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A remark on Serre's example of p-adic Eisenstein series

by

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1 Introduction.

In [Se], J. P. Serre developed the theory of p-adic modular forms and applied it to the construction of p-adic zeta function. In this paper, we shall try to generalize a formula for p-adic Eisenstein series which was originally given by Serre. A p-adic modular form is a formal power series

$$f = \sum_{t=0}^{\infty} a(t) q^t \in \mathbb{Q}_p[[q]]$$

which is the limit of a sequence of modular forms \(\{f_m\}\) with rational Fourier coefficients: \(\lim_{m \to \infty} f_m = f\).

If we denote by

$$f_m = \sum_{t=0}^{\infty} a^{(m)}(t) q^t \in \mathbb{Q}[[q]]$$

the Fourier expansion of \(f_m\) (q-expansion), this limit means that

$$v_p(f - f_m) := \inf_t v_p(a(t) - a^{(m)}(t)) \to +\infty \quad (m \to \infty),$$

where \(v_p\) is the valuation of \(\mathbb{Q}_p\) normalized as \(v_p(p) = 1\). If we denote by \(\{k_m\}\) the weight of \(\{f_m\}\), then Serre showed that \(\{k_m\}\) has the limit in the following set:

$$X := \lim_{m \to \infty} X/(p-1)p^{m-1}\mathbb{Z} = \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}.$$

Let \(E_k^{(n)}\) be the Siegel-Eisenstein series of degree \(n\) and weight \(k\) (for precise definition, see §2). Set

$$G_k := \frac{1}{2} \zeta(1-k) E_k^{(1)},$$

where \(\zeta(s)\) is the Riemann zeta function. For \(k \in X\), we take a sequence \(\{k_m\} \subset 2\mathbb{Z}\) such that \(\lim_{m \to \infty} k_m = k\) and \(|k_m| \to +\infty (m \to \infty)\). Serre defined the p-adic Eisenstein series \(G_k^{*}\) of weight \(k \in X\) by

$$G_k^{*} := \lim_{m \to \infty} G_{k_m}.$$

The right-hand side converges and it becomes a p-adic modular form. The following example is due to Serre:
**Example of $G^*_k$.** Let $p > 3$ be a prime number such that $p \equiv 3 \pmod{4}$ and $k = (1, \frac{p+1}{2}) \in X$. Then we have

$$G^*_k = h(-p) + \sum_{t=1}^{\infty} \sum_{0<d|t} \left( \frac{d}{p} \right) q^t,$$

where $h(-p)$ is the class number of the quadratic field $\mathbb{Q}(\sqrt{-p})$.

The main purpose of this paper is to give a generalization of this example. The Siegel modular form $f(Z)$ has a Fourier expansion of the form

$$f(Z) = \sum_T a_f(T) \exp[2\pi \sqrt{-1} \text{tr}(TZ)] = \sum_T a_f(T) q^T,$$

where $T$ runs over the set of half-integral, positive semi-definite symmetric matrices (see §2). For $T = (t_{ij})$ and $Z = (z_{ij})$, we set $q_{ij} := \exp(2\pi \sqrt{-1} z_{ij})$, $q_i = q_{ii}$, and $t_i = t_{ii}$. Then $f$ can be regarded as a power series in $\mathbb{C}[q_{ij}, q_{ij}^{-1}, [q_1, \ldots, q_n]]$. So we can define the $p$-adic Siegel modular form as an element of $\mathbb{Q}[q_{ij}, q_{ij}^{-1}, [q_1, \ldots, q_n]]$. Our result can be stated as follows:

**Theorem** Let $p > 3$ be a prime number such that $p \equiv 3 \pmod{4}$. If we put

$$k_m := 1 + \frac{p - 1}{2} \cdot p^{m-1} \in \mathbb{Z},$$

then the sequence $\{k_m\}$ has the limit $k = (1, \frac{p+1}{2}) \in X$ and

$$E^*_k := \lim_{m \to \infty} \left( \frac{1}{2} \zeta(1 - k_m) E^{(2)}_{k_m} \right)$$

$$= \frac{1}{2} h(-p) + \sum_{T \succeq 0} \text{rank}(T) \sum_{0<d|\varepsilon(T)} \left( \frac{d}{p} \right) q^T,$$

where $D(T)$ is the discriminant of the field $\mathbb{Q}(\sqrt{-\det(2T)})$ and we understand $D(T) = 0$ if $\det(T) = 0$, and $\varepsilon(T) := \text{g.c.d}(t_{11}, 2t_{12}, t_{22})$.

In the final section, we give an additional formula which is concerned with reduction mod $p$ of the Fourier coefficient of the Siegel-Eisenstein series.

