On the Dimension Formula for the Spaces of Siegel Cusp Forms of Half Integral Weight and Degree Two

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§1. Results

Let \( \mathfrak{S}_g = \{ Z \in M_g(\mathbb{C}) \mid {}^tZ = Z, \operatorname{Im} Z > 0 \} \) be the Siegel upper half plane of degree \( g \), \( \Gamma_g = \text{Sp}(g, \mathbb{Z}) \) the Siegel modular group of degree \( g \) and

\[
\Gamma_g^* = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \mid \text{diagonal elements of } A'B, C'D \text{ are even} \right\}.
\]

If \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), we denote \( (AZ + B)(CZ + D)^{-1} \) by \( M\langle Z \rangle \). Let \( \operatorname{e}(z) = \exp(2\pi i z) \) and for \( Z \in \mathfrak{S}_g \) put

\[
\theta(Z) = \sum_{\eta \in \mathbb{Z}^g} \operatorname{e} \left( \frac{1}{2} \eta Z \eta \right).
\]

If \( M \in \Gamma_g^* \), \( \theta(M\langle Z \rangle) / \theta(Z) \) is holomorphic on \( \mathfrak{S}_g \). Let \( \alpha = \begin{pmatrix} 2^{-1} & 1_g \\ 0 & 1_g \end{pmatrix} \) and let \( \Theta(Z) = \theta(2Z) = \theta(\alpha\langle Z \rangle) \). Let

\[
\Gamma_0^g(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \mid C \equiv O \pmod{N} \right\}.
\]

Then \( \alpha^{-1}\Gamma_g^* \alpha \cap \Gamma_g \) contains \( \Gamma_0^g(4) \). Hence if \( M \in \Gamma_0^g(4) \),

\[
J(M, Z) := \Theta(M\langle Z \rangle) / \Theta(Z)
\]

is holomorphic on \( \mathfrak{S}_g \) and satisfies the equality:

\[
J(M, Z)^2 = \det(CZ + D)\psi(\det D),
\]

where \( \psi : 1+2\mathbb{Z} \to \{ \pm 1 \} \) is the non-trivial Dirichlet character modulo 4. \( J(M, Z) \) is the automorphic factor of weight \( 1/2 \).

In the following we assume that \( g = 2 \). Let \( \text{Sym}^j : \text{GL}(2, \mathbb{C}) \to \text{GL}(j + 1, \mathbb{C}) \) be the symmetric tensor representation of degree \( j \). \( \text{Sym}^j(CZ + D) \) is also an automorphic factor (with respect to \( \Gamma_2 \)) and so is \( J(M, Z)^{2k+1} \text{Sym}^j(CZ + D) \) (with respect to \( \Gamma_0^2(4) \)). Let \( \Gamma \) be a subgroup of \( \Gamma_0^2(4) \) of finite index. A holomorphic mapping \( f : \mathfrak{S}_2 \to \mathbb{C}^{j+1} \) is called a Siegel modular form of half integral weight with respect to \( \Gamma \), if \( f \) satisfies the following equality for any \( M \in \Gamma \) and \( Z \in \mathfrak{S}_2 \):

\[
f(M\langle Z \rangle) = J(M, Z)^{2k+1} \text{Sym}^j(CZ + D) f(Z).
\]
We denote by $M_{j,k+1/2}(\Gamma)$ the \(C\)-vector space of all such mappings. \(f \in M_{j,k+1/2}(\Gamma)\) is called a cusp forms if \(f\) belongs to the kernels of the \(\Phi\)-operators. We denote the space of cusp forms by $S_{j,k+1/2}(\Gamma)$. Namely, \(f\) belongs to $S_{j,k+1/2}(\Gamma)$ if and only if

$$\lim_{\text{Im} z_2 \to \infty} f(M \langle Z \rangle) = 0,$$

for any $M \in \Gamma_2$, where $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$. It is known that $M_{j,k+1/2}(\Gamma)$ is finite-dimensional.

Let \(\chi\) be a character of \(\Gamma\) whose kernel is a subgroup of \(\Gamma\) of finite index. We denote by $M_{j,k+1/2}(\Gamma, \chi)$ the \(C\)-vector space of the holomorphic mappings of $\mathfrak{S}_2$ to $C^{j+1}$ which satisfy

$$f(M \langle Z \rangle) = J(M, Z)^{2k+1}(xM) \ Sym^j(CZ + D) f(Z),$$

for any $M \in \Gamma$ and $Z \in \mathfrak{S}_2$. We also denote by $S_{j,k+1/2}(\Gamma, \chi)$ its subspace of cusp forms.

Let $\psi$ be as before and let $j$ be odd. Then since $-1_4 \in \Gamma_0^2(4)$ and $\Sym^j(-1_2) = -1_{j+1}$, $M_{j,k+1/2}(\Gamma_0^2(4))$ and $M_{j,k+1/2}(\Gamma_0^2(4), \psi)$ are $\{0\}$. Therefore we assume $j$ is even in the following.

Our main results are the following two theorems.

**Theorem 1.1.** If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$, \(\dim S_{2j,+k1/2}(\Gamma_02(4))\) is given by the following Mathematica function:

```mathematica
SiegelHalf[j_, k_] := Block[{a, ljk},
  mod[x_, y_] := Mod[x, y] + 1;
  a = (2*j + 1)*(4*j + 2*k - 1)*(j + k - 1)*(2*k - 3)/2^5/3^2;
  a = a + (2*j + 1)*If[Mod[k, 2] == 0, 19 - 22*j, 25 - 22*j][2^{-6}];
  a = a + (2*j + 1)*If[Mod[k, 2] == 0, -1, 1]/2^6;
  a = a + (4*j + 2*k - 1)*(2*k - 3)/2^6;
  a = a + If[Mod[k, 2] == 0, 17 - 12*j - 12*j, 49 - 20*j - 20*j][2^{-6}];
  a = a + 7*(4*j + 2*k - 1)*(2*k - 3)/2^6/3;
  a = a + (35 - 48*k - 48*j)/2^5/3;
  a = a - 13/2^4/3;
  a = a + If[Mod[k, 2] == 0, 7, 15][2^6];
  a = a + If[Mod[k, 2] == 0, 2, 3][2^{-2};
  ljk = {1, -1};
  a = a + (j + k - 1)*ljk[[mod[j, 2]]][2^{-3};
  a = a - If[Mod[k, 2] == 0, 3, 5]*ljk[[mod[j, 2]]][2^{-4};
```
Theorem 1.2. If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$, $\dim S_{2j,k+1}/2(\Gamma_0^2(4), \psi)$ is given by the following Mathematica function:

