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A BASIS ON THE SPACE OF WHITTAKER FUNCTIONS FOR THE REPRESENTATIONS OF THE DISCRETE SERIES - THE CASE OF $Sp(2; \mathbb{R})$ -

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We investigate Whittaker functions of the discrete series of the real symplectic group $Sp(2; \mathbb{R})$. We determine a basis on the space of Whittaker functions and find integral expressions of their functions by classical special functions.

1. POWER SERIES SOLUTION

We consider the following system of differential equations for $\kappa_1, \kappa_2, \mu, \nu$ in $\mathbb{C}$:

\begin{align}
(1.1) \quad \{\partial_1 \partial_2 - \kappa_1 (a_1/a_2)^2\} \phi(a_1, a_2) &= 0, \\
(1.2) \quad \{(\partial_1 + \partial_2)^2 + 2\mu(\partial_1 + \partial_2) + \mu^2 - \nu^2 + 2\kappa_2 a_2^2 \partial_2\} \phi(a_1, a_2) &= 0.
\end{align}

This system has power series solutions for $(a_1, a_2)$ in a neighborhood of the origin. For $\rho_1, \rho_2$ in $\mathbb{C}$, we define the formal power series $\phi_{\rho_1, \rho_2}(a_1, a_2)$ by

\begin{equation}
\phi_{\rho_1, \rho_2}(a_1, a_2) = \left(\frac{a_1}{a_2}\right)^{\rho_1} a_2^{\rho_2} \sum_{m, n=0}^{\infty} c_{m, n} \left(\frac{a_1}{a_2}\right)^m a_2^n,
\end{equation}

We assume $c_{0,0} \neq 0$ and $\phi_{\rho_1, \rho_2}$ satisfies the system (1.1), (1.2). Then we have the following result:

**Proposition 1.1.** We put for any fixed $c \neq 0$ in $\mathbb{C}$,

\begin{equation}
c_{m, n} = \begin{cases} 
0, & \text{if } m \text{ or } n \text{ is odd,} \\
\frac{1}{\Gamma \left(\frac{\rho_1 + \rho_2}{2} + k + 1\right)} \frac{1}{\Gamma \left(\frac{\rho_1 + \rho_2}{2} + k - l + 1\right)} \frac{1}{\Gamma \left(\frac{\rho_1 + \mu + \nu}{2} + l + 1\right)} \frac{1}{\Gamma \left(\frac{\rho_2 + \mu + \nu}{2} + l + 1\right)}, & \text{if } (m, n) = (2k, 2l) \in 2\mathbb{Z} \times 2\mathbb{Z}.
\end{cases}
\end{equation}

Then for each $(\rho_1, \rho_2)$ in $\{(0, -\mu \pm \nu), (-\mu \pm \nu, -\mu \pm \nu)\}$, $\phi_{\rho_1, \rho_2}$ given in (1.3) is absolutely convergent for any $\kappa_1, \kappa_2, \mu, \nu$ in $\mathbb{C}$, in all $(a_1/a_2, a_2^2)$ in $\mathbb{C} \times \mathbb{C}$, and a solution of the system (1.1), (1.2).
Here if $\kappa_1 = 0$ (resp. $\kappa_2 = 0$), we put $\kappa_1^0$ (resp. $\kappa_2^0$) = 1.

For $(\kappa_1, \kappa_2)$ in $\mathbb{C}^2$ such that $\kappa_1 \kappa_2 = 0$, Proposition (1.1) means the following result:

**Corollary 1.1.** The system of differential equations (1.1), (1.2) has the following four solutions $f_{i,j}$ ($i, j = 0, 1$) for three cases:

1. if $\kappa_1 = \kappa_2 = 0$, $f_{i,j}(a_1, a_2) = \left( \frac{a_1}{a_2} \right)^i (-\mu + (-1)^j \nu + \frac{1}{2}(\mu + (-1)^j \nu - k) \left( \frac{\sqrt{-\kappa_1 a_1}}{a_2} \right)$,

2. if $\kappa_1 = 0$ and $\kappa_2 \neq 0$, $f_{i,j}(a_1, a_2) = \left( \frac{a_1}{a_2} \right)^i (-\mu + (-1)^j \nu) \left( \frac{2\sqrt{\kappa_2 a_2}}{2} \right)$,

3. if $\kappa_1 \neq 0$ and $\kappa_2 = 0$,

   \[ f_{i,j}(a_1, a_2) = (a_1 a_2)^{\frac{1}{2}} (-\mu + (-1)^j \nu) I_{\nu}(2\sqrt{\kappa_2 a_2}) \]

where we denote by $I_{\nu}(z)$ the modified Bessel function:

\[ I_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)}, \quad \text{for } |\arg(z)| < \pi. \]

