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On the Dimension Formula for the Spaces of Siegel Cusp Forms of Half Integral Weight and Degree Two

Ryuji Tsushima (Meiji Univ.)

§1. Results

Let $\mathfrak{S}_g = \{Z \in M_g(\mathbf{C}) \mid {}^tZ = Z,\ \mathrm{Im}\, Z > 0\}$ be the Siegel upper half plane of degree g, $\Gamma_g = Sp\,(g, \mathbf{Z})$ the Siegel modular group of degree g and

$$\Gamma_g^* = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \;\middle|\; \text{diagonal elements of } A\,{}^tB,\ C\,{}^tD \text{ are even} \right\}.$$

If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we denote $(AZ+B)(CZ+D)^{-1}$ by $M\langle Z\rangle$. Let $\mathbf{e}(z) = \exp(2\pi i z)$ and for $Z \in \mathfrak{S}_g$ put

$$\theta(Z) = \sum_{\eta \in \mathbf{Z}^g} \mathbf{e}\left(\frac{1}{2}{}^t\eta Z \eta\right).$$

If $M \in \Gamma_g^*$, $\theta(M\langle Z\rangle)/\theta(Z)$ is holomorphic on $\mathfrak{S}_g$. Let $\alpha = \begin{pmatrix} 2\cdot 1_g & O \\ O & 1_g \end{pmatrix}$ and let $\Theta(Z) = \theta(2Z) = \theta(\alpha\langle Z\rangle)$. Let

$$\Gamma_0^g(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \;\middle|\; C \equiv O \pmod{N} \right\}.$$

Then $\alpha^{-1}\Gamma_g^*\alpha \cap \Gamma_g$ contains $\Gamma_0^g(4)$. Hence if $M \in \Gamma_0^g(4)$,

$$J(M,Z) := \Theta(M\langle Z\rangle)/\Theta(Z)$$

is holomorphic on $\mathfrak{S}_g$ and satisfies the equality:

$$J(M,Z)^2 = \det(CZ+D)\psi(\det D),$$

where $\psi : 1+2\mathbf{Z} \to \{\pm 1\}$ is the non-trivial Dirichlet character modulo 4. $J(M,Z)$ is *the automorphic factor of weight* $1/2$.

In the following we assume that $g = 2$. Let $\mathrm{Sym}^j : GL(2,\mathbf{C}) \to GL(j+1,\mathbf{C})$ be the symmetric tensor representation of degree j. $\mathrm{Sym}^j(CZ+D)$ is also an automorphic factor (with respect to Γ_2) and so is $J(M,Z)^{2k+1}\,\mathrm{Sym}^j(CZ+D)$ (with respect to $\Gamma_0^2(4)$). Let Γ be a subgroup of $\Gamma_0^2(4)$ of finite index. A holomorphic mapping $f : \mathfrak{S}_2 \to \mathbf{C}^{j+1}$ is called *a Siegel modular form of half integral weight* with respect to Γ, if f satisfies the following equality for any $M \in \Gamma$ and $Z \in \mathfrak{S}_2$:

$$f(M\langle Z\rangle) = J(M,Z)^{2k+1}\,\mathrm{Sym}^j(CZ+D)\,f(Z).$$

We denote by $M_{j,k+1/2}(\Gamma)$ the **C**-vector space of all such mappings. $f \in M_{j,k+1/2}(\Gamma)$ is called *a cusp forms* if f belongs to the kernels of the Φ-operators. We denote the space of cusp forms by $S_{j,k+1/2}(\Gamma)$. Namely, f belongs to $S_{j,k+1/2}(\Gamma)$ if and only if

$$\lim_{\operatorname{Im} z_2 \to \infty} f(M\langle Z\rangle) = 0,$$

for any $M \in \Gamma_2$, where $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$. It is known that $M_{j,k+1/2}(\Gamma)$ is finite-dimensional.

Let χ be a character of Γ whose kernel is a subgroup of Γ of finite index. We denote by $M_{j,k+1/2}(\Gamma,\chi)$ the **C**-vector space of the holomorphic mappings of $\mathfrak{S}_2$ to $\mathbf{C}^{j+1}$ which satisfy

$$f(M\langle Z\rangle) = J(M,Z)^{2k+1}\,\chi(M)\,\mathrm{Sym}^j(CZ+D)\,f(Z),$$

for any $M \in \Gamma$ and $Z \in \mathfrak{S}_2$. We also denote by $S_{j,k+1/2}(\Gamma,\chi)$ its subspace of cusp forms.

Let ψ be as before and let j be odd. Then since $-1_4 \in \Gamma_0^2(4)$ and $\mathrm{Sym}^j(-1_2) = -1_{j+1}$, $M_{j,k+1/2}(\Gamma_0^2(4))$ and $M_{j,k+1/2}(\Gamma_0^2(4),\psi)$ are $\{0\}$. Therefore we assume j is even in the following. Our main results are the following two theorems.