2 **Siegel-Eisenstein series.**

Let $\mathbb{H}_n$ be the Siegel upper half space of degree $n$:

$$\mathbb{H}_n := \{ Z = X + \sqrt{-1} Y \in \text{Sym}_n(\mathbb{C}) \mid Y > 0 \}.$$

The real symplectic group $\text{Sp}_n(\mathbb{R})$ acts on $\mathbb{H}_n$ by

$$Z \mapsto M(Z) := (AZ + B)(CZ + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_n(\mathbb{R}).$$
The group $\Gamma_n := \text{Sp}_n(\mathbb{R}) \cap M_{2n}(\mathbb{Z})$ is called the Siegel modular group. Let $[\Gamma_n, k]$ denote the $\mathbb{C}$-vector space of Siegel modular forms of weight $k$ for $\Gamma_n$. Any element $f$ in $[\Gamma_n, k]$ admits a Fourier expansion of the form

\begin{equation}
(2.1) \quad f(Z) = \sum_{0 \leq T \in \Lambda_n} a_f(T) \exp[2\pi \mathrm{i} \text{tr}(TZ)],
\end{equation}

where the index set $\Lambda_n$ is defined by

\begin{equation}
(2.2) \quad \Lambda_n := \{(T = (t_{ij}) \in \text{Sym}_n(\mathbb{Q}) \mid t_{ii} \in \mathbb{Z}, 2t_{ij} \in \mathbb{Z}\}.
\end{equation}

Let $\Gamma_{n,0}$ be the subgroup of $\Gamma_n$ defined by

$$
\Gamma_{n,0} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C = O_n \right\}.
$$

For an even integer $k$, we define a series

\begin{equation}
(2.3) \quad E^{(n)}_k(Z) := \sum_{0 \leq T \in \Lambda_n} \det(CZ + D)^{-k}, \quad Z \in \mathbb{H}_n.
\end{equation}

This series is absolutely convergent if $k > n + 1$ and it becomes a Siegel modular form of weight $k$ for $\Gamma_n : E^{(n)}_k \in [\Gamma_n, k]$. Here we call this the Siegel-Eisenstein series of degree $n$ and weight $k$. We write the Fourier expansion of $E^{(n)}_k$ by

\begin{equation}
(2.4) \quad E^{(n)}_k(Z) = \sum_{0 \leq T \in \Lambda_n} a^{(n)}_k(T) \exp[2\pi \mathrm{i} \text{tr}(TZ)].
\end{equation}

It is known that any Fourier coefficient $a^{(n)}_k(T)$ is rational ([Si]). The explicit formula of $a^{(n)}_k(T)$ was studied by several authors ([Kau], [M], [Kat]). For later purpose, we shall introduce an abbreviation. For $T = (t_{ij}) \in \Lambda_n$ and $Z = (z_{ij}) \in \mathbb{H}_n$, we write

\begin{equation}
(2.5) \quad q^T := \exp[2\pi \mathrm{i} \text{tr}(TZ)] = \prod_{i<j} q^{2t_{ij}} \prod_{i=1}^n q^{t_{ii}},
\end{equation}

where $q_{ij} := \exp(2\pi \mathrm{i} z_{ij})$, and $q_i = q_{ii}$, $t_i = t_{ii}$. So the Fourier expansion (2.1) can be rewritten as

$$
f = \sum_{0 \leq T \in \Lambda_n} a_f(T) q^T \in \mathbb{C}[q_{ij}, q_{ij}^{-1}][[q_1, \ldots, q_n]],
$$

namely, $f$ is regarded as an element of the formal power series ring $\mathbb{C}[q_{ij}, q_{ij}^{-1}][[q_1, \ldots, q_n]]$. 

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3 Bernoulli numbers and generalized Bernoulli numbers.

In this section we review some of the basic facts about Bernoulli numbers and generalized Bernoulli numbers. The ordinary Bernoulli numbers $B_m$ are defined by

\[ \frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}. \]

As is well known, certain special values of the Riemann zeta function can be represented by the Bernoulli numbers: for any even positive integer $m$, we have

\[ \zeta(1-m) = -\frac{B_m}{m}. \]

**Theorem 3.1** (1) (Kummer) If $m$ and $n$ are positive even integers with $m \equiv n \pmod{p^{e-1}(p-1)}$ and $n \not\equiv 0 \pmod{p-1}$, then

\[ (1-p^{m-1}) \frac{B_m}{m} \equiv (1-p^{n-1}) \frac{B_n}{n} \pmod{p^e}. \]

(cf. [W], §5.3, Corollary 5.14).

(2) (von Staudt-Clausen) Let $m$ be even and positive. Then

\[ B_m + \sum_{\chi \neq \chi^0} \frac{1}{p} \in \mathbb{Z}. \]

Consequently, $pB_m$ is $p$-integral for all $m$ and all $p$. (cf. [W], Theorem 5.10).

(3) (Carlitz) If $p^{e-1}(p-1) | m$, then we have

\[ pB_m \equiv p-1 \pmod{p^e}. \]

(cf. [W], p. 86, 5.11 (b)).

(4) Let $p > 3$ be a prime number such that $p \equiv 3 \pmod{4}$. Then we have

\[ B_{\frac{p+1}{2}} \equiv -\frac{h(-p)}{2} \not\equiv 0 \pmod{p}. \]

(cf. [BS], Chap. 5, §8, Problem 4 and [W], p. 86, Exercise 5.9).