For the case $\kappa_1 \kappa_2 \neq 0$ , we have the following expressions of the power series solutions $\phi_{\rho_1, \rho_2}$:

**Definition 1.1.** We define for $i, j = 0, 1$, $|\arg \left( \sqrt{-\kappa_1 a_1} \right) | < \pi$,

\[ f_{i,j}(a_1, a_2) = \frac{2\pi \sqrt{-1}}{4^\mu} \sum_{k=0}^{\infty} \frac{(\sqrt{-\kappa_1} a_1 a_2 / 2)^{\frac{1}{2}(-\mu + (-1)^j \nu + k)}}{k! \Gamma((-1)^j \nu + k + 1)} I_{\nu}(2\sqrt{\kappa_2 a_2}) \left( \frac{\sqrt{-\kappa_1 a_1}}{a_2} \right), \]

and for each $(\rho_1, \rho_2) \in \{(0, -\mu \pm \nu), (-\mu \pm \nu, -\mu \pm \nu)\}$,

\[ \tilde{\phi}_{\rho_1, \rho_2} = \frac{2\pi \sqrt{-1}}{4^\mu} \left( \frac{-\kappa_1}{4} \right)^{\frac{1}{2}} \frac{\rho_1^{\frac{1}{2}} \rho_2^{\frac{1}{2}}}{\kappa_2} \phi_{\rho_1, \rho_2}. \]

Then we have the following result:

**Theorem 1.1.** (1) There are the following relations between $\{f_{i,j}|i, j = 0, 1\}$ and $\{\phi_{\rho_1, \rho_2}|(\rho_1, \rho_2) = (0, -\mu \pm \nu), (-\mu \pm \nu, -\mu \pm \nu)\}$:

\[ \tilde{\phi}_{\rho_1, \rho_2} = \begin{cases} f_{0,0} & \text{if } (\rho_1, \rho_2) = (0, -\mu + \nu), \\ f_{0,1} & \text{if } (\rho_1, \rho_2) = (0, -\mu - \nu), \\ f_{1,0} & \text{if } (\rho_1, \rho_2) = (-\mu + \nu, -\mu + \nu), \\ f_{1,1} & \text{if } (\rho_1, \rho_2) = (-\mu - \nu, -\mu - \nu). \end{cases} \]
(2) For each \((i,j)\), \(f_{i,j}\) has the following integral formula:

\[
 f_{i,j}(a_1, a_2) = \int (-1)^{-\frac{1}{2}(\mu+2)} \left( -1 \right)^{-\nu} \left( \frac{\sqrt{t}}{2} \right) \exp \left( \frac{t}{16\kappa_2a_2^2} - \frac{4\kappa_1\kappa_2a_1^2}{t} \right) dt
\]

Here we denote by \(C_0\) and \(C_1\) the following contour:

\[
 C_0 = \{-16\kappa_2a_2^2z \mid z \in C\}, \\
 C_1 = \{\frac{4\kappa_1\kappa_2a_1^2}{z} \mid z \in C\},
\]

where \(C\) is the contour which starts from a point \(+\infty\) on the real axis, proceeds along the real axis to 1, describes a circle counter-clockwise round the origin and returns to \(+\infty\) along the real axis.

By Theorem (1.1), we know when \(\phi_{\rho_1,\rho_2}, (\rho_1, \rho_2) = (0, -\mu \pm \nu), (-\mu \pm \nu, -\mu \pm \nu)\) are linearly independent.

**Corollary 1.2.** If and only if both \(\nu, \frac{-\mu + \nu}{2}\) and \(\frac{-\mu - \nu}{2}\) are not in \(\mathbb{Z}\), the set \(\{\phi_{\rho_1,\rho_2}\mid (\rho_1, \rho_2) = (0, -\mu \pm \nu), (-\mu \pm \nu, -\mu \pm \nu)\}\) is a basis on the space of solutions for the system (1.1), (1.2).

**2. ANOTHER BASIS ON THE SPACE OF SOLUTIONS**

The basis \(\{f_{i,j} \mid i, j = 0, 1\}\) does not contain a moderate growth function on \(\mathbb{R}_{>0} \times \mathbb{R}_{>0}\). Here \(\mathbb{R}_{>0}\) denotes the set of positive element in \(\mathbb{R}\). Now we construct another basis which contains a moderate growth function on \(\mathbb{R}_{>0} \times \mathbb{R}_{>0}\).