Theorem 1.1. *If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$, $\dim S_{2j,k+1/2}(\Gamma_0^2(4))$ is given by the following Mathematica function:*

```
SiegelHalf[j_,k_]:=Block[{a,ljk},

  mod[x_,y_]:=Mod[x,y]+1;

  a=(2*j+1)*(4*j+2*k-1)*(j+k-1)*(2*k-3)/2^5/3^2;
  a=a+(2*j+1)*If[Mod[k,2]==0,19-22*k-22*j,25-22*k-22*j]/2^6/3;
  a=a+3*(2*j+1)*If[Mod[k,2]==0,-1,1]/2^6;

  a=a+(4*j+2*k-1)*(2*k-3)/2^6;
  a=a+If[Mod[k,2]==0,17-12*k-12*j,49-20*k-20*j]/2^6;

  a=a+7*(4*j+2*k-1)*(2*k-3)/2^6/3;
  a=a+(35-48*k-48*j)/2^5/3;
  a=a-13/2^4/3;
  a=a+If[Mod[k,2]==0,7,15]/2^6;
  a=a+If[Mod[k,2]==0,2,3]/2^2;

  ljk={1,-1};
  a=a+(j+k-1)*ljk[[mod[j,2]]]/2^3;
  a=a-If[Mod[k,2]==0,3,5]*ljk[[mod[j,2]]]/2^4;
```

```
  a=a-If[Mod[k,2]==0,3,1]*ljk[[mod[j,2]]]/2^4;

  ljk={1,0,-1};
  a=a+2*ljk[[mod[j,3]]]*(j+k-1)/3^2;
  a=a-ljk[[mod[j,3]]]/2;

  ljk=(2*j+1)*{{1,0,-1},{0,-1,1},{-1,1,0}};
  a=a+ljk[[mod[j,3],mod[k,3]]]/2/3^2;

  ljk={{1,-2,1},{-2,1,1},{1,1,-2}};
  a=a+ljk[[mod[j,3],mod[k,3]]]/2/3^2;

  ljk={1,-2,1};
  a=a-ljk[[mod[j,3]]]/2/3^2;

  Return[a];

  ]
```

Theorem 1.2. *If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$,* $\dim S_{2j,k+1/2}(\Gamma_0^2(4), \psi)$ *is given by the following Mathematica function:*

```
SiegelHalfpsi[j_,k_]:=Block[{a,ljk},

  mod[x_,y_]:=Mod[x,y]+1;

  a=(2*j+1)*(4*j+2*k-1)*(j+k-1)*(2*k-3)/2^5/3^2;
  a=a+(2*j+1)*If[Mod[k,2]==0,25-22*k-22*j,19-22*k-22*j]/2^6/3;
  a=a-3*(2*j+1)*If[Mod[k,2]==0,-1,1]/2^6;

  a=a-(4*j+2*k-1)*(2*k-3)/2^6;
  a=a-If[Mod[k,2]==0,49-20*k-20*j,17-12*k-12*j]/2^6;

  a=a-7*(4*j+2*k-1)*(2*k-3)/2^6/3;
  a=a-(35-48*k-48*j)/2^5/3;
  a=a+13/2^4/3;
  a=a-If[Mod[k,2]==0,15,7]/2^6;
  a=a-If[Mod[k,2]==0,3,2]/2^2;

  ljk={1,-1};
  a=a+(j+k-1)*ljk[[mod[j,2]]]/2^3;
  a=a-If[Mod[k,2]==0,5,3]*ljk[[mod[j,2]]]/2^4;
```

```
a=a-If[Mod[k,2]==0,1,3]*ljk[[mod[j,2]]]/2^4;

ljk={1,0,-1};
a=a+2*ljk[[mod[j,3]]]*(j+k-1)/3^2;
a=a-ljk[[mod[j,3]]]/2;

ljk=(2*j+1)*{{1,0,-1},{0,-1,1},{-1,1,0}};
a=a+ljk[[mod[j,3],mod[k,3]]]/2/3^2;

ljk={{1,-2,1},{-2,1,1},{1,1,-2}};
a=a-ljk[[mod[j,3],mod[k,3]]]/2/3^2;

ljk={1,-2,1};
a=a+ljk[[mod[j,3]]]/2/3^2;

Return[a];

]
```

§2. Methods

Let $\Gamma_g(N)$ be the principal congruence subgroup of level N of Γ_g. Namely,

$$\Gamma_g(N) = \{M \in \Gamma_g \mid M \equiv 1_{2g} \pmod{N}\}.$$

This is a normal subgroup of Γ_g. If $N \geq 3$, $\Gamma_g(N)$ acts on $\mathfrak{S}_g$ without fixed points and the quotient space $X_g(N) := \Gamma_g(N)\backslash\mathfrak{S}_g$ is a (non-compact) manifold. $X_g(N)$ is a open subspace of a projective variety $\overline{X}_g(N)$ which was constructed by I. Satake (Satake compactification, [Sta]). If $g \geq 2$, $\overline{X}_g(N)$ has singularities along its cusps: $\overline{X}_g(N) - X_g(N)$. Cusps of $\overline{X}_g(N)$ is (as a set) a disjoint union of copies of $X_{g'}(N)$'s $(0 \leq g' < g)$. A desingularization $\widetilde{X}_g(N)$ of $\overline{X}_g(N)$ was constructed by J.-I. Igusa and Y. Namikawa ($g = 2$, 3, 4) ([Ig2], [N]) and more generally by D. Mumford and others (Toroidal compactification, [AMRT]).

Let $\mathcal{V}$ be $\mathfrak{S}_g \times \mathbf{C}^g$ and let $v \in \mathbf{C}^g$. $\Gamma_g(N)$ acts on $\mathcal{V}$ as follows:

$$M(Z,v) = (M\langle Z\rangle, (CZ+D)v).$$

If $N \geq 3$, $V := \Gamma_g(N)\backslash\mathcal{V}$ is non-singular and is a vector bundle over $X_g(N)$. V is extended to a vector bundle $\widetilde{V}$ over $\widetilde{X}_g(N)$. Let $\mathcal{H}_g$ be $\mathfrak{S}_g \times \mathbf{C}$ and let $v \in \mathbf{C}$. $\Gamma_g(4N)$ acts on $\mathcal{H}_g$ as follows:

$$M(Z,v) = (M\langle Z\rangle, J(M,Z)v).$$

$H_g := \Gamma_g(4N)\backslash\mathcal{H}_g$ is a line bundle over $X_g(4N)$. H_g is extended to a line bundle $\widetilde{H}_g$ over $\widetilde{X}_g(4N)$ and also to a line bundle $\overline{H}_g$ over $\overline{X}_g(4N)$.