Let $\chi$ be a Dirichlet character of conductor $f = f_\chi$. The generalized Bernoulli numbers $B_{m,\chi}$ are defined by

\[ \sum_{a=1}^{f} \chi(a) \frac{te^{at}}{e^{ft} - 1} = \sum_{m=0}^{\infty} B_{m,\chi} \frac{t^m}{m!}. \]

Let $L(s; \chi)$ be the Dirichlet $L$-function belonging to a Dirichlet character $\chi$:

\[ L(s; \chi) := \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}. \]
Then, for any integer $m \geq 1$, we have

\[ L(1-m; \chi) = -\frac{B_{m, \chi}}{m} \tag{3.9} \]

(e.g. cf. [I], §2, Theorem 1). In the following, we shall state Carlitz’s result about generalized Bernoulli numbers in the case that $\chi$ is quadratic.

**Theorem 3.2 (Carlitz [Ca])** Suppose that $\chi$ is a quadratic Dirichlet character of conductor $f_{\chi}$.

1. If $\chi \neq \chi^{0}$, then $f_{\chi}B_{m, \chi}$ is a rational integer for every $m \geq 0$ and if $f_{\chi}$ is not a power of a prime, then even $\frac{1}{m}B_{m, \chi}$ is a rational integer.

2. If $\chi$ is a rational prime such that $p^{e} \mid m$ but $p \nmid f_{\chi}$, then $p^{e}$ divides the numerator of $B_{m, \chi}$. If $f_{\chi}$ is divisible by at least two primes and $p$ is arbitrary prime, then again $p^{e}$ divides the numerator of $B_{m, \chi}$.

3. Suppose that $f_{\chi} = p$ is an odd prime, and $p^{e-1} \mid m$. Then

\[ pB_{m, \chi} \equiv p - 1 \pmod{p^{e}} \tag{3.10} \]

if $j(p - 1) = 2m$ for some odd $j$.

**Remark.** The original form of above statement (3) is as follows ([Ca], Theorem 3). Assume that $f_{\chi} = p$ is an odd prime and $p^{e-1} \mid m$. Let $\wp$ be a prime ideal in $\mathbb{Q}(\chi)$ defined by

\[ \wp = (p, 1 - \chi(g)g^{m}), \]

where $g$ is a primitive root mod $p$. If $\wp \neq (1)$, then

\[ pB_{m, \chi} \equiv p - 1 \pmod{\wp^{e}}. \]

In our case, $\chi$ is quadratic, namely, $\mathbb{Q}(\chi) = \mathbb{Q}$. Obviously, if $j(p - 1) = 2m$ for some odd $j$, then

\[ \chi(g)g^{m} \equiv 1 \pmod{p}. \]

Therefore, Theorem 3.2, (3) is a special case of Carlitz’s result.

4 Fourier coefficients of Siegel-Eisenstein series.

In this section, we shall introduce some explicit formulas of Fourier coefficient $a_{k}^{(n)}(T)$ of Siegel-Eisenstein series in the case $n \leq 2$.

It is well known that $a_{k}^{(1)}(t)$ ($4 \leq k \in 2\mathbb{Z}$) is given as follows:

\[ a_{k}^{(1)}(t) = \begin{cases} \frac{-2k}{B_k} \sigma_{k-1}(t) & \text{if } t > 0, \\ 1 & \text{if } t = 0, \end{cases} \tag{4.1} \]

where $\sigma_{m}(t) := \sum_{0 < d \mid t} d^{m}$.

In the case $n = 2$, G. Kauflold [Kau] and H. Maass [M] gave explicit formulas. Here we introduce a description of $a_{k}^{(2)}$ by M. Eichler and D. Zagier [EZ] in
which they used Cohen's function $H(r, N)$.

Let $r$ and $N$ be non-negative integers with $r \geq 1$. For $N \geq 1$, we define

$$h(r, N) := \begin{cases} (-1)^{\frac{r}{2}} (r - 1)! N^{r-\frac{1}{2}} 2^{1-r} \pi^{-r} L(r; \chi_{(-1)^rN}) & \text{if } (-1)^r N \equiv 0 \text{ or } 1 \pmod{4}, \\ 0 & \text{if } (-1)^r N \equiv 2 \text{ or } 3 \pmod{4}, \end{cases}$$

where $L(s; \chi)$ is the Dirichlet $L$-function and we write $\chi_D$ for the character $\chi_D(d) = (\frac{D}{d})$. Moreover, for $N \in \mathbb{R}$, we define

$$H(r, N) := \begin{cases} \sum_{d|N} h\left(\frac{r}{d}, \frac{N}{d}\right) & \text{if } (-1)^r N \equiv 0 \text{ or } 1 \pmod{4}, \quad N > 0, \\ \zeta(1 - 2r) & \text{if } N = 0, \\ 0 & \text{otherwise}. \end{cases}$$

The above defined function $H(r, N)$ is called Cohen's function. It is known that $H(r, N)$ has the following description.

**Lemma 4.1** ([Co], p.273, c)) If we set $(-1)^r N = D f^2$ with $D$ discriminant of a quadratic field, then we have

$$(4.2) \quad H(r, N) = L(1 - r; \chi_D) \sum_{0<d|f} \mu(d) \chi_D(d) d^{r-1} \sigma_{2r-1} \left(\frac{f}{d}\right),$$

where $\mu(d)$ is the Möbius function.

