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Koecher-Maaß Dirichlet series for Eisenstein series of Klingen type

Introduction

Let $f(Z)$ be a Siegel modular form of weight $k$ belonging to the symplectic group $\Gamma_n = Sp_n(\mathbb{Z})$. Then $f(Z)$ has a Fourier expansion of the form:

$$f(Z) = \sum_A a_f(A) exp(2\pi i tr(AZ)),$$

where $A$ runs over all semi-positive definite half-integral matrices over $\mathbb{Z}$ of degree $n$ and $tr(X)$ denotes the trace of a matrix $X$. We then define the Koecher-Maaß Dirichlet series $L(f, s)$ by

$$L(f, s) = \sum_A \frac{a_f(A)}{e(A)(\det A)^s},$$

where $A$ runs over a complete set of representatives of $GL_n(\mathbb{Z})$-equivalence classes of positive definite half-integral matrices of degree $n$, and $e(A)$ denotes the order of the orthogonal group of $A$. The Koecher-Maaß Dirichlet series can also be obtained as the Mellin transform of $F$, and therefore its analytic properties are relatively known. As for this, we refer to Maaß [M], and Arakawa [Ar1],[Ar2]. However we had little knowledge about its arithmetic properties. Thus we present the following problem:

**Problem 1**: Investigate the arithmetic properties of $L(f, s)$.

To this problem, Böcherer and Shulze-Pillot have made a large contribution. As for this, we refer to [B-R1],[B-R2], and [B-R3]. In those papers, they mainly
treat the case of Yoshida lifting. In this note, we take another approach to this problem. Namely we consider the Koecher-Maaß Dirichlet series for Eisensetin series of Klingen type; let $f$ be a cusp form of weight $k$ belonging to $\Gamma_r$ ($0 \leq r \leq n$) and define $[f]_r^n(Z)$ as

$$[f]_r^n(Z) = \sum_{M \in \Delta_{n,r} \setminus \Gamma_n} f(M < Z >^*) j(M, Z)^{-k},$$

where $\Delta_{n,r} = \{ (O_{n-r,n+r}, * ) \in \Gamma_n \}$, and for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ let $M < Z >^*$ denote the upper left $r \times r$-block of the matrix $(AZ + B)(CZ + D)^{-1}$ and $j(M, Z) = \det(CZ + D)$. We note that $[1]_0^n(Z)$ is nothing but the Siegel Eisenstein series $E_{n,k}(Z)$.

We then propose the following problem:

**Problem 2.** Let $0 \leq r < n$. Then give an explicit form of $L([f]_r^n, s)$ in terms of $f$.

In [B2] among others Böcherer gave an explicit form of $L([f]_r^2, s)$ for $r = 0, 1$. In [I-K1] we gave an explicit form of $L(E_{n,k}(Z), s)$ for an arbitrary $n$. We note that $L(E_{n,k}(Z), s)$ is also regarded as the zeta function of prehomogeneous vector space. From this point of view, Saito gave a generalization of our result (cf. [Sa]). In relation to Problem 2 we should remark that a certain Dirichlet series attached to $f$ appears in the explicit formula for $L([f]_1^2, s)$ by [B2]. This Dirichlet series is a modification of the Dirichlet series originally defined by Kohnen and Zagier [K-Z], and is of importance in its own right. Böcherer obtained a functional equation for the Dirichlet series from a general theory of the Koecher-Maaß Dirichlet series. Hence the following problem seems very interesting.

**Problem 3.** Investigate the analytic and arithmetic properties of the Dirichlet series related to $f$ appearing in an explicit formula for $L([f]_r^n, s)$.

In this note, we give an answer to Problems 2 and 3 for the case $[f]_1^n$ with $f$ a cusp form belonging to $\Gamma_1$ and $n$ even. This also gives a certain generalization of Böcherer's result in [B2].