Let Γ be a subgroup of $\Gamma_0^g(4)$ of finite index. If $g \geq 2$, Γ contains $\Gamma_g(4N)$ for some N ([BLS], [M]). In the following we assume that $g = 2$. The space of Siegel modular forms $M_{j,k+1/2}(\Gamma_2(4N))$ is canonically identified with the space

$$\Gamma(\widetilde{X}_2(4N), \mathcal{O}(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)})),$$

which is the space of the global holomorphic sections of $\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)}$. The divisor at infinity $D := \widetilde{X}_2(4N) - X_2(4N)$ is a divisor with simple normal crossings. The space of cusp forms $S_{j,k+1/2}(\Gamma_2(4N))$ is canonically identified with the space

$$\Gamma(\widetilde{X}_2(4N), \mathcal{O}(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)} - D)).$$

$\mathcal{O}(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)} - D)$ is the sheaf of germs of holomorphic sections which vanish along D and this is isomorphic to $\mathcal{O}(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)} \otimes [D]^{\otimes(-1)})$, where $[D]$ is the line bundle associated with D. We can prove the following

Theorem 2.1. *If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$, then*

$$H^p(\widetilde{X}_2(4N), \mathcal{O}(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)} \otimes [D]^{\otimes(-1)})) \simeq \{0\},$$

for $p > 0$.

By using this theorem and the theorem of Riemann-Roch-Hirzebruch we have

Theorem 2.2. *If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$,*

$$\begin{aligned} &\dim S_{j,k+1/2}(\Gamma_2(4N)) \\ =&2^3 3^{-1}(j+1)\{2(2k-3)(2j+2k-1)(j+2k-2)N^{10} - 30(j+2k-2)N^8 + 45N^7\} \\ &\times \prod_{p|N,\ p\,:\,odd\ prime} (1-p^{-2})(1-p^{-4}). \end{aligned}$$

Let Γ be a subgroup of $\Gamma_0^2(4)$ of finite index and let χ be a character of Γ whose kernel is a subgroup of Γ of finite index. We may assume that the kernel of χ contains $\Gamma_2(4N)$. Let $f \in S_{j,k+1/2}(\Gamma_2(4N))$ and $M \in \Gamma$. We define an action of M on $S_{j,k+1/2}(\Gamma_2(4N))$ as follows:

$$Mf\,(M\langle Z\rangle) = J(M,Z)^{2k+1}\chi(M)\,\mathrm{Sym}^j(CZ+D)\,f(Z).$$

Since $\Gamma_2(4N)$ acts trivially on $S_{j,k+1/2}(\Gamma_2(4N))$, this action induces an action of $\Gamma/\Gamma_2(4N)$ on $S_{j,k+1/2}(\Gamma_2(4N))$ and $S_{j,k+1/2}(\Gamma,\chi)$ is identified with the invariant subspace of $S_{j,k+1/2}(\Gamma_2(4N))$. Thus we have

$$S_{j,k+1/2}(\Gamma,\chi) = S_{j,k+1/2}(\Gamma_2(4N))^{\Gamma/\Gamma_2(4N)}.$$

Therefore $\dim S_{j,k+1/2}(\Gamma, \chi)$ is computed by using the holomorphic Lefschetz fixed point formula ([AS]).

To use the Lefschetz fixed point formula we have to classify the fixed points (sets). Let $N \geq 3$. Γ_2 and $\Gamma_2/\Gamma_2(N)$ act on $\widetilde{X}_2(N)$. We classify (the irreducible components of) the fixed points of Γ_2 in the following sense. Let Φ_1 and Φ_2 be the fixed points (sets). Φ_1 and Φ_2 is called *equivalent* if there is an element of Γ_2 which maps Φ_1 biholomorphically to Φ_2. The fixed points in the quotient space $X_2(N)$ were classified in [G]. The fixed points in the divisor at infinity are classified easily. In total there are 25 kinds of fixed points (sets). Among them 10 fixed points are not fixed by the elements of $\Gamma_0^2(4)$. But since the automorphic factor $J(M, Z)$ is defined with respect to $\Gamma_0^2(4)$, we have to classify the remaining 15 fixed points with respect to $\Gamma_0^2(4)$.

Let Φ be one of 15 fixed points and let

$$C(\Phi) = \{M \in \Gamma_2 \mid M \langle Z \rangle = Z \text{ for any } Z \in \Phi\},$$

$$C^p(\Phi) = \{M \in C(\Phi) \mid \Phi \text{ is closed in } \mathrm{Fix}(M)\},$$

$$N(\Phi) = \{M \in \Gamma_2 \mid M \text{ maps } \Phi \text{ into } \Phi\}.$$

What we have to do is to classify the double cosets $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi)$. Let $P_1, P_2, \ldots, P_n$ be the representatives of $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi)$. Next we have to check $P_i\, C^p(\Phi) P_i^{-1} \cap \Gamma_0^2(4)$ $(i = 1, 2, \ldots, n)$ is empty or not. Since Γ_2 is an infinite group, it is not an easy task to classify $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi)$. But since $\Gamma_0^2(4)$ contains $\Gamma_2(4)$, we can take the quotient by $\Gamma_2(4)$ and reduce the problem to a task in the finite group $\Gamma_2/\Gamma_2(4) \simeq Sp(2, \mathbf{Z}/4\mathbf{Z})$ and we can use the computer. We list the result in the following proposition. As to the notations of the fixed points (sets), see [T2]. Let ρ be $\exp(2\pi i/3)$.