Returning to the formula $a_k^{(2)}(T)$, for $O_2 \neq T \in \Lambda_2$ (cf. (2.2)), we define

$$(4.3) \quad \varepsilon(T) := \max\{l \in \mathbb{N} | l^{-1}T \in \Lambda_2\}.$$

**Theorem 4.2** ([EZ], p.80, Corollary 2) If $0 \leq T \in \Lambda_2 \setminus \{O_2\}$, then

$$(4.4) \quad a_k^{(2)}(T) = \frac{4k(k - 1)}{B_k \cdot B_{2k-2}} \sum_{0<d|\varepsilon(T)} d^{k-1} H \left(k - 1, \frac{\det(2T)}{d^2}\right).$$

Especially, if rank $T = 1$, then

$$(4.5) \quad a_k^{(2)}(T) = \frac{-2k}{B_k} \sum_{0<d|\varepsilon(T)} d^{k-1} = \frac{-2k}{B_k} \sigma_{k-1}(\varepsilon(T)).$$

**Remark.** It should be noted that the factor $4k(k - 1)/B_k \cdot B_{2k-2}$ in (4.4) is missing in the original formula of Eichler and Zagier.

By using (4.2), we can rewrite the formula (4.4). For $0 < T \in \Lambda_2$, we write

$$(4.6) \quad -\det(2T) = D(T) \cdot f(T)^2,$$

where $D(T)$ is the discriminant of the imaginary quadratic field $\mathbb{Q} \left(\sqrt{-\det(2T)}\right)$ and $f(T) \in \mathbb{N}$. It is quite obvious that the number $f(T)$ is divisible by $\varepsilon(T) : \varepsilon(T) \mid f(T)$. 
COROLLARY 4.3 (Explicit formula of $a_{k}^{(2)}(T)$) For $0 < T \in \Lambda_2$, we have

$$a_{k}^{(2)}(T) = -\frac{4k \cdot B_{k-1} \chi_D(T)}{B_k \cdot B_{2k-2}} F_k(T),$$

(4.7)

$$F_k(T) = \sum_{0 < d | \epsilon(T)} d^{k-1} \sum_{0 < d | \epsilon(T)} \mu(f) \chi_D(T)(f) f^{k-2} \sigma_{2k-3} \left( \frac{f(T)}{fd} \right).$$

5 $p$-adic Eisenstein series.

As we mentioned in Introduction, J. P. Serre developed the theory of $p$-adic modular form and applied it to the construction of $p$-adic zeta function. The $p$-adic Eisenstein series is a typical example of $p$-adic modular form. In this section, we shall briefly review Serre’s theory.

In the following, for simplicity, we assume that $p$ is an odd prime. Put

$$X_{m} := \mathbb{Z}/p^{m-1}(p - 1)\mathbb{Z} = \mathbb{Z}/p^{m-1}\mathbb{Z} \times \mathbb{Z}/(p - 1)\mathbb{Z}, \quad m \geq 1.$$ Then $\{X_{m}\}$ forms a projective system. Let $X$ be the limit of this system:

(5.1) $$X := \varliminf_{m \to \infty} X_{m} = \mathbb{Z}_p \times \mathbb{Z}/(p - 1)\mathbb{Z},$$

where $\mathbb{Z}_p$ is the ring of $p$-adic integers.

The $p$-adic modular form

(5.2) $$f = \sum_{t=0}^{\infty} a(t) q^t \in \mathbb{Q}_p[[q]]$$

is defined as the limit of a sequence of modular forms $\{f_{m}\}$ with rational Fourier coefficients. The limit means the following. Let $v_{p}$ be the valuation on $\mathbb{Q}_p$ (the field of $p$-adic numbers) normalized as $v_{p}(p) = 1$. We denote by

$$f_{m} = \sum_{t=0}^{\infty} a^{(m)}(t) q^t \in \mathbb{Q}[[q]]$$

the Fourier expansion of $f_{m}$. The convergence $\lim_{m \to \infty} f_{m} = f$ means that

$$v_{p}(f - f_{m}) = \inf_{t} v_{p}(a(t) - a^{(m)}(t)) \to +\infty \quad (m \to \infty).$$

We denote by $\{k_{m}\} \subset 2\mathbb{Z}$ the weight of $\{f_{m}\}$. Serre [Se] showed that $\{k_{m}\}$ has the limit $k$ in $X$. This element $k \in X$ is called the weight of $p$-adic modular form $f$. The $p$-adic Eisenstein series (in the sense of Serre) is defined as follows. Put

$$G_k := \frac{1}{2} \zeta(1 - k) E^{(1)}_k = -\frac{B_k}{2k} E^{(1)}_k,$$
where $E_{k}^{(1)}$ is the Siegel-Eisenstein series of degree 1 and weight $k$ ($4 \leq k \in 2\mathbb{Z}$).