\[
\text{SiegelHalfpsi}[j_-, k_-] := \text{Block}[[a, ljk],
\begin{align*}
\text{mod}[x_-, y_-] &= \text{Mod}[x, y]+1; \\
\text{a} &= (2*j+1)*(4*j+2*k-1)*(j+k-1)*(2*k-3)/2^5/3^2; \\
\text{a} &= a+(2*j+1)*\text{If}[\text{Mod}[k, 2]==0, 25-22*k-22*j, 19-22*k-22*j]/2^6/3; \\
\text{a} &= a-3*(2*j+1)*\text{If}[\text{Mod}[k, 2]==0, -1, 1]/2^6; \\
\text{a} &= a-(4*j+2*k-1)*(2*k-3)/2^6; \\
\text{a} &= a-\text{If}[\text{Mod}[k, 2]==0, 49-20*k-20*j, 17-12*k-12*j]/2^6; \\
\text{a} &= a+7*(4*j+2*k-1)*(2*k-3)/2^6/3; \\
\text{a} &= a-(35-48*k-48*j)/2^5/3; \\
\text{a} &= a+13/2^4/3; \\
\text{a} &= a-\text{If}[\text{Mod}[k, 2]==0, 15, 7]/2^6; \\
\text{a} &= a-\text{If}[\text{Mod}[k, 2]==0, 3, 2]/2^2; \\
\text{ljk} &= \{1,-1\}; \\
\text{a} &= a+(j+k-1)*\text{ljk}[[\text{Mod}[j, 2]]]/2^3; \\
\text{a} &= a-\text{If}[\text{Mod}[k, 2]==0, 5, 3]*\text{ljk}[[\text{Mod}[j, 2]]]/2^4; \\
\end{align*}
\]

Return[a];]
a=a-If[Mod[k,2]==0,1,3]*ljk[[mod[j,2]]]/2^4;

ljk={1,0,-1};

a=a+2*ljk[[mod[j,3]]]*(j+k-1)/3^2;
a=a-1*ljk[[mod[j,3]]]/2;

ljk=(2*j+1)*{{1,0,-1},{0,-1,1},{-1,1,0}};
a=a+ljk[[mod[j,3],mod[k,3]]]/2/3^2;

ljk={{1,-2,1},{-2,1,1},{1,1,-2}};
a=a-ljk[[mod[j,3],mod[k,3]]]/2/3^2;

ljk={1,-2,1};
a=a+ljk[[mod[j,3]]]/2/3^2;

Return[a];
]

§2. Methods

Let $\Gamma_g(N)$ be the principal congruence subgroup of level $N$ of $\Gamma_g$. Namely,

$$\Gamma_g(N) = \{ M \in \Gamma_g \mid M \equiv 1_{2g} \pmod{N} \}.$$ 

This is a normal subgroup of $\Gamma_g$. If $N \geq 3$, $\Gamma_g(N)$ acts on $\mathfrak{S}_g$ without fixed points and the quotient space $X_g(N) := \Gamma_g(N) \backslash \mathfrak{S}_g$ is a (non-compact) manifold. $X_g(N)$ is an open subspace of a projective variety $\overline{X}_g(N)$ which was constructed by I. Satake (Satake compactification, [Sta]). If $g \geq 2$, $\overline{X}_g(N)$ has singularities along its cusps: $\overline{X}_g(N) - X_g(N)$. Cusps of $\overline{X}_g(N)$ is (as a set) a disjoint union of copies of $X_g(N)$'s $(0 \leq g' < g)$. A desingularization $\tilde{X}_g(N)$ of $\overline{X}_g(N)$ was constructed by J.-I. Igusa and Y. Namikawa ($g = 2, 3, 4$) ([Ig2], [N]) and more generally by D. Mumford and others (Toroidal compactification, [AMRT]).

Let $V$ be $\mathfrak{S}_g \times \mathbb{C}^g$ and let $v \in \mathbb{C}^g$. $\Gamma_g(N)$ acts on $V$ as follows:

$$M(Z,v) = (M\langle Z \rangle,(CZ+D)v).$$

If $N \geq 3$, $V := \Gamma_g(N) \backslash V$ is non-singular and is a vector bundle over $X_g(N)$. $V$ is extended to a vector bundle $\tilde{V}$ over $\tilde{X}_g(N)$. Let $\mathcal{V}$ be $\mathfrak{S}_g \times \mathbb{C}$ and let $v \in \mathbb{C}$. $\Gamma_g(4N)$ acts on $\mathcal{V}$ as follows:

$$M(Z,v) = (M\langle Z \rangle,J(M,Z)v).$$
$H_g := \Gamma_g(4N) \backslash \mathcal{H}_g$ is a line bundle over $X_g(4N)$. $H_g$ is extended to a line bundle $\tilde{H}_g$ over $\tilde{X}_g(4N)$ and also to a line bundle $\overline{H}_g$ over $\overline{X}_g(4N)$.