Now to state our main result, for a non-zero integer $m$ such that $m \equiv 1 \mod 4$ or $\equiv 0 \mod 4$, let $\psi_m$ denote the character of the quadratic field $K$ whose discriminant is $m$. Here we understand that $\psi_1 = 1$. Put

$$F_n = \{ d_0 \in \mathbb{Z}_+; d_0 \text{ is the fundamental discriminant of a quadratic field or } 1 \}.$$
For a positive integer $D = D_0m^2$ with $D_0 \in \mathcal{F}_n$ and $m > 0$, put

$$L_D(s) = L(s, \psi_{(-1)^{n/2}D_0}) \sum_{d|m} \mu(d)\psi_{(-1)^{n/2}D_0}(d)d^{-s} \sum_{c|m} c^{1-2s},$$

where $L(s, \psi_{(-1)^{n/2}D_0})$ is Dirichlet L-function attached to $\psi_{(-1)^{n/2}D_0}$, and $\mu$ is the Möbius function. Write $L_D(s)$ as

$$L_D(s) = \sum_{m=1}^{\infty} \epsilon_D(m)m^{-s},$$

and for a modular form $f(z) = \sum_{m=1}^{\infty} a(m)\exp(2\pi imz)$ of weight $k$ with respect to $\Gamma_1$ put

$$L(f, s, D) = \sum_{m=1}^{\infty} a(m)\epsilon_D(m)m^{-s},$$

and

$$\mathcal{L}(f, \lambda, s) = \sum_D L(f, \lambda, s)D^{-s},$$

where $D$ runs over all positive integers such that $(-1)^{n/2}D \equiv 1, 0 \pmod{4}$. This type of Dirichlet series was originally introduced by Kohnen and Zagier [K-Z]. Further let $\zeta^+(f, s)$ denote the standard zeta function of $f$. Note that we have

$$\mathcal{L}(f, \lambda, s) = \zeta^+(f; 2s + 2\lambda - 1)\zeta(2s) \sum_{D_0 \in \mathcal{F}_n} D_0^{-s} \zeta(f; \psi_{(-1)^{n/2}D_0}; \lambda)$$

$$\times \prod_p \left\{ (1 + p^{-2s+k-1-2\lambda}\psi_{(-1)^{n/2}D_0}(p)^2)(1 + p^{-2s+k-2\lambda}) 
-a(p)\psi_{(-1)^{n/2}D_0}(p)p^{-2s-\lambda}(1 + p^{k-2\lambda}) \right\},$$

where $\zeta(f; \psi_{(-1)^{n/2}D_0}; s)$ denotes the twisted zeta function of $f$ by $\psi_{(-1)^{n/2}D_0}$.

**Theorem 1** Let $n$ be even. Then we have

$$L([f]_1^n, s) = 2^{ns}\gamma_n,k[\zeta(f; k - n/2)^{n/2}\zeta^+(f; k - 1)^{n/2}\zeta(2s - 2i + 1)\zeta(2s - 2k + 2i + 2)$$

$$\times \mathcal{L}(f, k - 1, s - k + 3/2)$$

$$+ (-1)^{n(n-2)/8}\zeta(f; k - 1)^{n/2-1}\zeta^+(f; k - 1)^{n/2-1}\zeta(2s - n + 1)\zeta(2s - 2i)\zeta(2s - 2k + 2i + 1)$$

$$\times \mathcal{L}(f, k - n/2, s - k + (n + 1)/2)].$$
where $\gamma_{n,k}$ is a constant depending only on $n$ and $k$.

By the above theorem combined with a general theory of $L([f]_1^n, s)$ by Maaß [M], we obtain

**Corollary.** Put

\[
L(f, n, s) = \pi^{-2s} \zeta(2s+2k-2n) \Gamma(s + k - (n+1)/2) \mathcal{L}(f; k - n/2, s).
\]

Then $L(f, n, s)$ can be continued analytically to a meromorphic function of $s$ in the whole complex plane, and has the following functional equation:

\[
L(f, n, n+1 - s - k) = L(f, n, s).
\]

**Remark 1.** If $n = 2$, the two terms inside the brackets in Theorem 1 coincide with each other, and unify in one term. This is nothing but Böcherer's result in [B2].

**Remark 2.** A similar formula holds for any $1 \leq r < n$. In particular we obtain an explicit formula for $r = 1$ and $n$ odd.