Proposition 2.3. *For each Φ the number of the elements of $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi)$ and the number of the double cosets such that $P_i\, C^p(\Phi) P_i^{-1} \cap \Gamma_0^2(4) \neq \phi$ is as follows.*

$\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$	1	1	$\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$	3	2	$\begin{pmatrix} z_1 & 1/2 \\ 1/2 & z_2 \end{pmatrix}$	5	3
$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$	11	2	$\begin{pmatrix} z & 1/2 \\ 1/2 & z \end{pmatrix}$	8	2	$\begin{pmatrix} z & z/2 \\ z/2 & z \end{pmatrix}$	6	1
$\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$	10	1	$\frac{\sqrt{-3}}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	24	2	$\begin{pmatrix} z_1 & z_2 \\ z_2 & \infty \end{pmatrix}$	4	4
$\begin{pmatrix} z & 0 \\ 0 & \infty \end{pmatrix}$	7	6	$\begin{pmatrix} z & 1/2 \\ 1/2 & \infty \end{pmatrix}$	10	7	$\begin{pmatrix} \infty & z \\ z & \infty \end{pmatrix}$	12	7
$\begin{pmatrix} \infty & 0 \\ 0 & \infty \end{pmatrix}$	15	13	$\begin{pmatrix} \infty & 1/2 \\ 1/2 & \infty \end{pmatrix}$	13	9	$\begin{pmatrix} \infty & \infty \\ \infty & \infty \end{pmatrix}$	8	8

Therefore there are 68 kinds of fixed points of $\Gamma_0^2(4)$ in total. By computing the contributions of these fixed points to the dimension of

$$S_{2j,k+1/2}(\Gamma_0^2(4)) = S_{2j,k+1/2}(\Gamma_2(4N))^{\Gamma_0^2(4)/\Gamma_2(4N)},$$

we can calculate $\dim S_{2j,k+1/2}(\Gamma_0^2(4))$ and similarly $\dim S_{2j,k+1/2}(\Gamma_0^2(4), \psi)$.

In this note I explain nothing about the computation of the theorem of Riemenn-Roch-Hirzebruch or the Lefschetz fixed point formula. As to the former, see [Y], [T4] and [T1]. As to the latter, see [T2].

§3. The case $j = 0$

In case $j = 0$, we denote the space $M_{0,k+1/2}(\Gamma_0^2(4))$ and $S_{0,k+1/2}(\Gamma_0^2(4))$ by $M_{k+1/2}(\Gamma_0^2(4))$ and $S_{k+1/2}(\Gamma_0^2(4))$, respectively. From Theorem 1.1 we have

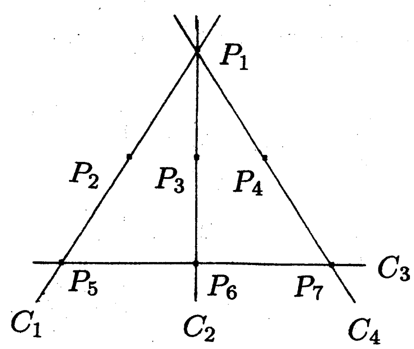
Proposition 3.1.

$$\begin{aligned}\sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^2(4))\, t^k &= \sum_{k=0}^{\infty} \texttt{SiegelHalf[0,k]}\, t^k + t^2 \\ &= \frac{2t^5 + 2t^6 - t^7 - 2t^8 - t^9 + t^{10}}{(1-t)(1-t^2)^2(1-t^3)}.\end{aligned}$$

Proof. If $f(Z) \in S_{k+1/2}(\Gamma_0^2(4))$, then $f(Z)\Theta(Z)^2 \in S_{k+3/2}(\Gamma_0^2(4))$. Since $\dim S_{7/2}(\Gamma_0^2(4))$ is equal to `SiegelHalf[0,3]` $= 0$, we have $S_{5/2}(\Gamma_0^2(4)) \simeq S_{3/2}(\Gamma_0^2(4)) \simeq S_{1/2}(\Gamma_0^2(4)) \simeq \{0\}$. But since `SiegelHalf[0,2]` $= -1$, `SiegelHalf[0,1]` $= 0$ and `SiegelHalf[0,0]` $= 0$, we have the equality of the first line. □

The cusps of the Satake compactification $\overline{\Gamma_0^2(4)\backslash\mathfrak{S}_2}$ of $\Gamma_0^2(4)\backslash\mathfrak{S}_2$ consists of 4 one-dimensional cusps and 7 zero-dimensional cusps. Each one-dimensional cusp is biholomorphic to $\overline{\Gamma_0^1(4)\backslash\mathfrak{S}_1}$.

Cusps of $\overline{\Gamma_0^2(4)\backslash\mathfrak{S}_2}$:



Let $\Phi = \left\{\begin{pmatrix} z_1 & z_2 \\ z_2 & \infty \end{pmatrix}\right\}$. $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi)$ consists of 4 double cosets. Let $M_1 = 1_4$ and let

$$M_2 = \begin{pmatrix} O & 1_2 \\ -1_2 & O \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&2&0&1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&2&1&0\\2&0&0&1 \end{pmatrix}, \quad g_n = \begin{pmatrix} 1&0&0&0\\0&1&0&n\\0&0&1&0\\0&0&0&1 \end{pmatrix}.$$

M_1, M_2, M_3 and M_4 are the representatives of $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi)$. Let C_i be the one-dimensional cusp corresponding to the double coset $\Gamma_0^2(4)M_iN(\Phi)$ $(i = 1,2,3,4)$, respectively. Put $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$. Let $i = 1$ or 4. Then we have

$$\lim_{\operatorname{Im} z_2\to\infty} J(M_i\, g_n M_i^{-1}, M_i\langle Z\rangle) = 1,$$

for any integer n. $M_2\, g_n M_2^{-1}$ belongs to $\Gamma_0^2(4)$ if and only if $4 \mid n$ and we have

$$\lim_{\operatorname{Im} z_2\to\infty} J(M_2\, g_{4n} M_2^{-1}, M_2\langle Z\rangle) = 1,$$

for any integer n. On the other hand we have

$$\lim_{\operatorname{Im} z_2\to\infty} J(M_3\, g_n M_3^{-1}, M_3\langle Z\rangle) = i^n,$$

where $i = \sqrt{-1}$. Hence if $f \in M(\Gamma_0^2(4))$, we have

$$\begin{aligned}
\lim_{\operatorname{Im} z_2\to\infty} f(M_3\langle Z\rangle) &= \lim_{\operatorname{Im} z_2\to\infty} f(M_3\langle g_n\langle Z\rangle\rangle) \\
&= \lim_{\operatorname{Im} z_2\to\infty} f((M_3\, g_n M_3^{-1})M_3\langle Z\rangle) \\
&= \lim_{\operatorname{Im} z_2\to\infty} J(M_3\, g_n M_3^{-1}, M_3\langle Z\rangle) f(M_3\langle Z\rangle) \\
&= i^n \lim_{\operatorname{Im} z_2\to\infty} f(M_3\langle Z\rangle).
\end{aligned}$$