Let $\Gamma$ be a subgroup of $\Gamma_0^g(4)$ of finite index. If $g \geq 2$, $\Gamma$ contains $\Gamma_g(4N)$ for some $N$ ([BLS], [M]). In the following we assume that $g = 2$. The space of Siegel modular forms $M_{j,k+1/2}(\Gamma_2(4N))$ is canonically identified with the space

$$\Gamma(\tilde{X}_2(4N), \mathcal{O}(\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{\otimes(2k+1)})),$$

which is the space of the global holomorphic sections of $\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{\otimes(2k+1)}$. The divisor at infinity $D := \tilde{X}_2(4N) - \tilde{X}_2(4N)$ is a divisor with simple normal crossings. The space of cusp forms $S_{j,k+1/2}(\Gamma_2(4N))$ is canonically identified with the space

$$\Gamma(\tilde{X}_2(4N), \mathcal{O}(\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{\otimes(2k+1)} - D)).$$

$\mathcal{O}(\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{\otimes(2k+1)} - D)$ is the sheaf of germs of holomorphic sections which vanish along $D$ and this is isomorphic to $\mathcal{O}(\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{\otimes(2k+1)} \otimes [D]^{\otimes(-1)})$, where $[D]$ is the line bundle associated with $D$. We can prove the following

**Theorem 2.1.** If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$, then

$$H^p(\tilde{X}_2(4N), \mathcal{O}(\text{Sym}^j(\tilde{V}) \otimes \tilde{H}_2^{\otimes(2k+1)} \otimes [D]^{\otimes(-1)})) \simeq \{0\},$$

for $p > 0$.

By using this theorem and the theorem of Riemann-Roch-Hirzebruch we have

**Theorem 2.2.** If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$,

$$\dim S_{j,k+1/2}(\Gamma_2(4N))$$

$$= 2^{3j-1}(j+1) \{2(2k-3)(2j+2k-1)(j+2k-2)N^{10} - 30(j+2k-2)N^8 + 45N^7 \}$$

$$\times \prod_{p \mid N, \ p \text{ odd prime}} (1 - p^{-2})(1 - p^{-4}).$$

Let $\Gamma$ be a subgroup of $\Gamma_0^g(4)$ of finite index and let $\chi$ be a character of $\Gamma$ whose kernel is a subgroup of $\Gamma$ of finite index. We may assume that the kernel of $\chi$ contains $\Gamma_2(4N)$. Let $f \in S_{j,k+1/2}(\Gamma_2(4N))$ and $M \in \Gamma$. We define an action of $M$ on $S_{j,k+1/2}(\Gamma_2(4N))$ as follows:

$$Mf(M \langle Z \rangle) = J(M, Z)^{2k+1} \chi(M) \text{Sym}^j(CZ + D) f(Z).$$

Since $\Gamma_2(4N)$ acts trivially on $S_{j,k+1/2}(\Gamma_2(4N))$, this action induces an action of $\Gamma/\Gamma_2(4N)$ on $S_{j,k+1/2}(\Gamma_2(4N))$ and $S_{j,k+1/2}(\Gamma, \chi)$ is identified with the invariant subspace of $S_{j,k+1/2}(\Gamma_2(4N))$. Thus we have

$$S_{j,k+1/2}(\Gamma, \chi) = S_{j,k+1/2}(\Gamma_2(4N))^\Gamma/\Gamma_2(4N).$$
Therefore \( \dim S_{j,k+1/2}(\Gamma, \chi) \) is computed by using the holomorphic Lefschetz fixed point formula ([AS]).

To use the Lefschetz fixed point formula we have to classify the fixed points (sets). Let \( N \geq 3 \). \( \Gamma_2 \) and \( \Gamma_2/\Gamma_2(N) \) act on \( \overline{X}_2(N) \). We classify (the irreducible components of) the fixed points of \( \Gamma_2 \) in the following sense. Let \( \Phi_1 \) and \( \Phi_2 \) be the fixed points (sets). \( \Phi_1 \) and \( \Phi_2 \) is called equivalent if there is an element of \( \Gamma_2 \) which maps \( \Phi_1 \) biholomorphically to \( \Phi_2 \). The fixed points in the quotient space \( X_2(N) \) were classified in [G]. The fixed points in the divisor at infinity are classified easily. In total there are 25 kinds of fixed points (sets). Among them 10 fixed points are not fixed by the elements of \( \Gamma_2^2(4) \). But since the automorphic factor \( J(M, Z) \) is defined with respect to \( \Gamma_2^2(4) \), we have to classify the remaining 15 fixed points with respect to \( \Gamma_2^2(4) \).

Let \( \Phi \) be one of 15 fixed points and let

\[
C(\Phi) = \{ M \in \Gamma_2 \mid M(Z) = Z \text{ for any } Z \in \Phi \},
\]
\[
C^p(\Phi) = \{ M \in C(\Phi) \mid \Phi \text{ is closed in } \text{Fix}(M) \},
\]
\[
N(\Phi) = \{ M \in \Gamma_2 \mid M \text{ maps } \Phi \text{ into } \Phi \}.
\]

What we have to do is to classify the double cosets \( \Gamma_2^2(4) \backslash \Gamma_2/N(\Phi) \). Let \( P_1, P_2, \ldots, P_n \) be the representatives of \( \Gamma_2^2(4) \backslash \Gamma_2/N(\Phi) \). Next we have to check \( P_i C^p(\Phi)P_i^{-1} \cap \Gamma_2^2(4) \) (\( i = 1, 2, \ldots, n \)) is empty or not. Since \( \Gamma_2 \) is an infinite group, it is not an easy task to classify \( \Gamma_2^2(4) \backslash \Gamma_2/N(\Phi) \). But since \( \Gamma_2^2(4) \) contains \( \Gamma_2(4) \), we can take the quotient by \( \Gamma_2(4) \) and reduce the problem to a task in the finite group \( \Gamma_2/\Gamma_2(4) \simeq Sp(2, \mathbb{Z}/4\mathbb{Z}) \) and we can use the computer. We list the result in the following proposition. As to the notations of the fixed points (sets), see [T2]. Let \( \rho \) be \( \exp(2\pi i/3) \).