Theorem 1 cannot be derived directly from the commutativity of Siegel operator and Hecke operators. The main idea of the proof is to relate the Koecher-Maaß Dirichlet series for a modular form $F$ to the standard zeta function for $F$. To be more precise, in Section 2 on the set of half-integral matrices we introduce a certain arithmetic function, which we call the squared Möbius function, and give a certain induction formula for the number of representations of half-integral matrices (cf. Theorem 2). In Section 3, we express the Koecher-Maaß Dirichlet series $L(F, s)$ in terms of the squared Möbius function, the standard zeta function, and the "primitive coefficients" of $F$. (cf. Theorem 3.1). The primitive coefficients of Eisenstein series of Klingen type is well-known (cf. Proposition 4.1). Thus, in Section 4, applying Theorem 3.1 to $F = [f]_1^n$ with $f$ a cusp form belonging to $\Gamma_1$, we express $L([f]_1^n, s)$ as a sum of Euler products (cf. Theorem 4.2), and complete the proof in the final section. For the detail, see [I-K2] and [I-K3].
2 Squared Möbius function for half-integral matrices

For an integral domain $R$ of characteristic 0 let $\mathcal{H}_n(R)$ denote the set of half-integral matrices over $R$. Further let $\mathcal{H}_n(\mathbb{Z})_{>0}$ (resp. $\mathcal{H}_n(\mathbb{Z})_{\geq 0}$) denote the set of positive definite (resp. semi-positive definite) half-integral matrices over $\mathbb{Z}$. Throughout this note, for two half-integral matrices $A$ and $B$ over $\mathbb{Z}_p$ of degree $n$ we write $A \sim B$ if there is a unimodular matrix $X$ of degree $n$ with entries in $\mathbb{Z}_p$ such that $^tXAX = B$. Further for two square matrices $U$ and $V$ we write $U \perp V = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$.

A half-integral matrix $A$ over $\mathbb{Z}_p$ is called non-degenerate modulo $p$ if the quadratic form $\overline{A}[x]$ over $\mathbb{Z}_p/p\mathbb{Z}_p$ associated with $A$ is non-degenerate. We should remark that $A$ is non-degenerate modulo $p$ if and only if $A$ is unimodular in the case of $p \neq 2$, where as it is non-degenerate modulo 2 if and only if $A = \frac{1}{2}U$ or $A \sim \frac{1}{2}U \perp c$ with $U$ an even-integral unimodular matrix and $c \in \mathbb{Z}_2^*$ in the case of $p = 2$. To define the arithmetic function in the introduction, first we define

$$\mathcal{K}_n'(\mathbb{Z}_p) = \{ A \in \mathcal{H}_n(\mathbb{Z}_p); A \sim V_0 \perp V_1 \text{ with } V_0, V_1 \text{ non–degenerate modulo } p \}.$$  

Next let $p = 2$. We then define a subset $\mathcal{K}_n''(\mathbb{Z}_2)$ of $\mathcal{H}_n(\mathbb{Z}_2)$ by

$$\mathcal{K}_n''(\mathbb{Z}_2) = \{ A \in \mathcal{H}_n(\mathbb{Z}_2); A \sim \frac{1}{2}V_0 \perp V \perp V_1 \text{ with } V_0, V_1 \text{ even–integral matrices}$$

and $V$ a diagonal unimodular matrix of degree 2 such that $\det V \equiv 1 \mod 4$},

and

$$\mathcal{K}_n(\mathbb{Z}_p) = \mathcal{K}_n'(\mathbb{Z}_2) \cup \mathcal{K}_n''(\mathbb{Z}_2) \text{ or } \mathcal{K}_n'(\mathbb{Z}_p)$$

according as $p = 2$ or not. For a $p$-adic number $c$ put

$$\chi_p(c) = 1, -1 \text{ or } 0$$

according as $Q_p(\sqrt{c}) = Q_p, Q_p(\sqrt{c})/Q_p$ is quadratic unramified, or $Q_p(\sqrt{c})/Q_p$ is quadratic ramified. Further for a symmetric matrix $A$ of even degree $n$ with entries in $Q_p$ we put

$$\xi_p(A) = \chi_p((-1)^{n/2} \det A).$$

For a non-degenerate half-integral matrix $A$ we define $\sigma_p(A)$ as follows; first assume that $A$ belongs to $\mathcal{K}_n'(\mathbb{Z}_p)$. Then we have $A \sim \frac{1}{2}V_0 \perp V_1 \text{ with } V_0, V_1 \text{ non-degenerate}$
matrices modulo \( p \) of degree \( n_0 \) and \( n_1 \), respectively. Then we put
\[
\sigma_p(A) = \begin{cases} 
(-1)^{n_1/2} \xi_p(V_1)p^{(n_1^2-2n_1)/4} & \text{if } n_1 \text{ is even} \\
(-1)^{(n_1-1)/2}p^{(n_1-1)^2/4} & \text{if } n_1 \text{ is odd.}
\end{cases}
\]