By (4.1), $G_{k}$ has a Fourier expansion of the form

$$G_{k} = -\frac{B_{k}}{2k} + \sum_{t=1}^{\infty} \sigma_{k-1}(t) q^{t} \in \mathbb{Q}[[q]].$$

Assume that $k \in X$. For an integer $t \geq 1$, we can define a $p$-adic integer $\sigma_{k-1}^{*}(t)$ by

$$\sigma_{k-1}^{*}(t) := \sum_{0<d|t} d^{k-1}.$$

If $k \in X$ is even, then we can choose a sequence of integers $\{k_{m}\}$ ($4 \leq k_{m} \in 2\mathbb{Z}$) such that $k_{m} \rightarrow k \in X$ and $|k_{m}| \rightarrow +\infty$ where $| \cdot |$ is the ordinary absolute value. For this $\{k_{m}\}$, we have

$$\lim_{m \rightarrow \infty} \sigma_{k_{m}-1}(t) = \sigma_{k-1}^{*}(t)$$

in $\mathbb{Z}_{p}$. The $p$-adic Eisenstein series (of degree 1) and weight $k \in X - \{0\}$ is defined by

$$G_{k}^{*} = \lim_{m \rightarrow \infty} G_{k_{m}}.$$

Namely,

$$G_{k}^{*} = \frac{1}{2} \zeta^{*}(1-k) + \sum_{t=1}^{\infty} \sigma_{k-1}^{*}(t) q^{t} \in \mathbb{Q}_{p}[[q]],$$

where the convergence of the constant term is guaranteed in [Se], 1.5, Cor. 2, and $\zeta^{*}$ is essentially the $p$-adic zeta function of Kubota and Leopoldt. Strictly speaking, if $(s, u) \in X = \mathbb{Z}_{p} \times \mathbb{Z}/(p-1)\mathbb{Z}$ ($(s, u) \neq 1$), then

$$\zeta^{*}(s, u) = L_{p}(s; \omega^{1-u}),$$

where $L_{p}(s; \chi)$ is the $p$-adic $L$-function with character $\chi$ and $\omega$ is the Teichmüller character (e.g. cf. [I], p.18).

**Example** (Serre). Let $p > 3$ be a prime number such that $p \equiv 3 \pmod{4}$. If $k = (1, \frac{p+1}{2}) \in X$, then

$$G_{k}^{*} = \frac{1}{2} h(-p) + \sum_{t=1}^{\infty} \sum_{0<d|t} \left(\frac{d}{p}\right) q^{t}.$$

As mentioned before, $h(-p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$. 


6 Main result.

One of the main purpose of this note is to give a generalization of the above-mentioned formula (5.6). It is interesting to us that the resulting formula has a simple form unexpectedly.

As was mentioned earlier, the Fourier expansion of Siegel modular form \( f \) can be written as

\[
f = \sum_{0 \leq T \in \Lambda_n} \alpha_f(T) q^T \in \mathbb{C}[q_{ij}, q_{ij}^{-1}][[q_1, \ldots, q_n]].
\]

As an analogy of the degree one case, one can define the notion of \( p \)-adic Siegel modular form \( f \) as the limit of a sequence of ordinary Siegel modular forms \( \{f_m\} \) with rational Fourier coefficients:

\[
f = \sum_{0 \leq T \in \Lambda_n} \alpha(T) q^T \in \mathbb{Q}[q_{ij}, q_{ij}^{-1}][[q_1, \ldots, q_n]],
\]

\[
f_m = \sum_{0 \leq T \in \Lambda_n} \alpha^{(m)}(T) q^T \in \mathbb{Q}[q_{ij}, q_{ij}^{-1}][[q_1, \ldots, q_n]],
\]

\[
v_p(f - f_m) := \inf_{0 \leq T \in \Lambda_n} v_p \left( \alpha(T) - \alpha^{(m)}(T) \right) \to +\infty \quad (m \to \infty).
\]

Our result is as follows:

**Theorem 6.1** Let \( p > 3 \) be a prime number such that \( p \equiv 3 \) (mod 4). If we put

\[
k_m := 1 + \frac{p-1}{2} \cdot p^{m-1} \in \mathbb{N},
\]

then the sequence \( \{k_m\}_{m=1}^{\infty} \) has the limit \( k = (1, \frac{p+1}{2}) \in X \) and

\[
E_k^* := \lim_{m \to \infty} \left( \frac{1}{2} \zeta(1-k_m) E_{k_m}^{(2)} \right)
\]

\[
= \frac{1}{2} h(-p) + \sum_{0 \leq T \in \Lambda_n} \text{rank}(T) \sum_{0 < d \in \epsilon(T) \cap \mathbb{Q}} \left( \frac{d}{p} \right) q^T,
\]

where we understand \( D(T) = 0 \) if \( \det(T) = 0 \).

To prove this theorem, we prepare some lemma.