**Proposition 2.3.** For each \( \Phi \) the number of the elements of \( \Gamma_2^2(4) \backslash \Gamma_2/N(\Phi) \) and the number of the double cosets such that \( P_i C^p(\Phi)P_i^{-1} \cap \Gamma_2^2(4) \neq \phi \) is as follows.

\[
\begin{array}{ccccccc}
( & z_1 & z_2 & ) & 1 & 1 & ( & z_1 & 0 & ) & 3 & 2 & ( & z_1 & 1/2 & ) & 5 & 3 \\
( & z_2 & z_3 & ) & & & ( & 0 & z_2 & ) & & & ( & 1/2 & z_2 & ) & & \\
( & z & 0 & ) & 11 & 2 & ( & z & 1/2 & z & ) & 8 & 2 & ( & z & z/2 & z & ) & 6 & 1 \\
( & 0 & z & ) & & & ( & 1/2 & z & ) & & & ( & z/2 & z & ) & & \\
( & \rho & 0 & ) & 10 & 1 & \sqrt{-3}/3 & ( & 2 & 1 & ) & 24 & 2 & ( & z_1 & z_2 & ) & 4 & 4 \\
( & 0 & \rho & ) & & & ( & 2 & 1 & ) & & & ( & z_2 & \infty & ) & & \\
( & z & 0 & ) & 7 & 6 & ( & z & 1/2 & ) & 10 & 7 & ( & z & z & ) & 12 & 7 \\
( & 0 & \infty & ) & & & ( & 1/2 & \infty & ) & & & ( & z & \infty & ) & & \\
( & \infty & 0 & ) & 15 & 13 & ( & \infty & 1/2 & ) & 13 & 9 & ( & \infty & \infty & ) & 8 & 8
\end{array}
\]
Therefore there are 68 kinds of fixed points of $\Gamma_0^2(4)$ in total. By computing the contributions of these fixed points to the dimension of

$$S_{2j,k+1/2}(\Gamma_0^2(4)) = S_{2j,k+1/2}(\Gamma_2(4N))\Gamma_2^2(4N)/\Gamma_2(4N),$$

we can calculate $\dim S_{2j,k+1/2}(\Gamma_0^2(4))$ and similarly $\dim S_{2j,k+1/2}(\Gamma_0^2(4),\psi)$.

In this note I explain nothing about the computation of the theorem of Riemann-Roch-Hirzebruch or the Lefschetz fixed point formula. As to the former, see [Y], [T4] and [T1]. As to the latter, see [T2].

§3. The case $j = 0$

In case $j = 0$, we denote the space $M_{0,k+1/2}(\Gamma_0^2(4))$ and $S_{0,k+1/2}(\Gamma_0^2(4))$ by $M_{k+1/2}(\Gamma_0^2(4))$ and $S_{k+1/2}(\Gamma_0^2(4))$, respectively. From Theorem 1.1 we have

Proposition 3.1.

$$\sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^2(4)) t^k = \sum_{k=0}^{\infty} \text{SiegelHalf}[0,k] t^k + t^2$$

$$= \frac{2t^5 + 2t^6 - t^7 - 2t^8 - t^9 + t^{10}}{(1-t)(1-t^2)^2(1-t^3)}.$$

Proof. If $f(Z) \in S_{k+1/2}(\Gamma_0^2(4))$, then $f(Z)\Theta(Z)^2 \in S_{k+3/2}(\Gamma_0^2(4))$. Since $\dim S_{7/2}(\Gamma_0^2(4))$ is equal to $\text{SiegelHalf}[0,3] = 0$, we have $S_{5/2}(\Gamma_0^2(4)) \simeq S_{3/2}(\Gamma_0^2(4)) \simeq S_{1/2}(\Gamma_0^2(4)) \simeq \{0\}$. But since $\text{SiegelHalf}[0,2] = -1$, $\text{SiegelHalf}[0,1] = 0$ and $\text{SiegelHalf}[0,0] = 0$, we have the equality of the first line. \qed

The cusps of the Satake compactification $\overline{\Gamma_0^2(4)\backslash \mathfrak{S}_2}$ of $\Gamma_0^2(4)\backslash \mathfrak{S}_2$ consists of 4 one-dimensional cusps and 7 zero-dimensional cusps. Each one-dimensional cusp is biholomorphic to $\overline{\Gamma_0^2(4)\backslash \mathfrak{S}_1}$.

Let $\Phi = \left\{ \left( \begin{array}{cc} z_1 & z_2 \\ z_2 & \infty \end{array} \right) \right\}$. $\Gamma_0^2(4)\backslash \Gamma_2/\Phi$ consists of 4 double cosets. Let $M_1 = 1_4$ and let

\[ M_2 = \left( \begin{array}{cc} O & 1_2 \\ -1_2 & O \end{array} \right), \quad M_3 = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array} \right), \quad M_4 = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array} \right), \quad g_n = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right). \]
$M_1, M_2, M_3$ and $M_4$ are the representatives of $\Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi)$. Let $C_i$ be the one-dimensional cusp corresponding to the double coset $\Gamma_0^2(4)M_iN(\Phi)$ ($i = 1, 2, 3, 4$), respectively. Put $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$.

Let $i = 1$ or 4. Then we have

$$\lim_{\text{Im } z_2 \to \infty} J(M_ig_nM_i^{-1}, M_i\langle z \rangle) = 1,$$

for any integer $n$. $M_2g_nM_2^{-1}$ belongs to $\Gamma_0^2(4)$ if and only if $4 \mid n$ and we have

$$\lim_{\text{Im } z_2 \to \infty} J(M_2g_nM_2^{-1}, M_2\langle z \rangle) = 1,$$

for any integer $n$. On the other hand we have

$$\lim_{\text{Im } z_2 \to \infty} J(M_3g_nM_3^{-1}, M_3\langle z \rangle) = i^n,$$

where $i = \sqrt{-1}$. Hence if $f \in M(\Gamma_0^2(4))$, we have

$$\lim_{\text{Im } z_2 \to \infty} f(M_3\langle z \rangle) = \lim_{\text{Im } z_2 \to \infty} f(M_3\langle g_n\langle z \rangle \rangle)f(M_3\langle z \rangle)$$

Therefore $\lim_{\text{Im } z_2 \to \infty} f(M_3\langle z \rangle)$ is identically 0. Namely, the $\Phi$-operators to the one-dimensional cusps $C_3$ and to the zero-dimensional cusps $P_5$, $P_6$ and $P_7$ are 0-maps. From this we have