Therefore $\lim_{\operatorname{Im} z_2\to\infty} f(M_3\langle Z\rangle)$ is identically 0. Namely, the Φ-operators to the one-dimensional cusp C_3 and to the zero-dimensional cusps P_5, P_6 and P_7 are 0-maps. From this we have

Proposition 3.2.

$$\begin{aligned}
&\sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma_0^2(4))\, t^k \\
&= \sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^2(4))\, t^k + 3\sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^1(4))\, t^k + 4\sum_{k=0}^{\infty} t^k - (3+3t+t^2) \\
&= \frac{2t^5+2t^6-t^7-2t^8-t^9+t^{10}}{(1-t)(1-t^2)^2(1-t^3)} + \frac{3(t^4+t^5)}{(1-t^2)^2} + \frac{4}{(1-t)} - (3+3t+t^2) \\
&= \frac{1}{(1-t)(1-t^2)^2(1-t^3)} = \frac{1+t+t^3+t^4}{(1-t^2)^3(1-t^6)}.
\end{aligned}$$

Proof. In general the Eisenstein series of Klingen type of degree n attached to a cusp form of degree r and weight k converges if $k > n+r+1$ ([K]). In case k is a half integer, this is also proved similarly as in the case of integral weight. Hence Φ-operators to the one-dimensional cusps C_1, C_2 and C_4 are surjective ($\dim S_{k+1/2}(\Gamma_0^1(4)) = 0$, if $k \le 3$). Φ-operators to the zero-dimensional cusps P_i $(i = 1,2,3,4)$ are surjective if $k \ge 3$. Hence the assertion was proved for $k \ge 3$. We can prove $\dim M_{1/2}(\Gamma_0^2(4)) = 1$, $\dim M_{3/2}(\Gamma_0^2(4)) = 1$ and $\dim M_{5/2}(\Gamma_0^2(4)) = 3$ by using the knowledge of the cases of higher weights ([T6]). So we have the proposition. □

Proposition 3.3.

$$M_{k+1/2}(\Gamma_0^2(4), \psi) = S_{k+1/2}(\Gamma_0^2(4), \psi).$$

Proof. Let $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$ and $f \in M_{k+1/2}(\Gamma_0^2(4), \psi)$. We have to prove that

$$(*) \qquad \lim_{\operatorname{Im} z_2 \to \infty} f(M\langle Z\rangle) = 0$$

for any $M \in \Gamma_2$. Let M_i $(i = 1, 2, 3, 4)$ be as before and let

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

To prove the assertion, it suffices to prove $(*)$ for M_1, M_2, M_3 and M_4. From $P\langle Z\rangle = Z$, we have

$$M\langle Z\rangle = MP\langle Z\rangle = (MPM^{-1})M\langle Z\rangle.$$

Since $M_i P M_i^{-1} = P$ for $i = 1$, 2 and 3, we have

$$\begin{aligned} f(M_i\langle Z\rangle) &= J(P, M_i\langle Z\rangle)^{2k+1}\,\psi(-1)\, f(M_i\langle Z\rangle) \\ &= -f(M_i\langle Z\rangle). \end{aligned}$$

Hence $f(M_i\langle Z\rangle) = 0$. Next let $i = 4$. Then we have

$$M_4 P M_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 4 & 0 & 0 & -1 \end{pmatrix}$$

and $J(M_4 P M_4^{-1}, M_4\langle Z\rangle) = 1$. Therefore similarly as above we have $f(M_4\langle Z\rangle) = 0$. □

Remark 3.4. Note that $f(M_i\langle Z\rangle)$ is identically zero before $\operatorname{Im} z_2$ goes to ∞. So it may be natural to ask that for any $M \in \Gamma_2$, $f(M\langle Z\rangle)$ is identically zero or not. But this is not true in general. Let Φ be $\left\{\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right\}$ and let

$$M_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

$\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi)$ consists of 3 double cosets. Their representatives are M_1, M_4 and M_5.

$$M_5 P M_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}$$

does not belong to $\Gamma_0^2(4)$ but belongs to $\alpha^{-1}\Gamma_2^*\alpha \cap \Gamma_2$ and satisfies $J(M_5 P M_5^{-1}, M_5\langle Z\rangle) = 1$. Therefore if $f(Z) \in S_{k+1/2}(\alpha^{-1}\Gamma_2^*\alpha \cap \Gamma_2, \psi)$, it holds that $f(M\langle Z\rangle) = 0$ for any $M \in \Gamma_2$ and $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$. ($\psi$ is extended to a character of $\alpha^{-1}\Gamma_2^*\alpha \cap \Gamma_2$.)

Proposition 3.5.

$$\sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma_0^2(4),\psi)\, t^k = \sum_{k=0}^{\infty} \texttt{SiegelHalfpsi[0,k]}\, t^k + (3+t+t^2)$$

$$= \frac{t^{10}}{(1-t)(1-t^2)^2(1-t^3)} = \frac{t^{10}(1+t+t^3+t^4)}{(1-t^2)^3(1-t^6)}.$$

Proof. Since we have $\dim S_{7/2}(\Gamma_0^2(4),\psi) = \texttt{SiegelHalfpsi[0,3]} = 0$, it follows that $S_{5/2}(\Gamma_0^2(4),\psi) \simeq S_{3/2}(\Gamma_0^2(4),\psi) \simeq S_{1/2}(\Gamma_0^2(4),\psi) \simeq \{0\}$. On the other hand since we have $\texttt{SiegelHalfpsi[0,2]} = -1$, $\texttt{SiegelHalfpsi[0,1]} = -1$ and $\texttt{SiegelHalfpsi[0,0]} = -3$, we have the equality of the first line. □

Let $M(\Gamma_0^2(4))$, $M(\Gamma_0^2(4),\psi)$ and $A(\Gamma_0^2(4),\psi)$ be $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4))$, $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4),\psi)$ and $\bigoplus_{k=0}^{\infty} M_k(\Gamma_0^2(4),\psi^k)$, respectively. Then $A(\Gamma_0^2(4),\psi)$ is a graded ring and since it holds $J(M,Z)^2 = \det(CZ+D)\psi(\det D)$, $M(\Gamma_0^2(4))$ and $M(\Gamma_0^2(4),\psi)$ are $A(\Gamma_0^2(4),\psi)$-modules. From the result of J.-I. Igusa ([Ig1]), we have the following proposition. (We can also prove them by dimension formula.)