**Lemma 6.2** For non-negative integers \( k, N \), we define \( S_k(N) := \sum_{a=1}^{N} a^k \).

Then, for any prime \( p > 3 \) and integer \( h \geq 1 \), the following congruence relation holds:

\[
\frac{S_{k_m}(p^h)}{p^h} \equiv B_{k_m} \pmod{p^h},
\]

where \( B_{k_m} \) is the \( k_m \)-th Bernoulli number and \( k_m \) is the integer defined in Theorem 6.1.
PROOF. Let $B_n(x)$ be the n-th Bernoulli polynomial. The following identity is well known:

$$S_k(N) = \frac{1}{k+1} (B_{k+1}(N) - B_{k+1}(0))$$

(e.g. cf. [I], p.15). Since

$$B_{k+1}(x) - B_{k+1}(0) = (k + 1) \cdot B_k \cdot x + \binom{k+1}{2} \cdot B_{k-1} \cdot x^2 + \cdots,$$

we have

$$\frac{S_{k_m}(p^h)}{p^h} = B_{k_m} + \frac{k_m}{2} \cdot B_{k_m-1} \cdot p^h + \frac{k_m(k_m-1)}{2 \cdot 3} \cdot B_{k_m-2} \cdot p^{2h} + \cdots.$$

The prime $p$ does not appear in the denominator of $B_{k_m-1}$ and appears at most once those of $B_{k_m-j}$ ($j \geq 2$). This shows (6.2). \qed

PROOF of Theorem 6.1. Put

(6.3) $$E_{k_m} := \frac{1}{2} \zeta(1-k_m) E_{k_m}^{(2)}.$$ We write the Fourier expansion of $E_{k_m}$ by

(6.4) $$E_{k_m} = \sum_{0 \leq \tau \in \Lambda_2} a((m)\tau) q \in \mathbb{Q}[q_{12}, q_{12}^{-1}].$$

Moreover, put

(6.5) $$a(T) := \begin{cases} \frac{1}{2} h(-p) & \text{if } T = O_2, \\ \sum_{0 < d|\varepsilon(T)} \left( \frac{d}{p} \right) & \text{if } \text{rank}(T) = 1, \\ 2 \sum_{0 < d|\varepsilon(T)} \left( \frac{d}{p} \right) & \text{if } \text{rank}(T) = 2 \text{ and } D(T) = -p, \\ 0 & \text{otherwise}. \end{cases}$$

Our aim is to show the following:

(6.6) $$\inf_{0 \leq T \in \Lambda_2} v_p \left( a^{(m)}(T) - a(T) \right) \to +\infty \quad (m \to \infty).$$

As a first step, we shall show that

(6.7) $$\lim_{m \to \infty} a^{(m)}(O_2) = \lim_{m \to \infty} \left( \frac{B_{km}}{2km} \right) = \frac{1}{2} h(-p).$$

Although this is a part of the result (5.6), we shall give a direct proof. By Kummer's congruence (3.3),

$$(1 - p^{-k_m-1}) \frac{B_{km}}{k_m} \equiv (1 - p^{-k_l-1}) \frac{B_{kl}}{k_l} \pmod{p^l}.$$
for $m > l$ (note that $p > 3$). This means that the sequence $\{(1-p^{k-1})B_{k_m}/k_m\}$, hence $\{B_{k_m}/k_m\}$ converges in $\mathbb{Q}_p$. By Euler's criterion,

$$a^{k_m} = \left(a^{\frac{p-1}{2}}\right)^{p^{m-1}} \cdot a \equiv \left(\frac{a}{p}\right) a \quad (\text{mod } p^m).$$

Hence we have

$$S_{k_m}(p^h) = \sum_{a=1}^{p^h} a^{k_m} \equiv \sum_{a=1}^{p^h} \left(\frac{a}{p}\right) a = \left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right)p^{h-1} \quad (\text{mod } p^m)$$

for any positive integers $m, h$ with $m > h$, equivalently,

$$S_{k_m}(p^h) = \frac{1}{p^h} \left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right) \quad (\text{mod } p^{m-h}).$$

From this, we have

$$\lim_{m \to \infty} \frac{S_{k_m}(p^h)}{p^h} = \frac{1}{p} \left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right)$$

for any fixed integer $h$. Using (6.2), we obtain

$$\lim_{m \to \infty} \frac{B_{k_m}}{k_m} = \lim_{m \to \infty} B_{k_m} \equiv \lim_{m \to \infty} \frac{S_{k_m}(p^h)}{p^h} = \frac{1}{p} \left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right) \quad (\text{mod } p^h).$$

This shows

$$\lim_{m \to \infty} \frac{B_{k_m}}{k_m} = \frac{1}{p} \left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right).$$

From the general formula for $h(D)$ ($D$: fundamental discriminant), we get the following identity:

$$h(-p) = -\frac{1}{p} \left(\sum_{a=1}^{p-1} \chi_{p}(a) a\right) = -\frac{1}{p} \left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right)$$

(e.g. cf. [Z], §9, Satz 3). Combining (6.11) and (6.12), we get (6.7). The second step is to prove the following: for $T \neq O_2$,

$$a^{(m)}(T) \equiv a(T) \quad (\text{mod } p^m).$$

or equivalently,

$$\inf_{O_2 \neq T \in \Lambda_2} v_p \left(a^{(m)}(T) - a(T)\right) \geq m.$$
First assume that $T$ is rank 1. In this case, by (4.5), we have
\[ a^{(m)}(T) = -\frac{B_{k_m}}{2k_m} \cdot a^{(2)}_{k_m}(T) = \sigma_{km^{-1}}(\varepsilon(T)). \]

