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Conditional Cyclic Base Change

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1 – Introduction

Base change is a major topic in the modern theory of automorphic form. The quadratic base change for modular forms, from $\mathbb{Q}$ to a real quadratic field $\mathbb{Q}(\sqrt{d})$, was first obtained by Doi and Naganuma using L-functions [DN]. Then Saito [Sa] discovered that cyclic base change could be deduced from the comparison between a trace formula and a twisted trace formula; but technical limitations to his result arose from the fact he was working in the classical language. Shintani reformulated Saito's method in the representations theoretic and adelic framework and understood it should yield unconditional results for the cyclic base change of automorphic representations for $GL(2)$; this was worked out by Langlands [Lan]. From this and other quite deep results (due to Gelbart, Jacquet, Piatetskii-Shapiro and Shalika), Langlands deduced non trivial cases of Artin's conjecture. This last result, as completed by Tunnell, is an essential step in Wiles' proof of Fermat last theorem.

Some further important contributions should be quoted. In [AC], Arthur and Clozel proved the existence of base change for $GL(n)$, and Rogawski [Rog] established base change between unitary groups in 3 variables and their split (outer) form $GL(3)$. Partial results have also been obtained for more general unitary groups (see below).

In this note I would like to describe how much one can prove as regards the existence of base change for arbitrary reductive groups. We cannot get an unconditional theorem for arbitrary reductive groups since a key ingredient in the proof – the so-called fundamental lemma – is only available (but for a few exceptions) for the endoscopic group attached to the trivial endoscopic character. To circumvent this lack of knowledge we have to restrict to representations of a certain type at some specific places.

2 – Weak base change

Let E/F be a cyclic extension of a number field F, and let θ be a generator of the Galois group $\mathrm{Gal}(E/F)$.

Let G_0 be a reductive group over F; let G be the restriction of scalars from E to F of G_0; this is the group scheme such that for any F-algebra A

$$G(A) = G_0(E \otimes_F A)$$

Observe that θ defines an automorphism of G over F. Denote by H the quasi-split inner form of G_0. The group H is the endoscopic group attached to the trivial endoscopic character for G twisted by θ.

For almost all places v of F, the group scheme G is defined over $\mathfrak{O}_v$ the ring of integers of F_v; then we consider $K_v^G = G(\mathfrak{O}_v)$. For almost all places v the groups K_v^G and K_v^H are hyperspecial maximal compact subgroups in $G_v := G(F_v)$ and H_v and one may choose Borel F_v-subgroups $B_v^G \subset G_v$ and $B_v^H \subset H_v$.

For such places we have the notion of unramified representations i.e. irreducible admissible representations with a non zero invariant vector under the hyperspecial maximal compact subgroup. Recall that an unramified representations is the unique subquotient with a non zero invariant vector under the hyperspecial maximal compact subgroup, of a principal series representations induced by an unramified characters of the Borel subgroup.

There is a norm map between the tori quotients of the Borel subgroups B_v^G and B_v^H by their unipotent radicals. For tori, base change is nothing but the composition of characters with the norm map. For unramified representation the local base change is defined using local base change for characters of the tori, quotients of the Borel subgroups by their unipotent radicals.

Definition. *Let π_v^H be an unramified representations of $H(F_v)$ that is a subquotient of the principal series attached to an unramified characters*

η_v of a Borel F-subgroup B_v^H. We say that π_v^G is the local base change of π_v^H if π_v^G is the subquotient with a non zero invariant vector under K_v^G of the principal series representation induced by the character

$$\xi_v = \eta_v \circ N_{E/F} \ .$$

We shall not try to define local base change in general. Unramified local base change is enough to define weak (global) base change:

Definition. *Let π^H and π^G be two admissible irreducible representations of the adelic groups $H(\mathbb{A}_F)$ and $G(\mathbb{A}_F)$. We say that π^G is a weak base change of π^H if for almost all places v the representation π_v^G is the local base change of π_v^H.*

3 – θ-stable representations

Let π^G be an admissible irreducible representation of $G(\mathbb{A}_F)$ which is θ-stable

$$\pi^G \circ \theta \simeq \pi^G \ .$$

Denote by $A(\pi^G)$ the isotypic component of π^G in the cuspidal spectrum. The automorphism θ acts on $A(\pi^G)$ and we thus get a natural representation of the semi-direct product $G(\mathbb{A}_F) \rtimes \mathrm{Gal}(E/F)$ in this space.

Definition. *We say that a cuspidal automorphic representation π^G contributes non trivially to the θ-twisted trace formula if the trace of operators on $A(\pi^G)$ defined by smooth compactly supported functions on the coset $G(\mathbb{A}_F) \rtimes \theta$, is not identically zero.*

Observe that θ-stable cuspidal representations that occur with multiplicity one contributes non trivially to the θ-twisted trace formula.

4 – The main theorems

It would be too technical to state the results for the most general situation; we shall assume that the derived group H_{der} is simply connected and that the co-center $D_H = H/H_{der}$ is a torus split over an extension E_0 of F contained in E. In particular G satisfies the Hasse principle. Let $\mathfrak{V}$ be a finite set of finite places such that for at least one place $v \in \mathfrak{V}$ the algebra $F_v \otimes E_0$ is a field. We assume moreover that $\mathfrak{V}$ contains at least two places. We first state a descent theorem.