**Proposition 3.2.**

$$\sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma_0^2(4)) t^k$$

$$= \sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^2(4)) t^k + 3 \sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^2(4)) t^k + 4 \sum_{k=0}^{\infty} t^k - (3 + 3t + t^2)$$

$$= \frac{2t^5 + 2t^6 - t^7 - 2t^8 - t^9 + t^{10}}{(1-t)(1-t^2)^2(1-t^3)} + \frac{3(t^4 + t^5)}{(1-t)^2} + \frac{4}{(1-t)} - (3 + 3t + t^2)$$

$$= \frac{1}{(1-t)(1-t^2)^2(1-t^3)} = \frac{1 + t + t^3 + t^4}{(1-t^2)^3(1-t^6)}.$$ 

**Proof.** In general the Eisenstein series of Klingen type of degree $n$ attached to a cusp form of degree $r$ and weight $k$ converges if $k > n + r + 1$ ([K]). In case $k$ is a half integer, this is also proved similarly as in the case of integral weight. Hence $\Phi$-operators to the one-dimensional cusps $C_1$, $C_2$ and $C_4$ are surjective ($\dim S_{k+1/2}(\Gamma_0^2(4)) = 0$, if $k \leq 3$). $\Phi$-operators to the zero-dimensional cusps $P_i$ ($i = 1, 2, 3, 4$) are surjective if $k \geq 3$. Hence the assertion was proved for $k \geq 3$. We can prove $\dim M_{1/2}(\Gamma_0^2(4)) = 1$, $\dim M_{3/2}(\Gamma_0^2(4)) = 1$ and $\dim M_{5/2}(\Gamma_0^2(4)) = 3$ by using the knowledge of the cases of higher weights ([T6]). So we have the proposition. \qed
Proposition 3.3. 

\[ M_{k+1/2}(\Gamma_0^2(4), \psi) = S_{k+1/2}(\Gamma_0^2(4), \psi). \]

Proof. Let \( Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \) and \( f \in M_{k+1/2}(\Gamma_0^2(4), \psi) \). We have to prove that

\[ \lim_{\text{Im} z_2 \to \infty} f(M \langle Z \rangle) = 0 \]

for any \( M \in \Gamma_2 \). Let \( M_i \) (\( i = 1, 2, 3, 4 \)) be as before and let

\[ P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]

To prove the assertion, it suffices to prove (*) for \( M_1, M_2, M_3 \) and \( M_4 \). From \( P \langle Z \rangle = Z \), we have

\[ M \langle Z \rangle = MP \langle Z \rangle = (MPM^{-1})M \langle Z \rangle. \]

Since \( M_i PM^{-1} = P \) for \( i = 1, 2 \) and \( 3 \), we have

\[ f(M_i \langle Z \rangle) = J(P, M_i \langle Z \rangle)^{2k+1} \psi(-1) f(M_i \langle Z \rangle) \]

\[ = -f(M_i \langle Z \rangle). \]

Hence \( f(M_i \langle Z \rangle) = 0 \). Next let \( i = 4 \). Then we have

\[ M_4 PM_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 4 & 0 & 0 & -1 \end{pmatrix} \]

and \( J(M_4 PM_4^{-1}, M_4 \langle Z \rangle) = 1 \). Therefore similarly as above we have \( f(M_4 \langle Z \rangle) = 0 \). \( \square \)

Remark 3.4. Note that \( f(M_i \langle Z \rangle) \) is identically zero before \( \text{Im} z_2 \) goes to \( \infty \). So it may be natural to ask that for any \( M \in \Gamma_2 \), \( f(M \langle Z \rangle) \) is identically zero or not. But this is not true in general. Let \( \Phi \) be \( \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \right\} \) and let

\[ M_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

\( \Gamma_0^2(4) \backslash \Gamma_2 / N(\Phi) \) consists of 3 double cosets. Their representatives are \( M_1, M_4 \) and \( M_5 \).

\[ M_5 PM_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \]

does not belong to \( \Gamma_0^2(4) \) but belongs to \( \alpha^{-1}\Gamma_2^* \cap \Gamma_2 \) and satisfies \( J(M_5 PM_5^{-1}, M_5 \langle Z \rangle) = 1 \). Therefore if \( f(Z) \in S_{k+1/2}(\alpha^{-1}\Gamma_2^* \cap \Gamma_2, \psi) \), it holds that \( f(M \langle Z \rangle) = 0 \) for any \( M \in \Gamma_2 \) and \( Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \). (\( \psi \) is extended to a character of \( \alpha^{-1}\Gamma_2^* \cap \Gamma_2 \).)
Proposition 3.5.
\[
\sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma_0^2(4), \psi) t^k = \sum_{k=0}^{\infty} \text{SiegelHalfpsi}[0,k] t^k + (3 + t + t^2) = \frac{t^{10}(1 + t + t^3 + t^4)}{(1 - t)^3(1 - t^2)^3(1 - t^6)}.
\]

Proof. Since we have \( \dim S_{3/2}(\Gamma_0^3(4), \psi) = \text{SiegelHalfpsi}[0,3] = 0 \), it follows that \( S_{3/2}(\Gamma_0^3(4), \psi) \simeq S_{1/2}(\Gamma_0^3(4), \psi) \simeq \{0\} \). On the other hand since we have \( \text{SiegelHalfpsi}[0,2] = -1 \), \( \text{SiegelHalfpsi}[0,1] = -1 \) and \( \text{SiegelHalfpsi}[0,0] = -3 \), we have the equality of the first line.

Let \( M(\Gamma_0^3(4)), M(\Gamma_0^3(4), \psi) \) and \( A(\Gamma_0^3(4), \psi) \) be \( \bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^3(4)), \bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^3(4), \psi) \) and \( \bigoplus M_k(\Gamma_0^3(4), \psi^k) \), respectively. Then \( A(\Gamma_0^3(4), \psi) \) is a graded ring and since it holds \( J(M, Z)^2 = \det(CZ + D)\psi(\det D), M(\Gamma_0^3(4)) \) and \( M(\Gamma_0^3(4), \psi) \) are \( A(\Gamma_0^3(4), \psi) \)-modules. From the result of J.-I. Igusa ([Ig1]), we have the following proposition. (We can also prove them by dimension formula.)

