 2s + 1 - k)K(f, s)$$

An explicit formula of $M(A)$ is well known (cf. [Ki2, Theorem 5.6.3]). To give an explicit formula of $G(A, B)$ for $A, B \in \mathcal{K}_n(\mathbb{Z}) \cap \mathcal{H}_n(\mathbb{Z})_{>0}$, let $\alpha_p(A, B)$ be the local density representing $B$ by $A$ over $\mathbb{Z}_p$, and put

$$G_p(A, B) = \frac{\alpha_p(A, B)}{\alpha(A, A)} p^{-\nu_p(\det B) + \nu_p(\det A)/2}.$$ 

Then by Siegel's main theorem on quadratic forms we have

$$G(A, B) = \prod_p G_p(A, B)$$
Now for a non-degenerate matrix $U$ modulo $p$ of degree $n$ put

$$J(i, U, p) = \begin{cases} (p^{n/2} - \xi_p(U))(p^{n/2-i} + \xi_p(U)) \prod_{j=1}^{i-1}(p^{n-2j} - 1) & \text{if } n \text{ is even} \\ \prod_{j=1}^{i}(p^{n-2j+1} - 1) & \text{if } n \text{ is odd}. \end{cases}$$

Further put $\phi_i(x) = \prod_{j=1}^{i}(x^j - 1)$. Then the following proposition gives us an explicit formula of $G_p(A, B)$, and therefore that of $G(A, B)$:

**Proposition 3.2** Let $A, B \in \mathcal{K}_n(\mathbb{Z}_p)$, and $i = (\nu_p(\det B) - \nu_p(\det A))/2$. Assume that $A$ dominates $B$ over $\mathbb{Z}_p$.

1. Let $B \sim \frac{1}{2}U_0 \perp \frac{1}{2}U_1$ with $U_0, U_1$ non-degenerate modulo $p$. Then we have

$$G_p(A, B) = \frac{J(i; \frac{1}{2}U_1; p)}{\phi_i(p)}.$$

2. Let $p = 2$ and $B \sim \frac{1}{2}U_0 \perp V \perp U_1$ with $U_0, U_1$ even unimodular and $V$ a diagonal unimodular matrix of degree 2 such that $\det V \equiv 1 \mod 4$. Then we have

$$G_p(A, B) = \frac{J(i; \frac{1}{2}U_1 \perp 1; p)}{\phi_i(p)}.$$

Thus, if we get an explicit form of $G_f^*(C_0)$, we will know a lot of information on $L(f, s)$. In fact, in the case where $f$ is Klingen-Eisenstein series, by [B-R] or [Kil], we know an explicit form of $G_f^*(C_0)$, and therefore give an explicit form of $L(f, s)$ by the above theorem. We also remark that we have given an explicit form of $L(f, s)$ for Siegel-Eisenstein series $f$ by a different method from this note (cf. [I-K1]).

**4 Koecher-Maaß Dirichlet series for Eisenstein series of Klingen type**

Let $f$ be a Siegel cusp form of weight $k$ belonging to $\Gamma_r$ and $[f]^n_r$ the Klingen's Eisenstein series of degree $n$ attached to $f$. Then $f$ and $[f]^n_r$ have the following Fourier expansions:

$$f(z) = \sum_{C \in \mathcal{H}_r(\mathbb{Z}) \geq 0} b(C) \exp(2\pi i r(Cz)).$$
\[ f^n_r(Z) = \sum_{T \in \mathcal{H}_n(Z)} a_{n,f}(T) \exp(2\pi i \text{tr}(TZ)). \]

For two positive definite half-integral matrices \( B \) and \( C \) of degree \( m \) and \( n \), respectively, over \( \mathbb{Z} \) put

\[ G(B, C)^* = \sum_{B' \in \mathcal{H}(B)} \frac{a(B', C)^*}{a(B', B')}, \]

where \( a(B', C)^* \) denotes the number of primitive representations of \( C \) by \( B \). Then rewriting [B-R, Theorem 1] we have

**Proposition 4.1.** We have

\[ G_{n,f}(B)^* = a_{n,k}(B)^* \sum_{c} \frac{G(B, C)^* b(C)^*}{(\det C)^{k-1/2} a(C, C) a_{r,k}(C)^*}, \]

where \( a_{n,k}(B)^* \) and \( a_{r,k}(C)^* \) denote the primitive Fourier coefficients of Siegel-Eisenstein series of degree \( n \) and \( r \), respectively.