Proposition 3.6.

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^2(4))\, t^k = \frac{1+t^4+t^{11}+t^{15}}{(1-t^2)^3(1-t^6)},$$

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^2(4),\psi)\, t^k = \frac{t+t^3+t^{12}+t^{14}}{(1-t^2)^3(1-t^6)},$$

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^2(4),\psi^k)\, t^k = \frac{1+t+t^3+t^4}{(1-t^2)^3(1-t^6)}.$$

From this we have

Corollary 3.7. *$M(\Gamma_0^2(4))$ and $M(\Gamma_0^2(4),\psi)$ are free $A(\Gamma_0^2(4),\psi)$-modules of rank* 1.

The generator of $M(\Gamma_0^2(4))$ is $\Theta(Z)$. Let $f_{21/2}(Z)$ be the generator of $M(\Gamma_0^2(4),\psi)$. Then $f_{21/2}(Z)\Theta(Z)$ is an automorphic form with respect to $J(M,Z)^{22}\psi(\det D) = \det(CZ+D)^{11}$. Hence this belongs to $M_{11}(\Gamma_0^2(4))$. Let $f_{11}(Z)$ be the base of $M_{11}(\Gamma_0^2(4))$ ($\dim M_{11}(\Gamma_0^2(4)) = 1$). Then $f_{11}(Z)/\Theta(Z)$ is holomorphic and we can assume that $f_{21/2}(Z) = f_{11}(Z)/\Theta(Z)$. Since $A(\Gamma_0^2(4),\psi)$ is contained in $\bigoplus_{k=0}^{\infty} M_k(\Gamma_2(4))$ and $\bigoplus_{k=0}^{\infty} M_k(\Gamma_2(4))$ is contained in the ring of theta constants ([Ig1]), every elements of $M(\Gamma_0^2(4))$ and $M(\Gamma_0^2(4),\psi)$ are representable by theta constants.

Remark 3.8. T. Ibukiyama represented the generators of $A(\Gamma_0^2(4),\psi)$ and $f_{21/2}(Z)$ explicitly by theta constants ([Ib]). Especially $A(\Gamma_0^2(4),\psi)$ is generated by algebraically independent modular forms f_1, X, g_2 and f_3 whose weights are 1, 2, 2 and 3, respectively. $f_{21/2}(Z)$ is divisible by 9 theta constants. Let $Z \in \mathfrak{S}_2$. Then there exists $M \in \Gamma_2$ such that $M\langle Z\rangle = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$, if and only if one of 10 theta constants vanishes at Z (J.-I. Igusa, [H]). Hence $f_{21/2}(Z) \notin S_{21/2}(\alpha^{-1}\Gamma_2^*\alpha \cap \Gamma_2, \psi)$.

§4. The case $j = 2$

If $j > 0$, the Φ-operator to one-dimensional cusp maps $M_{2j,k+1}(\Gamma_0^2(4))$ to $S_{2j+k+1/2}(\Gamma_0^1(4))$ and the Φ-operators to zero-dimensional cusps are 0-maps. Let C_i $(i = 1, 2, 3, 4)$ be as before. The following proposition for the case of integral weight was proved in [A]. The case of half integral weight can be similarly proved.

Proposition 4.1. *If $k \geq 4$, the Φ-operator to C_i $(i = 1, 2, 4)$*

$$\Phi : M_{2j,k+1/2}(\Gamma_0^2(4)) \to S_{2j+k+1/2}(\Gamma_0^1(4))$$

is surjective.

For two series $\sum a_k t^k$ and $\sum b_k t^k$ we write

$$\sum a_k t^k \equiv \sum b_k t^k \quad (k \geq m),$$

if $a_k = b_k$ for any $k \geq m$. From Theorem 1.1 and the above proposition we have

Proposition 4.2.

$$\begin{aligned}\sum_{k=0}^{\infty} \dim S_{2,k+1/2}(\Gamma_0^2(4))\, t^k &\equiv \sum_{k=0}^{\infty} \texttt{SiegelHalf[1,k]}\, t^k \quad (k \geq 4)\\ &= \frac{-t^2 + t^3 + 3t^4 + 3t^5 - 3t^7}{(1-t)(1-t^2)^2(1-t^3)},\\ \sum_{k=0}^{\infty} \dim M_{2,k+1/2}(\Gamma_0^2(4))\, t^k &\equiv \frac{-t^2 + t^3 + 3t^4 + 3t^5 - 3t^7}{(1-t)(1-t^2)^2(1-t^3)} + 3\frac{(t^2+t^3)}{(1-t^2)^2} \quad (k \geq 4)\\ &= \frac{2t^2 + t^3}{(1-t)(1-t^2)^2(1-t^3)}.\end{aligned}$$

We study the structure of the $A(\Gamma_0^2(4), \psi)$-module $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4))$ by a similar method in [Sto] where T. Satoh studied the the space of vector valued modular forms of integral weight with respect to Γ_2.