Again by Euler's criterion, we obtain
\[(6.15) \quad a^{(m)}(T) = \sum_{0<d|\varepsilon(T)} d^{k_m-1} = \sum_{0<d|\varepsilon(T)} d^{\frac{p-1}{2}} \cdot \frac{1}{p} \equiv \sum_{0<d|\varepsilon(T)} \left( \frac{d}{p} \right) \pmod{p^m}. \]

Finally we assume that $T \in \Lambda_2$ is rank 2. By Corollary 4.3, $a^{(m)}(T)$ can be written as
\[(6.16) \quad a^{(m)}(T) = -\frac{B_{k_m}}{2k_m} \cdot a^{(2)}_{k_m}(T) = \frac{2B_{k_m-1,\chi D(T)}}{B_{2k_m-2}} \cdot F_{k_m}(T), \]
\[ F_{k_m}(T) = \sum_{0<d|\varepsilon(T)} d^{k_m-1} \sum_{0<f|\varepsilon(T)} \mu(f) \chi_D(T)(f) f^{k_m-2} \sigma_{2k_m-3}(\frac{f(T)}{fd}). \]

We shall prove the following:
\[(6.17) \quad \frac{B_{k_m-1,\chi D(T)}}{B_{2k_m-2}} \equiv \begin{cases} 1 & \text{if } D(T) = -p \\ 0 & \text{otherwise} \end{cases} \pmod{p^m}. \]

By definition, the factor of Bernoulli numbers becomes
\[ \frac{B_{k_m-1,\chi D(T)}}{B_{2k_m-2}} = \frac{B_{k_m-1,\chi D(T)}}{B_{p-1}p^m-1}. \]

Suppose that $D(T) \neq -p$. By Theorem 3.2, (1), (2) and (3.5), we have
\[ B_{\frac{p-1}{2} \cdot p^m-1,\chi D(T)} \equiv 0 \pmod{p^m}, \quad pB_{p-1}p^m-1 \equiv p-1 \pmod{p^m}. \]

From these formulas, we get
\[ \frac{B_{\frac{p-1}{2} \cdot p^m-1,\chi D(T)}}{B_{p-1}p^m-1} \equiv 0 \pmod{p^m}. \]

Suppose that $D(T) = -p$. By (3.5) and Theorem 3.2, (3), we have
\[ pB_{\frac{p-1}{2} \cdot p^m-1,\chi -p} \equiv p-1 \pmod{p^m}, \quad pB_{p-1}p^m-1 \equiv p-1 \pmod{p^m}. \]

From these formulas, we obtain
\[ \frac{B_{\frac{p-1}{2} \cdot p^m-1,\chi -p}}{B_{p-1}p^m-1} \equiv 1 \pmod{p^m}, \]
and this completes the proof of (6.17). Next we shall show that, if \( D(T) = -p \), then

\[
F_{k_m}(T) \equiv \sum_{0<d|\varepsilon(T)} \left( \frac{d}{p} \right) \pmod{p^m}. 
\]

In our case, we have \( \chi_{D(T)}(a) = \chi_{-p}(a) = \left( \frac{a}{p} \right) \). Therefore

\[
F_{k_m}(T) \equiv \sum_{0<d|\varepsilon(T)} \left( \frac{d}{p} \right) \sum_{0<f|\varepsilon(T)} \mu(f) f^{-1} \sigma_{-1}^{-1} \left( \frac{f(T)}{fd} \right) \pmod{p^m},
\]

where \( \sigma_{-1}^{-1}(l) = \sum_{0<d|l, (d, p) = 1} d^{-1} \) (cf. §5). To prove (6.18), it suffices to show that

\[
\sum_{0<d|\varepsilon(T)} \mu(f) f^{-1} \sigma_{-1}^{-1} \left( \frac{f(T)}{fd} \right) = 1
\]

for any \( d \) with \( d | \varepsilon(T) \). In general, we can prove

\[
\sum_{0<l|m} \mu(l) l^{-1} \sigma_{-1}^{-1} \left( \frac{m}{l} \right) = 1
\]

for any \( m \in \mathbb{N} \). For any \( m \in \mathbb{N} \) with \( p^e \mid m \), we put \( m_0 := m/p^e = p_1^{e_1} \cdots p_r^{e_r} \) (\( p_i \) prime \( \neq p \)). Then

\[
\sum_{0<l|m \atop (l, p) = 1} \mu(l) l^{-1} \sigma_{-1}^{-1} \left( \frac{m_0}{l} \right) = \prod_{i=1}^{r} \left( \sum_{0<l|p_i} \mu(l) l^{-1} \sigma_{-1}^{-1} \left( \frac{p_i}{l} \right) \right).
\]

The inner sum of the last formula is trivially equal to 1. This shows (6.20). Combining (6.17) and (6.18), we obtain

\[
a^{(m)}(T) \equiv \begin{cases} 
2 \sum_{0<d|\varepsilon(T)} \left( \frac{d}{p} \right) & \text{if } D(T) = -p \\
0 & \text{otherwise}
\end{cases} \pmod{p^m}.
\]