Next let \( p = 2 \) and assume that \( A \) belongs to \( \mathcal{K}_n^\prime(\mathbb{Z}_2) \). Then we have \( A \sim 1/2V_0 \perp V \perp V_1 \) with \( V_0, V_1 \) even-integral unimodular matrices of degree \( n_0 \) and \( n_1 \), respectively, and \( V \) a unimodular diagonal matrix of degree 2 such that \( \det V \equiv 1 \mod 4 \). Then we put
\[
\sigma_p(A) = (-1)^{n_1}2pn^{2}/14.
\]

Finally if \( A \) does not belong to \( \mathcal{K}_n(\mathbb{Z}_p) \) we put \( \sigma_p(A) = 0 \). For a non-degenerate half-integral matrix \( A \) over \( \mathbb{Z} \) put
\[
\sigma(A) = \prod_p \sigma_p(A).
\]

By definition \( \sigma(A) \) depends only on the genus of \( A \). Put \( \mathcal{K}_n(\mathbb{Z}) = \mathcal{H}_n(\mathbb{Z}) \cap \prod_p \mathcal{K}_n(\mathbb{Z}_p) \). Then by definition we have \( \sigma(A) = 0 \) for \( A \not\in \mathcal{K}_n(\mathbb{Z}) \). We remark that \( \mathcal{H}_1(\mathbb{Z}) \cap GL_1(\mathbb{Q}) \) can be identified with the set of all non-zero integers. Further by definition we have \( \sigma(a) = 1 \) or 0 according as \( a \) is square free or not, and therefore, \( \sigma \) is nothing but the square of the usual Möbius function in case \( n = 1 \). Thus we call \( \sigma \) the squared Möbius function over \( \mathcal{H}_n(\mathbb{Z}) \). Now for a non-degenerate positive definite half-integral matrices \( A \) and \( B \) of degree \( n \) over \( \mathbb{Z} \) put
\[
G(A, B) = \sum_{A' \in \mathcal{G}(A)} \frac{a(A', B)}{a(A', A')},
\]

where \( \mathcal{G}(A) \) denotes the set of equivalence classes belonging to the genus of \( A \), and \( a(A, B) \) the representation number of \( B \) by \( A \). As is well-known \( G(A, B) \) is determined by \( \mathcal{G}(A) \) and \( \mathcal{G}(B) \). Then we have

**Theorem 1.** Let \( A \) be a positive definite half-integral matrix of degree \( n \) over \( \mathbb{Z} \). Then we have
\[
\sum_{A_0} \sigma(A_0)G(A_0, A) = 1,
\]
where \( A_0 \) runs over all genera of positive definite half-integral matrices of degree \( n \).

For a proof, see [I-K2].
3 Koecher-Maaß Dirichlet series and the standard zeta function

From now on for a $p$-adic number $c$ let $\nu(c) = \nu_p(c)$ denote the normalized additive valuation on $\mathbb{Q}_p$. Now for a Siegel modular form $f$ of weight $k$ belonging to $\Gamma_n$ we define the Koecher-Maaß Dirichlet series $L(f, s)$ for $f$ as in Introduction. For a non-degenerate half-integral matrix $A$ over $\mathbb{Z}_p$ let $r = r_p(A)$ denote the rank of a maximal totally singular subspace of the quadratic space over $\mathbb{Z}_p/p\mathbb{Z}_p$ associated with $A$. If $n - r$ is even, we have $A \sim \frac{1}{2} U_0 \perp \frac{1}{2} U_1$ with $U_0$ an even integral unimodular matrix of degree $n - r$ and $U_1$ an even integral matrix. We then put $\eta_p(A) = \xi_p(\frac{1}{2} U_0)$.

Now we define a polynomial $B_p(v; A)$ by

$$B_p(v, A) = \begin{cases} (1+v)(1-\eta_p(A)p^{-(n-r)/2}v) \prod_{i=1}^{(n-r)/2-1}(1 - p^{-2i}v^2) & \text{if } n - r \text{ is even} \\ (1+v) \prod_{i=1}^{(n-r-1)/2}(1 - p^{-2i}v^2) & \text{if } n - r \text{ is odd} \end{cases}$$