Let V be $\{S \in M_2(\mathbf{C}) \mid {}^tS = S\}$. We define the action of $M \in GL(2, \mathbf{C})$ on V by $S \mapsto MS\,{}^tM$. This action defines a representation of $GL(2, \mathbf{C})$ which is equivalent to Sym^2. Let F be a C^∞-function on $\mathfrak{S}_2$ and let

$$\Delta F = \begin{pmatrix} \dfrac{\partial F}{\partial Z_{11}} & \dfrac{1}{2}\dfrac{\partial F}{\partial Z_{12}} \\ \dfrac{1}{2}\dfrac{\partial F}{\partial Z_{12}} & \dfrac{\partial F}{\partial Z_{22}} \end{pmatrix}.$$

If $M \in \Gamma_2$, it holds that

$$(CZ+D)\Delta(F(M\langle Z\rangle))\,{}^t(CZ+D) = (\Delta F)(M\langle Z\rangle).$$

Hence if F satisfies $F(M\langle Z\rangle) = F(Z)$, we have

$$(\Delta F)(M\langle Z\rangle) = (CZ+D)\Delta(F(Z))\,{}^t(CZ+D).$$

Let $f \in M_k(\Gamma_0^2(4), \psi^k)$ and $g \in M_{\ell+1/2}(\Gamma_0^2(4))$. Then $g^{2k}/f^{2\ell+1}$ is a (meromorphic) modular form of weight 0. Therefore $\Delta(g^{2k}/f^{2\ell+1})$ is a (meromorphic) modular form with respect to Sym^2. $f^{2\ell+2}/g^{2k-1}$ is a (meromorphic) modular form of weight $k+\ell+1/2$. Hence

$$\begin{aligned}[f,g] &:= \frac{1}{k(2\ell+1)}(f^{2\ell+2}/g^{2k-1})\Delta(g^{2k}/f^{2\ell+1}) \\ &= \frac{1}{\ell+1/2} f\Delta g - \frac{1}{k} g\Delta f\end{aligned}$$

becomes a holomorphic modular form and belongs to $M_{2,k+\ell+1/2}(\Gamma_0^2(4))$. In general we have

Proposition 4.3. *Let $f \in M_k(\Gamma_0^2(4), \psi^{k+\alpha})$ and $g \in M_{\ell+1/2}(\Gamma_0^2(4), \psi^\beta)$. Then*

$$[f,g] = \frac{1}{\ell+1/2} f\Delta g - \frac{1}{k} g\Delta f$$

belongs to $M_{2,k+\ell+1/2}(\Gamma_0^2(4), \psi^{\alpha+\beta})$.

From this we have

Theorem 4.4. $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4))$ *is a free $A(\Gamma_0^2(4), \psi)$-module of rank 3 and the generators are $[X,\Theta]$, $[g_2,\Theta]$ and $[f_3,\Theta]$.*

Proof. Let h_1, $h_2 \in M_{k-2}(\Gamma_0^2(4), \psi^{k-2})$ and $h_3 \in M_{k-3}(\Gamma_0^2(4), \psi^{k-3})$. Assume that

$$h_1[X,\Theta] + h_2[g_2,\Theta] + h_3[f_3,\Theta]$$

is identically zero. We may assume that h_1, h_2 or h_3 is not divisible by $f_1 = \Theta^2$. Then we have

$$(*) \qquad 2(h_1X + h_2g_2 + h_3f_3)\Delta(\Theta) = \Theta\left(\frac{1}{2}h_1\Delta(X) + \frac{1}{2}h_2\Delta(g_2) + \frac{1}{3}h_3\Delta(f_3)\right).$$

Let the quotient of h_i by f_1 be q_i and the remainder r_i $(i=1,2,3)$. Assume that $r_1X + r_2g_2 + r_3f_3$ is identically 0[1]. Then we have

$$\begin{aligned}&2\Theta(q_1X + q_2g_2 + q_3f_3)\Delta(\Theta) \\ &= \left(\frac{1}{2}r_1\Delta(X) + \frac{1}{2}r_2\Delta(g_2) + \frac{1}{3}r_3\Delta(f_3)\right) + f_1\left(\frac{1}{2}q_1\Delta(X) + \frac{1}{2}q_2\Delta(g_2) + \frac{1}{3}q_3\Delta(f_3)\right).\end{aligned}$$

So $\frac{1}{2}r_1\Delta(X) + \frac{1}{2}r_2\Delta(g_2) + \frac{1}{3}r_3\Delta(f_3)$ is identically 0 on $H_\Theta := \{Z \in \mathfrak{S}_2 \mid \Theta(Z) = 0\}$. Therefore we have

[1]In the talk at RIMS, I said that $h_1X + h_2g_2 + h_3f_3$ is not divisible by f_1, But this was false.

$$\begin{pmatrix} \frac{\partial X}{\partial Z_{11}} & \frac{\partial g_2}{\partial Z_{11}} & \frac{\partial f_3}{\partial Z_{11}} \\ \frac{\partial X}{\partial Z_{12}} & \frac{\partial g_2}{\partial Z_{12}} & \frac{\partial f_3}{\partial Z_{12}} \\ \frac{\partial X}{\partial Z_{22}} & \frac{\partial g_2}{\partial Z_{22}} & \frac{\partial f_3}{\partial Z_{22}} \end{pmatrix} \begin{pmatrix} \frac{r_1}{2} \\ \frac{r_2}{2} \\ \frac{r_3}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

on H_Θ. But we can show that the determinant $D(Z)$ of the matrix in the left-hand side of the above equation is not divisible by $\Theta(Z)$ as follows. Let

$$M = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then from the transformation formula of theta constants we have

$$\begin{aligned} \Theta\left(M\left\langle\begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix}\right\rangle\right) &= \theta\left(M\left\langle\begin{pmatrix} 2Z_{11} & 0 \\ 0 & 2Z_{22} \end{pmatrix}\right\rangle\right) \\ &= \theta_{0000}\left(M\left\langle\begin{pmatrix} 2Z_{11} & 0 \\ 0 & 2Z_{22} \end{pmatrix}\right\rangle\right) \\ &= \kappa(M)\mathbf{e}(\phi_{1111}(M))\det(2CZ+D)^{1/2}\theta_{1111}\begin{pmatrix} 2Z_{11} & 0 \\ 0 & 2Z_{22} \end{pmatrix} \\ &= 0, \end{aligned}$$

where $\kappa(M)$ and $\mathbf{e}(\phi_{1111}(M))$ are eighth root of unity and θ_{0000} and θ_{1111} are theta constants of characteristic ${}^t(0,0,0,0)$ and ${}^t(1,1,1,1)$, respectively.