Proposition 3.6.
\[
\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^3(4)) t^k = \frac{1 + t^4 + t^{11} + t^{15}}{(1 - t^2)^3(1 - t^6)},
\]
\[
\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^3(4), \psi) t^k = \frac{t + t^3 + t^{12} + t^{14}}{(1 - t^2)^3(1 - t^6)},
\]
\[
\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^3(4), \psi^k) t^k = \frac{1 + t + t^3 + t^4}{(1 - t^2)^3(1 - t^6)}.
\]

From this we have

Corollary 3.7. \( M(\Gamma_0^3(4)) \) and \( M(\Gamma_0^3(4), \psi) \) are free \( A(\Gamma_0^3(4), \psi) \)-modules of rank 1.

The generator of \( M(\Gamma_0^3(4)) \) is \( \Theta(Z) \). Let \( f_{21/2}(Z) \) be the generator of \( M(\Gamma_0^3(4), \psi) \). Then \( f_{21/2}(Z) \Theta(Z) \) is an automorphic form with respect to \( J(M, Z)^2 \psi(\det D) = \det(CZ + D)^{11} \). Hence this belongs to \( M_{11}(\Gamma_0^3(4)) \). Let \( f_{11}(Z) \) be the base of \( M_{11}(\Gamma_0^3(4)) \) (\( \dim M_{11}(\Gamma_0^3(4)) = 1 \)). Then \( f_{11}(Z) / \Theta(Z) \) is holomorphic and we can assume that \( f_{21/2}(Z) = f_{11}(Z) / \Theta(Z) \). Since \( A(\Gamma_0^3(4), \psi) \) is contained in \( \bigoplus M_k(\Gamma_2(4)) \) and \( \bigoplus M_k(\Gamma_2(4)) \) is contained in the ring of theta constants ([Ig1]), every elements of \( M(\Gamma_0^3(4)) \) and \( M(\Gamma_0^3(4), \psi) \) are representable by theta constants.

Remark 3.8. T. Ibukiyama represented the generators of \( A(\Gamma_0^3(4), \psi) \) and \( f_{21/2}(Z) \) explicitly by theta constants ([Ib]). Especially \( A(\Gamma_0^3(4), \psi) \) is generated by algebraically independent modular forms \( f_1, X, g_2 \) and \( f_3 \) whose weights are 1, 2, 2 and 3, respectively. \( f_{21/2}(Z) \) is divisible by 9 theta constants. Let \( Z \in \mathcal{G}_2 \). Then there exists \( M \in \Gamma_2 \) such that \( M(Z) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \), if and only if one of 10 theta constants vanishes at \( Z \) (J.-I. Igusa, [H]). Hence \( f_{21/2}(Z) \not\in S_{21/2}(\alpha^{-1} \Gamma_2^* \alpha \cap \Gamma_2, \psi) \).
§4. The case \( j = 2 \)

If \( j > 0 \), the \( \Phi \)-operator to one-dimensional cusp maps \( M_{2j,k+1}(\Gamma_{0}^{2}(4)) \) to \( S_{2j+k+1/2}(\Gamma_{0}^{1}(4)) \) and the \( \Phi \)-operators to zero-dimensional cusps are 0-maps. Let \( C_{i} (i = 1, 2, 3, 4) \) be as before. The following proposition for the case of integral weight was proved in [A]. The case of half integral weight can be similarly proved.

**Proposition 4.1.** If \( k \geq 4 \), the \( \Phi \)-operator to \( C_{i} (i = 1, 2, 4) \)

\[ \Phi : M_{2j,k+1/2}(\Gamma_{0}^{2}(4)) \to S_{2j+k+1/2}(\Gamma_{0}^{1}(4)) \]

is surjective.

For two series \( \sum a_{k}t^{k} \) and \( \sum b_{k}t^{k} \) we write

\[ \sum a_{k}t^{k} \equiv \sum b_{k}t^{k} \quad (k \geq m), \]

if \( a_{k} = b_{k} \) for any \( k \geq m \). From Theorem 1.1 and the above proposition we have

**Proposition 4.2.**

\[
\sum_{k=0}^{\infty} \dim S_{2,k+1/2}(\Gamma_{0}^{2}(4)) t^{k} \equiv \sum_{k=0}^{\infty} \text{SiegelHalf}[1,k] t^{k} \quad (k \geq 4) \\
= \frac{-t^{2} + t^{3} + 3t^{4} + 3t^{5} - 3t^{7}}{(1-t)(1-t^{2})^{2}(1-t^{3})},
\]

\[
\sum_{k=0}^{\infty} \dim M_{2,k+1/2}(\Gamma_{0}^{2}(4)) t^{k} \equiv \frac{-t^{2} + t^{3} + 3t^{4} + 3t^{5} - 3t^{7}}{(1-t)(1-t^{2})^{2}(1-t^{3})} + 3\frac{(t^{2} + t^{3})}{(1-t^{2})^{2}} \quad (k \geq 4) \\
= \frac{2t^{2} + t^{3}}{(1-t)(1-t^{2})^{2}(1-t^{3})}.
\]

We study the structure of the \( A(\Gamma_{0}^{2}(4), \psi) \)-module \( \bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_{0}^{2}(4)) \) by a similar method in [Sto] where T. Satoh studied the the space of vector valued modular forms of integral weight with respect to \( \Gamma_{2} \).