Here we make the convention that $B_p(v, A) = 1$ if $r = n$. For a non-degenerate half-integral matrix $A$ over $\mathbb{Z}$ put

$$B(s; A) = \prod_p B_p(p^{-s}; A).$$

For a positive definite half-integral matrix $A$ of degree $n$ over $\mathbb{Z}$, put

$$M(A) = \sum_{A' \in \Theta(A)} \frac{1}{a(A', A')}.$$

Now let $A$ and $B$ be non-degenerate half-integral matrices of degree $n$ over $\mathbb{Z}$. We say $A$ dominates $B$ over $\mathbb{Z}$ if there is a square matrix $D$ with entries in $R$ such that $B = {}^t D A D$, and define a finite Euler product $T(s; A, B)$ by

$$T(s; A, B) = \prod_{p} \prod_{i=1}^{m_p} (1 - p^{-s-n+i}).$$

where $m_p = 1/2(\nu_p(\det B) - \nu_p(\det A))$. We also put $T(s; A, B) = 1$ if $A$ does not dominate $B$ over $\mathbb{Z}$.

Now following [B-R], we define a "primitive" Fourier coefficient $a_f^*(A)$ by means of the relation:

$$a_f(A) = \sum_D a_f^*({}^t D^{-1} A D^{-1}).$$
where $D$ runs over a complete set of representatives of left $GL_n(\mathbb{Z})$-equivalence classes of non-degenerate square matrices of degree $n$, and put

$$G^*_f(C_0) = \sum_{C \in \mathcal{G}(C_0)} \frac{a_f^*(C)}{a(C,C)}.$$ 

Put

$$K(f, s) = \sum_{A_0} \frac{\sigma(A_0)B(2s + 1 - k, A_0)}{(\det A_0)^s}$$

$$\times M(A_0) \sum_{C_0} \frac{G(C_0, A_0)G^*_f(C_0)}{M(C_0)} T(2s + 2 - 2k; C_0, A),$$

where $A_0$ and $C_0$ run over all genera of positive definite matrices of degree $n$.

Now let $L_{np} = L(GSp_n(\mathbb{Q}_p), Sp_n(\mathbb{Z}_p))$ be the Hecke algebra associated with the pair $(GSp_n(\mathbb{Q}_p), Sp_n(\mathbb{Z}_p))$ for each prime $p$. Assume that $f$ is an eigen function for all the Hecke operators, and for each prime $p$ let $\alpha_{0,p}, \alpha_{1,p}, ..., \alpha_{n,p}$ denote the Satake parameters of $L_{np}$ determined by $f$. We then define the standard zeta function $\zeta^+(f, s)$ of $f$ by

$$\zeta^+(f, s) = \prod_p \left( \prod_{i=1}^n (1 - \alpha_{i,p} p^{-s})(1 - \alpha_{i,p}^{-1} p^{-s}) \right)^{-1}.$$ 

We note that the analytic and arithmetic properties of $\zeta^+(f, s)$ are fairly well known (cf. [An2],[B1],[Sh]). Then by [An1, Theorem 1],[An2, Theorem 4.3.19] and Theorem 1 we obtain

**Theorem. 3.1** Let $A \in \mathcal{K}_n(\mathbb{Z})$. We have

$$L(f, s) = \zeta^+(f, 2s + 1 - k)K(f, s)$$

An explicit form of $M(A)$ is well known (cf. [Ki2, Theorem 5.6.3]). To give an explicit formula of $G(A, B)$ for $A, B \in \mathcal{K}_n(\mathbb{Z}) \cap \mathcal{H}_n(\mathbb{Z})_{>0}$, let $\alpha_p(A, B)$ be the local density representing $B$ by $A$ over $\mathbb{Z}_p$, and put

$$G_p(A, B) = \frac{\alpha_p(A, B)}{\alpha(A, A)} p^{(-\nu_p(\det B) + \nu_p(\det A))/2}.$$ 

Then by Siegel's main theorem on quadratic forms we have

$$G(A, B) = \prod_p G_p(A, B)$$
Now for a non-degenerate matrix $U$ modulo $p$ of degree $n$ put

$$J(i, U, p) = \begin{cases} (p^{n/2} - \xi_p(U))(p^{n/2-i} + \xi_p(U)) \Pi_{j=1}^{i-1}(p^{n-2j} - 1) & \text{if } n \text{ is even} \\ \Pi_{j=1}^{i}(p^{n-2j+1} - 1) & \text{if } n \text{ is odd.} \end{cases}$$

Further put $\phi_i(x) = \Pi_{j=1}^{i}(x^{j}-1)$. Then the following proposition gives us an explicit formula of $G_p(A, B)$, and therefore that of $G(A, B)$:

**Proposition 3.2** Let $A, B \in \mathcal{K}_n(\mathbb{Z}_p)$, and $i = (\nu_p(\det B) - \nu_p(\det A))/2$. Assume that $A$ dominates $B$ over $\mathbb{Z}_p$.