Since X, g_2 and f_3 are represented by theta constants, we can prove that $D(M\langle Z\rangle)$ is not divisible by Z_{12} from the transformation formula of theta constants and explicit Fourier expansions of theta constants ([T7]). Hence r_i $(i = 1,2,3)$ is identically 0 on H_Θ. This contradicts to the assumption that h_1, h_2 or h_3 is not divisible by f_1. Therefore $h_1X + h_2g_2 + h_3f_3$ in $(*)$ is not divisible by Θ. On the other hand, $\Delta(\Theta)$ in $(*)$ is also not divisible by Θ. Otherwise all of the points in H_Θ are singular points of H_Θ. These facts contradict to the assumption that $h_1[X,\Theta]+h_2[g_2,\Theta]+h_3[f_3,\Theta]$ is identically zero.

From Proposition 4.2 theorem was proved for $k \geq 4$. The case $k \leq 3$ is easily proved from the result of the case $k \geq 4$. □

Remark 4.5. If $f \in M_k(\Gamma_0^2(4), \psi^{k+1})$ and $g \in M_{\ell+1/2}(\Gamma_0^2(4), \psi)$, then $[f,g] \in M_{2,k+\ell+1/2}(\Gamma_0^2(4))$. Where is this part? $\bigoplus_{k=0}^{\infty} M_k(\Gamma_0^2(4), \psi^{k+1})$ is a free $A(\Gamma_0^2(4), \psi)$-module of rank 1 and the generator is f_{11}. Since

$$[f_{11}, f_{21/2}] = -\frac{1}{22}[f_{21/2}^2, \Theta],$$

this part is already contained in $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4))$.

Similarly as before we have

Proposition 4.6.

$$\sum_{k=0}^{\infty} \dim M_{2,k+1/2}(\Gamma_0^2(4),\psi)\, t^k = \sum_{k=0}^{\infty} \dim S_{2,k+1/2}(\Gamma_0^2(4),\psi)\, t^k = \frac{t^5+2t^6}{(1-t)(1-t^2)^2(1-t^3)}.$$

From this we present

Conjecture 4.7. $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4),\psi)$ is a free $A(\Gamma_0^2(4),\psi)$-module of rank 3.

Remark 4.8. The form of type $[f,g]$ in $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4),\psi)$ of the lowest weight is

$$[f_{11},\Theta] = -\frac{21}{22}[\Theta^2, f_{21/2}].$$

Hence $M_{2,k+1/2}(\Gamma_0^2(4),\psi)$ is not spanned by the forms of this type. T. Satoh proved that the space $M_{2,2k}(\Gamma_2)$ is spanned by the forms of the above type but the space $M_{2,2k+1}(\Gamma_2)$ is not spanned by the forms of the above type in [Sto] using the dimension formula ([T3]). This is natural since $\Theta M_{2,2k}(\Gamma_2) \subset M_{2,2k+1/2}(\Gamma_0^2(4))$ and $\Theta M_{2,2k+1}(\Gamma_2) \subset M_{2,2k+3/2}(\Gamma_0^2(4),\psi)$.

So we would like to present

Problem 4.9. Find the generators of the module $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4),\psi)$.

§5. The case of general level

For example we can compute $\dim S_{2j,k+1/2}(\Gamma_0^2(4p),\chi)$ (p : odd prime). This has been already reduced to a routine work (cf. [T5] for the case of integral weight) but will be a hard job.

Appendix

We list here the generating functions of `SiegelHalf[j,k]` and `SiegelHalfpsi[j,k]`.

Table A.1. $\sum_{j,k=0}^{\infty}$ `SiegelHalf[j,k]`$s^j t^k$ *is a rational function of* s *and* t *whose denominator is*

$$(1-s^2)^2(1-s^3)^2(1-t)(1-t^2)^2(1-t^3).$$

The coefficients of $s^j t^k$ $(0 \le j \le 9,\ 0 \le k \le 7)$ *in the numerator are given by the following matrix.*

0	0	−3	−6	−6	−3	4	3	−3	−4
0	0	1	1	1	3	3	1	1	1
−1	−1	7	17	20	8	−12	−8	8	10
1	1	2	7	7	−2	−9	−4	1	2
2	3	−2	−12	−20	−9	8	4	−8	−8
1	3	−5	−21	−23	−5	12	6	−7	−9
0	0	−1	−1	2	2	1	3	4	2
−2	−3	4	14	13	0	−8	−2	7	7

Table A.2. $\sum_{j,k=0}^{\infty}$ `SiegelHalfpsi[j,k]`$s^j t^k$ *is a rational function of* s *and* t *whose denominator is*

$$(1-s^2)^2(1-s^3)^2(1-t)(1-t^2)^2(1-t^3).$$

The coefficients of $s^j t^k$ $(0 \le j \le 9,\ 0 \le k \le 7)$ *in the numerator are given by the following matrix.*

−3	0	6	6	−6	−21	−11	3	6	2
2	0	−4	−5	1	12	10	1	−3	−2
6	0	−12	−11	17	47	23	−6	−12	−4
0	0	0	5	10	4	−5	−6	−3	1
−5	0	13	15	−12	−41	−25	−1	9	5
−6	1	15	9	−21	−46	−24	6	14	4
3	2	−6	−12	−3	13	14	6	−2	−3
4	0	−9	−8	8	26	17	0	−6	−2

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e-mail address: `tsushima@math.meiji.ac.jp`