This proves (6.13). If we put \( b_m := v_p(a^{(m)}(O_2) - a(O_2)) \), then, by (6.5) and (6.7), we have \( b_m \to +\infty \) (\( m \to \infty \)). Therefore we obtain

\[
\inf_{0 \leq T \in \Lambda_2} v_p \left( a^{(m)}(T) - a(T) \right) \geq \min(m, b_m) \to +\infty \quad (m \to \infty).
\]

This shows (6.6) and completes the proof of Theorem 6.1. \( \square \)
7 Reduction mod $p$ of Fourier coefficient of Siegel-Eisenstein series.

By similar argument used in §6, we can present an additional formula for the Fourier coefficient of Siegel-Eisenstein series of degree 2.

The following result is due to Yamaguchi.

**Theorem 7.1 (Yamaguchi [Y])** Let $p > 3$ be a prime number such that $p \equiv 3 \pmod{4}$. For any $0 < T \in \Lambda_2$ with $f(T) = 1$, we have

$$(7.1) \quad a_{\frac{R\pm 1}{2},2}^{(2)}(T) \equiv -\frac{4pB_{\frac{p-1}{2},x_{D(T)}}}{h(-p)} \pmod{p}$$

(for the definition of $f(T)$, see (4.6)).

**Remark.** The right-hand side does not necessarily vanish because there is a possibility that prime $p$ appears in the denominator of $B_{\frac{p-1}{2},x_{D(T)}}$.

We can genralize the above result.

**Theorem 7.2** Let $p > 3$ be a prime number such that $p \equiv 3 \pmod{4}$. For any $0 < T \in \Lambda_2$, we have

$$(7.2) \quad a_{\frac{R\pm 1}{2},2}^{(2)}(T) \equiv \frac{4\alpha_T}{h(-p)} \sum_{0<d|\epsilon(T)} \left(\frac{d}{p}\right) \pmod{p},$$

where

$$\alpha_T := \begin{cases} 1 & \text{if } D(T) = -p, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** By Corollary 4.3, we can write as

$$a_{\frac{R\pm 1}{2},2}^{(2)}(T) = -\frac{2(p+1)B_{\frac{p-1}{2},x_{D(T)}}}{B_{\frac{p-1}{2}} \cdot B_{p-1}} \cdot F_{\frac{R\pm 1}{2}}(T).$$

Recall

$$B_{\frac{p+1}{2}} \equiv -\frac{h(-p)}{2} \not\equiv 0 \pmod{p}, \quad \text{(Theorem 3.1, (4)).}$$

This implies

$$(7.3) \quad a_{\frac{R\pm 1}{2},2}^{(2)}(T) \equiv \frac{4(p+1)B_{\frac{p-1}{2},x_{D(T)}}}{h(-p) \cdot B_{p-1}} \cdot F_{\frac{R\pm 1}{2}}(T) \pmod{p}.$$

First suppose that $D(T) \neq -p$. In this case, $p$ does not appear in the denominator of $B_{\frac{p-1}{2},x_{D(T)}}$ (cf. Theorem 3.2, (1)). Then, by the theorem of von Staudt-Clausen (Theorem 3.1, (2)), the right-hand side of (7.3) is divisible by $p$. Secondly suppose that $D(T) = -p$. In this case, we have

$$pB_{p-1} \equiv -1 \pmod{p}, \quad pB_{\frac{p-1}{2},x_{-p}} \equiv -1 \pmod{p}.$$
Therefore, we get

\[ \frac{B_{\frac{z_{\frac{-1}{2},x-p}}}{B_{p-1}}} \equiv 1 \pmod{p}. \]

So we can rewrite (7.3) as

\[ d^{(2)}_{\frac{e_{2}}{2}}(T) \equiv \frac{4\alpha_{T}}{h(-p)} F_{\frac{e_{2}}{2}}(T) \pmod{p}. \]

We shall show

\[ F_{\frac{e_{2}}{2}}(T) \equiv \sum_{0<d|\epsilon(T)} \left( \frac{d}{p} \right) \pmod{p}. \]

The proof of this formula is the same as that of (6.18). In fact, we have

\[
F_{\frac{e_{2}}{2}}(T) = \sum_{0<d|\epsilon(T)} d^{\frac{e_{2}}{2}} \sum_{0<f|\frac{\epsilon(T)}{d}} \mu(f) \chi_{-p}(f) f^{\frac{e_{2}}{2}} \sigma_{p-2} \left( \frac{f(T)}{f \Delta} \right) \\
\equiv \sum_{0<d|\epsilon(T)} \left( \frac{d}{p} \right) \sum_{0<f|\frac{\epsilon(T)}{d}} \mu(f) f^{-1} \sigma_{-1} \left( \frac{f(T)}{f \Delta} \right) \quad \pmod{p}.
\]

We can show by (6.20) that the inner sum is equal to 1. This proves (7.5), and consequently, we get (7.2).

\[ \square \]

References


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