1. Let $B \sim \frac{1}{2}U_0 \perp \frac{1}{2}U_1$ with $U_0, U_1$ non-degenerate modulo $p$. Then we have

$$G_p(A, B) = \frac{J(i; \frac{1}{2}U_1; p)}{\phi_i(p)}.$$

2. Let $p = 2$ and $B \sim \frac{1}{2}U_0 \perp V \perp U_1$ with $U_0, U_1$ even unimodular and $V$ a diagonal unimodular matrix of degree 2 such that $\det V \equiv 1 \mod 4$. Then we have

$$G_p(A, B) = \frac{J(i; \frac{1}{2}U_1 \perp 1; p)}{\phi_i(p)}.$$

Thus, if we get an explicit form of $G_f^*(C_0)$, we will know a lot of information on $L(f, s)$. In fact, in the case where $f$ is Klingen-Eisenstein series, by [B-R] or [Kil], we know an explicit form of $G_f^*(C_0)$, and therefore give an explicit form of $L(f, s)$ by the above theorem. We also remark that we have given an explicit form of $L(f, s)$ for Siegel-Eisenstein series $f$ by a different method from this note (cf. [I-K1]).

### 4 Koecher-Maaß Dirichlet series for Eisenstein series of Klingen type

Let $f$ be a Siegel cusp form of weight $k$ belonging to $\Gamma_r$, and $[f]_n^r$ the Klingen’s Eisenstein series of degree $n$ attached to $f$. Then $f$ and $[f]_n^r$ have the following Fourier expansions:

$$f(z) = \sum_{C \in \mathfrak{H}_r(\mathbb{Z}) \geq 0} b(C) \exp(2\pi itr(Cz)),$$
For two positive definite half-integral matrices $B$ and $C$ of degree $m$ and $n$, respectively, over $\mathbb{Z}$ put

$$G(B, C)^* = \sum_{B^J(B)} \frac{a(B', C)^*}{a(B', B')}$$

where $a(B', C)^*$ denotes the number of primitive representations of $C$ by $B$. Then rewriting [B-R, Theorem 1] we have

Proposition 4.1. We have

$$G_{n,f}(B)^* = a_{n,k}(B)^* \sum_{c} \frac{G(B, C)^* b(C)^*}{(\det C)^{(r+1)/2} a(C, C) a_{r,k}(c)^*},$$

where $a_{n,k}(B)^*$ and $a_{r,k}(c)^*$ denote the primitive Fourier coefficients of Siegel-Eisenstein series of degree $n$ and $r$, respectively.

Now let $r = 1$. For an element $A$ of $\mathcal{H}_n(\mathbb{Z}_p)$ and a non-zero $p$-adic integer, put

$$H_p(s; A; e) = \frac{p^{((n+1)/2-s)\nu(\det A)} \sigma_p(A) B_p(p^{-(2s-k+1)} A)}{\alpha_p(A, A)} \sum_{C_0} p^{(2k-1-n)\nu(\det C_0)/2} G_p(C_0, A)$$

$$\times T_p(p^{-(2s-k+1)}; C_0, A) \alpha_p(H_k, C_0)^* p^{-\nu(\det C_0)/2} \alpha_p(C_0, e)^*,$$

and for a non-zero $p$-adic number $d_0$, and a function $\omega$ on $\mathcal{H}_n(\mathbb{Z}_p)$ put

$$H_p(s; d_0; \omega, e) = \sum_{r=0}^{\nu(\det A)^*} \sum_{d_0 f^2 \equiv 2^{n/2} \text{ mod } 4} \omega(A) H_p(s; A; e),$$