Now let \( r = 1 \). For an element \( A \) of \( \mathcal{H}_n(\mathbb{Z}_p) \) and a non-zero \( p \)-adic integer, put

\[ H_p(s; A; e) \]

\[ = \frac{p^{((n+1)/2-s)\nu(\det A)\sigma_p(A)B_p(p^{-2s-k+1}; A)}}{\alpha_p(A, A)} \sum_{C_0} p^{(2k-1-n)\nu(\det C_0)/2} G_p(C_0, A) \]

\[ \times T_p(p^{-2s-k+1}; C_0, A) \alpha_p(H_k, C_0)^* p^{-\nu(\det C_0)/2} \alpha_p(C_0, e)^*, \]

and for a non-zero \( p \)-adic number \( d_0 \), and a function \( \omega \) on \( \mathcal{H}_n(\mathbb{Z}_p) \) put

\[ H_p(s; d_0; \omega; e) \]

\[ = \sum_{r=0}^{\infty} \sum_{\nu(\det A = p^{2s-2n/2}) \mid r d_0} \omega(A) H_p(s; A; e), \]

where for a half-integral matrix \( U \) and \( V, \alpha_p(U, V)^* \) denotes the primitive local density representing \( V \) by \( U \), and \( G_p(C_0, A) \) is the one defined in Section 1. Let \( \iota_p \) be a constant function on \( \mathcal{H}_n(\mathbb{Z}_p) \) taking the value 1, and \( h_p \) the Hasse invariant on \( \mathcal{H}_n(\mathbb{Z}_p) \). We note that \( h_p(C_0) \) for \( C_0 \in \mathcal{H}_n(\mathbb{Z}_p) \) is the same as that of \( A \) if \( C_0 \) dominates \( A \) over \( \mathbb{Z}_p \). Let \( A \) be a positive definite half-integral matrix of degree \( n \) over \( \mathbb{Z} \). If \( n \) is even, then \( \det A \) can be expressed as \( d_0 f^2 \) with positive integers \( d_0 \) and \( f \) such that \( \nu_p(d_0) \leq 1 \) for \( p \neq 2 \), and \((-1)^{n/2}d_0 \equiv 1 \) or \( \equiv 0 \) mod 4. If \( n \) is odd, \( \det A \) can be expressed as \( d_0 f^2 \) with a positive integer \( f \) and a square free positive integer \( d_0 \). Thus by Theorem 3.1 and Proposition 4.1, we have
Theorem 4.2. (1) Let $n$ be even. Then we have
\[ K([f]^n, s) = \alpha_{nk} \sum_{e=1}^{\infty} b(e) \cdot B(k-1, e) \cdot e^{n/2} \sum_{d_0 \in \mathcal{F}_n} \prod_{p} H_p(s; d_0; \iota_p; e) + \prod_{p} H_p(s; d_0; h_p; e), \]
where $\mathcal{F}_n$ is the set defined in Section 1, and $\alpha_{nk}$ is a constant depending only on $n$ and $k$.

(2) Let $n$ be odd. Then we have
\[ K([f]^n, s) = \beta_{nk} \sum_{e=1}^{\infty} b(e) \cdot B(k-1, e) \cdot e^{n/2} \sum_{d_0} \prod_{p} H_p(s; d_0; \iota_p; e) + \prod_{p} H_p(s; d_0; h_p; e), \]
where $d_0$ runs over all square free positive integers, and $\beta_{nk}$ is a constant depending only on $n$ and $k$.

5 Proof of Theorem 1

In this section let $n$ be even. Then by Proposition 3.2, Theorem 4.2, and [Ki2, Theorem 5.6.3] combined with some combinatorial technique, we obtain

**Theorem 5.1** Let $n$ be even, and $D_0 \in \mathbb{Z}_p^*$ with $p$ odd, or $D_0 \in \mathbb{Z}_2^*$ such that $(-1)^{n/2}D_0 \equiv 1 \mod 4$. Put $Q(e, D_0) = 1 - p^{-n+2}$ or 1 according as $e \equiv 0 \mod p^2$ or not, and $R(e, D_0) = 1 + \delta p^{-n/2+1}$ or 1 according as $e \equiv 0 \mod p$ or not, where $\delta = \chi_p((-1)^{n/2}D_0)$. Further put $\Phi_{nk} = \frac{1 - p^{-k}}{\phi(n/2)}(1 - p^{-2})^k$. Then we have

(1)
\[ H_p(s; D_0; \iota_p; e) = 2^{\delta_p} p^n \cdot p^{n/2} \cdot p^{(n-2)n/2} \prod_{i=0}^{n/2-2} (1 - p^{2i-n+1+2i-2s})(1 - p^{2i+2-2s}) \]
\[ \times [Q(e, D_0) \cdot p^{n/2-k} \cdot (1 - p^{2i-n+1+2k-2s})(1 - p^{2i+2-2s})] \]
\[ + R(e, D_0)(1 + \delta p^{n/2-k}) \prod_{i=0}^{n/2-1} (1 - p^{2i-n+1+2k-2s})(1 - p^{2i+2-2s}). \]

(2)
\[ H_p(s; D_0; h_p; e) = (-1, -1) p^{n/2} \cdot p^{(n-2)n/2} \prod_{i=0}^{n/2-2} (1 - p^{2i-n+1+2i-2s})(1 - p^{2i+2-2s}) \]
\[ \times [Q(e, D_0) \cdot \delta p^{n/2-k} \cdot (1 - p^{2i-n+1+2k-2s})(1 - p^{2i+2-2s})] \]
\[ + R(e, D_0) \prod_{i=0}^{n/2-1} (1 - p^{2i-n+1+2k-2s})(1 - p^{2i+2-2s}). \]
\[(1+p^{-k+1})p^{-2s+2k-2}(1-p^{n/2-k})\delta (1-\delta p^{-n/2}) \prod_{i=0}^{n/2-2} \left(1-p^{2i-n+2k-2s}(1-p^{2i+1-2s})\right)\].