Let \( V \) be \( \{ S \in M_{2}(\mathbb{C}) \mid ^{t}S = S \} \). We define the action of \( M \in GL(2, \mathbb{C}) \) on \( V \) by \( S \mapsto MS'M \). This action defines a representation of \( GL(2, \mathbb{C}) \) which is equivalent to \( \text{Sym}^{2} \). Let \( F \) be a \( \mathbb{C}^\infty \)-function on \( \mathfrak{S}_{2} \) and let

\[
\Delta F = \begin{pmatrix}
\frac{\partial F}{\partial Z_{11}} & 1 & \frac{\partial F}{\partial Z_{12}} \\
1 & \frac{\partial F}{\partial Z_{12}} & 2 \frac{\partial F}{\partial Z_{22}} \\
2 \frac{\partial F}{\partial Z_{12}} & \frac{\partial F}{\partial Z_{22}} & 0
\end{pmatrix}
\]

If \( M \in \Gamma_{2} \), it holds that

\[ (CZ + D)\Delta(F(M(Z)))^{t}(CZ + D) = (\Delta F)(M(\mathcal{Z})). \]
Hence if $F$ satisfies $F(M(Z)) = F(Z)$, we have
\[(\Delta F)(M(Z)) = (CZ + D)\Delta(F(Z)) + (CZ + D).\]

Let $f \in M_k(\Gamma^0_0(4), \psi^k)$ and $g \in M_{\ell+1/2}(\Gamma^0_0(4))$. Then $g^{2k}/f^{2\ell+1}$ is a (meromorphic) modular form of weight 0. Therefore $\Delta(g^{2k}/f^{2\ell+1})$ is a (meromorphic) modular form with respect to $\text{Sym}^2$. $f^{2\ell+2}/g^{2k-1}$ is a (meromorphic) modular form of weight $k + \ell + 1/2$. Hence
\[ [f, g] := \frac{1}{k(2\ell + 1)}(f^{2\ell+2}/g^{2k-1})\Delta(g^{2k}/f^{2\ell+1}) = \frac{1}{\ell + 1/2}f\Delta g - \frac{1}{k}g\Delta f \]
becomes a holomorphic modular form and belongs to $M_{2,k+\ell+1/2}(\Gamma^0_0(4))$. In general we have

**Proposition 4.3.** Let $f \in M_k(\Gamma^0_0(4), \psi^{k+\alpha})$ and $g \in M_{\ell+1/2}(\Gamma^0_0(4), \psi^\beta)$. Then
\[ [f, g] = \frac{1}{\ell + 1/2}f\Delta g - \frac{1}{k}g\Delta f \]
belongs to $M_{2,k+\ell+1/2}(\Gamma^0_0(4), \psi^{\alpha+\beta})$.

From this we have

**Theorem 4.4.** $\bigoplus_{k=0}^{\infty} M_{2,k+\ell+1/2}(\Gamma^0_0(4))$ is a free $A(\Gamma^0_0(4), \psi)$-module of rank 3 and the generators are $[X, \Theta], [g_2, \Theta]$ and $[f_3, \Theta]$.

**Proof.** Let $h_1, h_2 \in M_{k-2}(\Gamma^0_0(4), \psi^{k-2})$ and $h_3 \in M_{k-3}(\Gamma^0_0(4), \psi^{k-3})$. Assume that
\[ h_1[X, \Theta] + h_2[g_2, \Theta] + h_3[f_3, \Theta] \]
is identically zero. We may assume that $h_1, h_2$ or $h_3$ is not divisible by $f_1 = \Theta^2$. Then we have
\[ (*) \quad 2(h_1 X + h_2 g_2 + h_3 f_3)\Delta(\Theta) = \Theta \left( \frac{1}{2}h_1\Delta(X) + \frac{1}{2}h_2\Delta(g_2) + \frac{1}{3}h_3\Delta(f_3) \right). \]
Let the quotient of $h_i$ by $f_1$ be $q_i$ and the remainder $r_i$ (i = 1, 2, 3). Assume that $r_1X + r_2g_2 + r_3f_3$ is identically 0. Then we have
\[ 2\Theta(q_1 X + q_2 g_2 + q_3 f_3)\Delta(\Theta) = \left( \frac{1}{2}r_1\Delta(X) + \frac{1}{2}r_2\Delta(g_2) + \frac{1}{3}r_3\Delta(f_3) \right) + f_1 \left( \frac{1}{2}q_1\Delta(X) + \frac{1}{2}q_2\Delta(g_2) + \frac{1}{3}q_3\Delta(f_3) \right). \]
So $\frac{1}{2}r_1\Delta(X) + \frac{1}{2}r_2\Delta(g_2) + \frac{1}{3}r_3\Delta(f_3)$ is identically 0 on $H_\Theta := \{ Z \in \mathfrak{S}_2 \mid \Theta(Z) = 0 \}$. Therefore we have

\[ 1^\text{In the talk at RIMS, I said that } h_1X + h_2g_2 + h_3f_3 \text{ is not divisible by } f_1. \text{ But this was false.} \]
on $H_{\Theta}$. But we can show that the determinant $D(Z)$ of the matrix in the left-hand side of the above equation is not divisible by $\Theta(Z)$ as follows. Let

$$M = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$ 

Then from the transformation formula of theta constants we have

$$\Theta\left(M \left\langle \frac{Z_{11}}{R_{1}}, \frac{0}{R_{2}}, \frac{0}{R_{3}} \right\rangle \right) = \theta\left(M \left\langle \frac{2Z_{11}}{R_{1}}, \frac{0}{R_{2}}, \frac{2Z_{22}}{R_{3}} \right\rangle \right)$$

$$= \theta_{0000}\left(M \left\langle \frac{2Z_{11}}{R_{1}}, \frac{0}{R_{2}}, \frac{2Z_{22}}{R_{3}} \right\rangle \right)$$

$$= \kappa(M)\mathrm{e}^{(\phi_{1111}(M))}\det(2CZ+D)^{1/2}\theta_{1111}\left(\frac{2Z_{11}}{R_{1}}, \frac{0}{R_{2}}, \frac{2Z_{22}}{R_{3}} \right)$$

$$= 0,$$

where $\kappa(M)$ and $\mathrm{e}^{(\phi_{1111}(M))}$ are eighth root of unity and $\theta_{0000}$ and $\theta_{1111}$ are theta constants of characteristic $(0,0,0,0)$ and $(1,1,1,1)$, respectively.