where for a half-integral matrix $U$ and $V, \alpha_p(U, V)^*$ denotes the primitive local density representing $V$ by $U$, and $G_p(C_0, A)$ is the one defined in Section 1. Let $\iota_p$ be a constant function on $\mathcal{H}_n(\mathbb{Z}_p)$ taking the value 1, and $h_p$ the Hasse invariant on $\mathcal{H}_n(\mathbb{Z}_p)$. We note that $h_p(C_0)$ for $C_0 \in \mathcal{H}_n(\mathbb{Z}_p)$ is the same as that of $A$ if $C_0$ dominates $A$ over $\mathbb{Z}_p$. Let $A$ be a positive definite half-integral matrix of degree $n$ over $\mathbb{Z}$. If $n$ is even, then $\det A$ can be expressed as $d_0 f^2$ with positive integers $d_0$ and $f$ such that $\nu_p(d_0) \leq 1$ for $p \neq 2$, and $(-1)^{n/2} d_0 \equiv 1$ or $\equiv 0$ mod 4. If $n$ is odd, $\det A$ can be expressed as $d_0 f^2$ with a positive integer $f$ and a square free positive integer $d_0$. Thus by Theorem 3.1 and Proposition 4.1, we have
Theorem 4.2. (1) Let \( n \) be even. Then we have

\[
K([f]^n, s) = \alpha_{nk} \sum_{e=1}^{\infty} b(e) B(k-1, e) e^{n/2-k} \sum_{d_0 \in \mathcal{F}_n} \left( \prod_p H_p(s; d_0; \iota_p; e) + \prod_p H_p(s; d_0; h_p; e) \right),
\]

where \( \mathcal{F}_n \) is the set defined in Section 1, and \( \alpha_{nk} \) is a constant depending only on \( n \) and \( k \).

(2) Let \( n \) be odd. Then we have

\[
K([f]^n, s) = \beta_{nk} \sum_{e=1}^{\infty} b(e) B(k-1, e) e^{n/2-k} \sum_{d_0 \in \mathcal{F}_n} \left( \prod_p H_p(s; d_0; \iota_p; e) + \prod_p H_p(s; d_0; h_p; e) \right),
\]

where \( d_0 \) runs over all square free positive integers, and \( \beta_{nk} \) is a constant depending only on \( n \) and \( k \).

5 Proof of Theorem 1

In this section let \( n \) be even. Then by Proposition 3.2, Theorem 4.2, and [Ki2, Theorem 5.6.3] combined with some combinatorial technique, we obtain

**Theorem 5.1** Let \( n \) be even, and \( D_0 \in \mathbb{Z}_p^* \) with \( p \) odd, or \( D_0 \in \mathbb{Z}_2^* \) such that \((-1)^{n/2}D_0 \equiv 1 \) mod 4. Put \( Q(e, D_0) = 1 - p^{-n+2} \) or 1 according as \( e \equiv 0 \) mod \( p^2 \) or not, and \( R(e, D_0) = 1 + \delta p^{-n/2+1} \) or 1 according as \( e \equiv 0 \) mod \( p \) or not, where \( \delta = \chi_p((-1)^{n/2}D_0) \). Further put \( \Phi_{nk} = \frac{(1-p^{-k}) \prod_{i=0}^{n/2-1} (1-p^{-2i+2}) \nu(e)}{\phi n/2(p^{-2})} \). Then we have

(1) \[
H_p(s; D_0; \iota_p, e) = 2^{\delta_2, p^n s} p^{(n-2)\nu(e)/2} \Phi_{nk} \times [Q(e, D_0)p^{-2s+2k-3}(1-p^{-n-2k})(1+p^{-k+2}) \prod_{i=0}^{n/2-2} \left( 1 - p^{2i-n-1+2k-2s}(1-p^{2i+2-2s}) \right) + R(e, D_0)(1 + \delta p^{n/2-k}\delta) \prod_{i=0}^{n/2-1} (1-p^{2i-n-1+2k-2s})(1-p^{2i+2-2s})].
\]

(2) \[
H_p(s; D_0; h_p, e) = (-1, -1)^p (n+2)^{\delta_2, p^n s} p^{(n-2)\nu(e)/2} (1 + \delta p^{n/2-k}) \Phi_{nk} \times [Q(e, D_0)\delta p^{-2s+2k-n/2-2}(1-p^{n/2-k}\delta)(1+p^{-k}) \prod_{i=0}^{n/2-2} \left( 1 - p^{2i-n+2k-2s}(1-p^{2i+1-2s}) \right) + R(e, D_0)\prod_{i=0}^{n/2-1} (1-p^{2i-n+2k-2s})(1-p^{2i+1-2s})].
\]
Theorem 5.2 Let n be even, and \( D_0 \in p\mathbb{Z}_p^* \) with \( p \) odd, or \( D_0 \in 4\mathbb{Z}_2^* \) such that \((-1)^{n/2+1}D_0 \equiv 3 \mod 4 \) or \( D_0 \in 8\mathbb{Z}_2^* \). Put \( \nu = \nu(D_0) \) and \( d_0 = 2^{-\delta_{2,p}n}p^{-\nu}D_0 \). Further put \( \Psi_{nk} = \frac{(1-p^{-k})\prod_{i=1}^{n/2}1-p^{2i+k}}{\phi_{n/-1}(2^{i+k})} \).