**Theorem 5.2** Let \( n \) be even, and \( D_0 \in p\mathbb{Z}_p^* \) with \( p \) odd, or \( D_0 \in 4\mathbb{Z}_2^* \) such that \((-1)^{n/2}4^{-1}D_0 \equiv 3 \mod 4 \) or \( D_0 \in 8\mathbb{Z}_2^* \). Put \( l_0 = \nu(D_0) \) and \( d_0 = 2^{-\delta_2 p} p^{-l_0} D_0 \).

Further put \( \Psi_{nk} = \frac{(1-p^{-k})\prod i=1(1-n/2p)-2k+2i}{\phi_{n/2-1}(p^{-2})} \cdot \)

(1) Put \( Q(e, D_0) = 1 - p^{-n+2} \) or 1 according as \( e \equiv 0 \mod p^2 \) or not. Then we have

\[ H_p(s; D_0; e_p, e) = 2^{\delta_2 p} p_{n-k} p^{(-s+k-3/2)l_0}(1 + p^{-2s+k-1})\Psi_{nk} \times \prod i=0^{n/2-1} \left(1-p^{2i-n-1+2k-2s}(1-p^{2i+1-2s})\right). \]

(2.1) Let \( p \neq 2 \), and \( R(e, D_0) = (\frac{(-1)^n e}{p}), \frac{(-p^{-2}eD_0)}{p} \) or 0 according as \( e \in \mathbb{Z}_p^* \), \( e \in p\mathbb{Z}_p^* \) or not, where \( (\frac{\cdot}{p}) \) denotes Legendre symbol. Then we have

\[ H_p(s; D_0; h_p, e) = p^{-s+k-(n+1)/2} R(e, D_0)(1-p^{-n-2k})(1+p^{-n-k})\Psi_{nk} \times (1+p^{-2s+k-1}) \prod i=0^{n/2-2} \left(1-p^{2i-n+2k-2s}(1-p^{2i+1-2s})\right). \]

(2.2) Let \( p = 2 \) and \( e = 2^r e_0 \) with \( (2, e_0) = 1 \). Put

\[ R(e, D_0) = (-1)^{n(n-2)/2} \frac{2^r (-1)^{n/2}}{e_0} \frac{(-1)^{n/2}(-1)(e-1)/2} {d_0} \) or 0 according as \( m_0 \leq 1 \) or not, where \( (\cdot\cdot\cdot)^{\cdot\cdot}\cdot \) denotes the Jacobi symbol. Then we have

\[ H_p(s; D_0; h_p, e) = 2^{n+k-(s+k-(n+1)/2)r} R(e, D_0)(1-p^{-n-2k})(1+p^{-n-k})\Psi_{nk} \times (1+p^{-2s+k-1}) \prod i=0^{n/2-2} \left(1-p^{2i-n+2k-2s}(1-p^{2i+1-2s})\right). \]

**Proof of Theorem 1.** By Theorems 5.1 and 5.2 combined with Theorem 4.2, we have

\[ K([f]_1^n, s) = 2^{n} \gamma_{nk} \frac{(-1)^{n(n-2)/2} \zeta(f; k - n/2)\zeta^+(f; k - 1) \prod i=0^{n/2-2} \zeta(2s - 2i + 2) \prod i=1^{n/2-1} \zeta(2s - 2k + n - 2i + 1)}{\zeta^+(f; k - 1) \prod i=0^{n/2-2} \zeta(2s - 2i + 2) \prod i=1^{n/2-1} \zeta(2s - 2k + n - 2i + 1)}. \]
$$\times \sum_{D_0} D_0^{-s+k-3/2} \zeta(f; \psi(-1)^{n/2}D_0; k-1)$$
$$\times \prod_p \left\{ (1+p^{-2s+k-2} \psi(-1)^{n/2}D_0(p)^2)(1+p^{-2s+k-1}) - a(p) \psi(-1)^{n/2}D_0(p)p^{-2s+k-2}(1+p^{2-k}) \right\}$$
$$+ \frac{\zeta(f; k-1)}{\zeta^{+}(f \cdot k-1) \prod_{i=0}^{n/2} \zeta(2s-2i-1) \prod_{i=1}^{n/2} \zeta(2s-2k-2i+n)}$$
$$\times \sum_{D_0} D_0^{-s+k-(n+1)/2} \zeta(f; \psi(-1)^{n/2}D_0; k-n/2)$$
$$\times \prod_p \left\{ (1+p^{-2s+k-2} \psi(-1)^{n/2}D_0(p)^2)(1+p^{-2s+k-1}) - a(p) \psi(-1)^{n/2}D_0(p)p^{-2s+k-n/2-1}(1+p^{n-k}) \right\}.$$