Since $X$, $g_2$ and $f_3$ are represented by theta constants, we can prove that $D(M(Z))$ is not divisible by $Z_{12}$ from the transformation formula of theta constants and explicit Fourier expansions of theta constants ([T7]). Hence $r_i$ ($i = 1, 2, 3$) is identically $0$ on $H_{\Theta}$. This contradicts to the assumption that $h_1$, $h_2$ or $h_3$ is not divisible by $f_1$. Therefore $h_1X + h_2g_2 + h_3f_3$ in (*) is not divisible by $\Theta$. On the other hand, $\Delta(\Theta)$ in (*) is also not divisible by $\Theta$. Otherwise all of the points in $H_{\Theta}$ are singular points of $H_{\Theta}$. These facts contradict to the assumption that $h_1[X, \Theta] + h_2[g_2, \Theta] + h_3[f_3, \Theta]$ is identically zero.

From Proposition 4.2 theorem was proved for $k \geq 4$. The case $k \leq 3$ is easily proved from the result of the case $k \geq 4$.

\[ \square \]

**Remark 4.5.** If $f \in M_k(\Gamma_0^2(4), \psi^{k+1})$ and $g \in M_{k+1/2}(\Gamma_0^2(4), \psi)$, then $[f, g] \in M_{2, k+1/2}(\Gamma_0^2(4))$.

Where is this part? $\bigoplus_{k=0}^{\infty} M_k(\Gamma_0^2(4), \psi^{k+1})$ is a free $A(\Gamma_0^2(4), \psi)$-module of rank 1 and the generator is $f_{11}$. Since

$$[f_{11}, f_{21/2}] = -\frac{1}{22}[f_{21/2}, \Theta],$$

this part is already contained in $\bigoplus_{k=0}^{\infty} M_{2, k+1/2}(\Gamma_0^2(4))$.

Similarly as before we have
Proposition 4.6.

$$\sum_{k=0}^{\infty} \dim M_{2,k+1/2}(\Gamma_{0}(4),\psi) t^{k} = \sum_{k=0}^{\infty} \dim S_{2,k+1/2}(\Gamma_{0}(4),\psi) t^{k} = \frac{t^{5} + 2t^{6}}{(1-t)(1-t^{2})^{2}(1-t^{3})}.$$  

From this we present

Conjecture 4.7. $$\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_{0}(4),\psi)$$ is a free $$A(\Gamma_{0}(4),\psi)$$-module of rank 3.

Remark 4.8. The form of type $[f,g]$ in $$\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_{0}(4),\psi)$$ of the lowest weight is

$$[f_{11},\Theta] = -\frac{21}{22}\Theta_{2}^{2}, f_{21/2}.$$  

Hence $M_{2,k+1/2}(\Gamma_{0}(4),\psi)$ is not spanned by the forms of this type. T. Satoh proved that the space $M_{2,2k}(\Gamma_{2})$ is spanned by the forms of the above type but the space $M_{2,2k+1}(\Gamma_{2})$ is not spanned by the forms of the above type in [Sto] using the dimension formula ([T3]). This is natural since $\Theta M_{2,2k}(\Gamma_{2}) \subset M_{2,2k+1/2}(\Gamma_{0}(4))$ and $\Theta M_{2,2k+1}(\Gamma_{2}) \subset M_{2,2k+3/2}(\Gamma_{0}(4),\psi)$.

So we would like to present

Problem 4.9. Find the generators of the module $$\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_{0}(4),\psi).$$

§5. The case of general level

For example we can compute $\dim S_{2j,k+1/2}(\Gamma_{0}(4p),\chi)$ ($p :$ odd prime). This has been already reduced to a routine work (cf. [T5] for the case of integral weight) but will be a hard job.

APPENDIX

We list here the generating functions of SiegelHalf[j,k] and SiegelHalfpsi[j,k].

Table A.1. $$\sum_{j,k=0}^{\infty} \text{SiegelHalf}[j,k] s^{j} t^{k}$$ is a rational function of $s$ and $t$ whose denominator is

$$(1-s^{2})^{2}(1-s^{3})^{2}(1-t)(1-t^{2})^{2}(1-t^{3}).$$

The coefficients of $s^{j} t^{k}$ ($0 \leq j \leq 9$, $0 \leq k \leq 7$) in the numerator are given by the following matrix.

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</table>
Table A.2. $\sum_{j,k=0}^{\infty} \text{SiegelHalfpsi}[j,k] s^j t^k$ is a rational function of $s$ and $t$ whose denominator is

$$(1 - s^2)^2(1 - s^3)(1 - t)(1 - t^2)^2(1 - t^3).$$

The coefficients of $s^j t^k$ ($0 \leq j \leq 9, 0 \leq k \leq 7$) in the numerator are given by the following matrix.

$$\begin{array}{cccccccc}
-3 & 0 & 6 & 6 & -6 & -21 & -11 & 3 & 6 & 2 \\
2 & 0 & -4 & -5 & 1 & 12 & 10 & 1 & -3 & -2 \\
6 & 0 & -12 & -11 & 17 & 47 & 23 & -6 & -12 & -4 \\
0 & 0 & 0 & 5 & 10 & 4 & -5 & -6 & -3 & 1 \\
-5 & 0 & 13 & 15 & -12 & -41 & -25 & -1 & 9 & 5 \\
-6 & 1 & 15 & 9 & -21 & -46 & -24 & 6 & 14 & 4 \\
3 & 2 & -6 & -12 & -3 & 13 & 14 & 6 & -2 & -3 \\
4 & 0 & -9 & -8 & 8 & 26 & 17 & 0 & -6 & -2
\end{array}$$

REFERENCES


[Ib] T. Ibukiyama, On Siegel modular forms of half integral weight of $\Gamma_0(4)$ of degree two (in preparation).


[T6] _____, Dimension formula for the spaces of Siegel cusp forms of half integral weight and degree two (in preparation).


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