Theorem 1. *Let $\pi^G = \otimes\pi_v^G$ be an automorphic cuspidal representation for G over F such that for all $v \in \mathfrak{V}$ the representation π_v^G is the Steinberg representation and assume that the representation π^G contributes non trivially to the θ-twisted trace formula. Then, there exist an automorphic cuspidal representation π^H for H such that π^G is a weak base change of π^H.*

In the particular case where $E = F$ we obtain a (conditional) transfer for cuspidal representations from an inner form to its quasi split form "à la Jacquet-Langlands". For the converse theorem, i.e. the lifting theorem we have to assume that G_0 is quasi-split: $G_0 = H$.

Theorem 2. *Assume that G_0 is quasi-split. Let π^H be an automorphic cuspidal representation for G over F such that for all $v \in \mathfrak{V}$ the representation π_v^H is the Steinberg representation. Then, there exist an automorphic cuspidal representation π^G for G such that π^G is a weak base change of π^H.*

5 – Application to cuspidal cohomology

The compatibility of weak base change with the local base change, when defined, can be checked in many cases. This is in particular the case for representation with cohomology: one can show that a cuspidal representation as in theorem 2, with non trivial cohomology at some places has a weak base change with non trivial cohomology at those places.

Let F be a number field and let S be a finite set of places containing all archimedean ones. Consider an S-arithmetic subgroup Γ of

$$G_S := G(F_S)\,.$$

One defines its cuspidal cohomology as follows:

$$H^*_{cusp}(\Gamma, \mathbb{C}) := H^*_d(G_S, L^2_{cusp}(\Gamma\backslash G_S)^\infty)$$

where H^*_d is the differentiable cohomology and $L^2_{cusp}(\Gamma\backslash G_S)^\infty$ is the space of smooth vectors in the space of square integrable functions generated by cusp forms. As is shown in [BLS] this is a direct summand of the usual cohomology of Γ.

The compatibility of base change with cohomology allows to prove for any semi-simple simply connected split group G a theorem proved only for $G = SL(n)$ in [BLS].

Theorem 3. *Let G be a simply connected semi-simple split group over a totally real field F_0. Consider a tower of cyclic extensions*

$$F_0 \subset F_1 \subset \ldots \subset F_n = E .$$

Let S be a finite set of places of E containing all archimedean places. Then, any S-arithmetic subgroup Γ of $G(E_S)$ contains an S-arithmetic subgroup Γ' of finite index with non trivial cuspidal cohomology:

$$H^*_{cusp}(\Gamma', \mathbb{C}) \neq 0 .$$

6 – About the proof

When H is a unitary group, theorems similar to theorems 1 and 2 are already in the literature (see for example [Clo2] and [Clo3]). The key ingredients of the proofs, which rely on a trace formula comparison, are:

1 – the fundamental lemma for units in the base change situation [Ko1];

2 – the extension of the fundamental lemma for base change to the full unramified Hecke algebra using [Clo1] (or [Lab1])

3 – the stabilization of the trace formula using Euler-Poincare functions at finite places following [Ko2].

Our proof follows the same familiar patterns. We observe that some of the arguments in the papers quoted above are incomplete or incorrect. For example the proof of the vanishing assertion in the fundamental lemma for elements that are not a norm is not correct in [Clo1] (and the same error is reproduced in [Lab1]). This can be corrected following a suggestion of Kottwitz. In the proof of the existence of lifting, as outlined in [Clo3], it does not seem to be enough to deal with functions of regular support and some control on the singular set seems necessary.

In [Lab2] we carry out the necessary work to correct and complete these proofs and we extend the arguments to the general case. This preprint is presently available under the internet address:

http://www.math.jussieu.fr/~labesse/publications.html

REFERENCES

[AC] J. Arthur L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Math. Studies **120** Princeton Univ. Press (1989).

[BLS] A. BOREL J.-P. LABESSE J. SCHWERMER, *On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields*, Compositio Math. **102** (1996), pp. 1-40.

[Clo1] L. CLOZEL, *The fundamental lemma for stable base change*, Duke Math. J. **61** No. 1 (1990), pp. 255-302.

[Clo2] L. CLOZEL, *représentations galoisiennes associées aux représentations automorphes autoduales de $GL(n)$*, Public. Math. IHES **73** (1991), pp. 97-145.

[Clo3] L. CLOZEL, *On the cohomology of Kottwitz's arithmetic varieties*, Duke Math. J. **72** (1993), pp. 757-795.

[DN] K. DOI H. NAGANUMA, *On the functional equation of certain Dirichlet series*, Invent. Math. **9** 1969/1970 p.1–14

[Ko1] R. KOTTWITZ, *Base change for units of Hecke algebras*, Compositio Math. **60** (1986), pp. 237-250.

[Ko2] R. KOTTWITZ, *Tamagawa numbers*, Annals of Math. **127** (1988), pp. 629-646.

[Lab1] J.-P. LABESSE, *Le lemme fondamental pour le changement de base stable*, Duke Math. J. **61** No.2 (1990), pp. 519-530.

[Lab2] J.-P. LABESSE, *Changement de base conditionnel pour les groupes réductifs*, preprint (January 1998)

[Lan] R.P. LANGLANDS, *Base Change for $GL(2)$*, Annals of Math. Studies **96** Princeton Univ. Press (1980).

[Rog] J. ROGAWSKI, *Automorphic Representations of Unitary Groups in Three Variables*, Annals of Math. Studies **123** Princeton Univ. Press (1990).

[Sa] H. SAITO, *Automorphic forms and algebraic extensions of number fields*, Kyoto Univ., Lectures in Math. **8** Kinokuniya Book Store Co. Ltd. Tokyo, Japan (1975).

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