(1) Put \( Q(e, D_0) = 1 - p^{-n+2} \) or 1 according as \( e \equiv 0 \mod p^2 \) or not. Then we have

\[
H_p(s; D_0; e) = 2^{\delta_{2,p}n}p^{-s+k-n/2}R(e, D_0)(1 - p^{-n-k})\Psi_{nk}
\]

\[\times Q(e, D_0) \prod_{i=0}^{n/2-1} (1 - p^{2i-n+2k-2s})(1 - p^{2i+1-2s}).\]

(2.1) Let \( p \neq 2 \), and \( R(e, D_0) = \frac{(-1)^{n/2}e}{p} \left( \frac{-p^{-2}eD_0}{p} \right) \) or 0 according as \( e \in \mathbb{Z}_p^* \), \( e \in p\mathbb{Z}_p^* \) or not, where \( \left( \frac{\cdot}{p} \right) \) denotes Legendre symbol. Then we have

\[
H_p(s; D_0; e) = p^{-s+k-n/2}R(e, D_0)(1 - p^{-n-k})(1 + p^{n-k})\Psi_{nk}
\]

\[\times (1 + p^{-2s+k-1}) \prod_{i=0}^{n/2-1} (1 - p^{2i-n+2k-2s})(1 - p^{2i+1-2s}).\]

(2.2) Let \( p = 2 \) and \( e = 2^r e_0 \) with \( (2, e_0) = 1 \). Put

\[ R(e, D_0) = (-1)^{n(n-2)/8} \left( \frac{2^r(-1)^{n/2}}{e_0} \right) \left( \frac{2^{m_0}(-1)^{n/2}(-1)^{e-1/2}}{d_0} \right) \] or 0 according as \( m_0 \leq 1 \) or not, where \( \left( \frac{\cdot}{2} \right) \) denotes the Jacobi symbol. Then we have

\[
H_p(s; D_0; e) = 2^{ns+(-s+k-(n+1)/2)}R(e, D_0)(1 - p^{-n-2k})(1 + p^{n-k})\Psi_{nk}
\]

\[\times (1 + p^{-2s+k-1}) \prod_{i=0}^{n/2-1} (1 - p^{2i-n+2k-2s})(1 - p^{2i+1-2s}).\]

Proof of Theorem 1. By Theorems 5.1 and 5.2 combined with Theorem 4.2, we have

\[ K([f]_n^+, s) = 2^{ns} \gamma_{nk} \frac{(-1)^{(n-1)/8} \zeta(f; k - n/2)}{\zeta(2s - 2k + n - 2i + 1)} \]
\begin{align*}
\times \sum_{D_0} D_0^{-s+k-3/2} \zeta(f; \psi_{(-1)^n/2D_0}; k - 1) \\
\times \prod_p \{(1+p^{-2s+k-2} \psi_{(-1)^n/2D_0}(p)^2)(1+p^{-2s+k-1}) - a(p) \psi_{(-1)^n/2D_0}(p)p^{-2s+k-2}(1+p^{2-k}) \}
+ \frac{\zeta(f; k - 1)}{\zeta^+(f; k - 1) \prod_{i=0}^{n/2-2} \zeta(2s - 2k - 2i + n)}
\times \sum_{D_0} D_0^{-s+k-(n+1)/2} \zeta(f; \psi_{(-1)^n/2D_0}; k - n/2) \\
\times \prod_p \{(1+p^{-2s+k-2} \psi_{(-1)^n/2D_0}(p)^2)(1+p^{-2s+k-1}) - a(p) \psi_{(-1)^n/2D_0}(p)p^{-2s+k-n/2-1}(1+p^{n-k}) \}].
\end{align*}

We note that

\[ \zeta^+([f]_n; 2s - k + 1) = \zeta^+(f, 2s - k + 1) \prod_{i=1}^{n-1} \zeta(2s - i) \zeta(2s - 2k + i + 2). \]

Thus we complete the assertion by Theorem 3.1 keeping the remark before Theorem 1 in mind.

References