**Definition 2.1.** We set for each \(l = 0, 1\),

\[
 f_l = \begin{cases} 
 1, & \text{if } \nu \notin \mathbb{Z}, \\
 \frac{1}{2\sqrt{-1}} \sin \left\{ -\frac{1}{2}(-\mu + (-1)^l\nu)\pi \right\}, & \text{if } \frac{1}{2}\{-\mu + (-1)^l\nu\} \notin \mathbb{Z}, \\
 \lim_{\nu \to m} \frac{1}{2\sqrt{-1}} \sin \left\{ -\frac{1}{2}(-\mu + (-1)^l\nu)\pi \right\}, & \text{if } \frac{1}{2}\{-\mu + (-1)^l\nu\} = m \in \mathbb{Z},
\end{cases}
\]

\(\phi_1 = f_{0,0}, \phi_2 = f_0, \phi_3 \text{(resp. } \phi_4) = \frac{\pi f_{0,1} - f_{0,0}}{2 \sin \nu \pi} \left( \text{resp. } \frac{\pi f_1 - f_0}{2 \sin \nu \pi} \right), \text{ if } \nu \notin \mathbb{Z}, \)

\(\frac{\pi f_{0,1} - f_{0,0}}{2 \sin \nu \pi} \left( \text{resp. } \frac{\pi f_1 - f_0}{2 \sin \nu \pi} \right), \text{ if } \nu = m \in \mathbb{Z},\)

Then we have the following:
Theorem 2.1. For any \( \kappa_1, \kappa_2, \mu, \nu \in \mathbb{C} \), the set \( \{ \phi_i \mid i = 1, 2, 3 \text{ or } 4 \} \) is a basis on the space of solutions for the system (1.1), (1.2). Moreover we have the following integral formula of \( \phi_3 \):

\[
\phi_3(a_1, a_2) = \int_{C_0} t^{-\frac{1}{2}\mu} K_{\nu}\left( \frac{\sqrt{t}}{2} \right) \exp\left( \frac{t}{16\kappa_2 a_2^2} - \frac{4\kappa_1 \kappa_2 a_1^2}{t} \right) \frac{dt}{t},
\]

and when \( \left| \arg\left( \frac{-\kappa_1 a_1}{a_2} \right) \right| < \frac{\pi}{4} \), we have the following integral formula of \( \phi_2 \) and \( \phi_4 \):

\[
\phi_2(a_1, a_2) = \int_{0}^{(-16\kappa_2 a_2^2) \cdot \infty} t^{-\frac{1}{2}\mu} I_{\nu}\left( \frac{\sqrt{t}}{2} \right) \exp\left( \frac{t}{16\kappa_2 a_2^2} - \frac{4\kappa_1 \kappa_2 a_1^2}{t} \right) \frac{dt}{t},
\]

\[
\phi_4(a_1, a_2) = \int_{0}^{(-16\kappa_2 a_2^2) \cdot \infty} t^{-\frac{1}{2}\mu} K_{\nu}\left( \frac{\sqrt{t}}{2} \right) \exp\left( \frac{t}{16\kappa_2 a_2^2} - \frac{4\kappa_1 \kappa_2 a_1^2}{t} \right) \frac{dt}{t}.
\]

Here we denote by \( K_{\nu} \) the Bessel function:

\[
K_{\nu}(z) = \begin{cases} 
\pi \frac{I_{-\nu}(z) - I_{\nu}(z)}{2 \sin \nu \pi}, & \text{if } \nu \notin \mathbb{Z}, \\
\lim_{\nu \rightarrow m} \pi \frac{I_{-\nu}(z) - I_{\nu}(z)}{2 \sin \nu \pi}, & \text{if } \nu = m \in \mathbb{Z}.
\end{cases}
\]

and \( \int_{0}^{(-16\kappa_2 a_2^2) \cdot \infty} dt \) implies that we exchange the variable \( s \) in the usual integral \( \int_{0}^{\infty_{s}} ds \) on \((0, \infty)\) for \( s = -16\kappa_2 a_2^2 t \).

Next we shall obtain some evaluations of \( |\phi_i(a_1, a_2)| \) \((1 \leq i \leq 4)\). We need some evaluations of the Bessel functions \( I_{\nu}(z) \) and \( K_{\nu}(z) \):

**Lemma 2.1.** We assume that \( \nu \in \mathbb{R} \). Then, for any \( \epsilon > 0 \), there exist constants \( C_\epsilon, C_\epsilon' > 0 \) such that:

\[
\frac{K_{\nu}(z)}{\Gamma\left( \delta_{\nu} + \frac{1}{2} \right)} \leq C_\epsilon \left( \frac{z}{2} \right)^{\delta_{\nu}} \exp(-z), \quad \text{for } z \in \mathbb{R} \text{ and } z \geq \epsilon,
\]

\[
\frac{|I_{\nu}(z)|}{\Gamma\left( \delta_{\nu} + \frac{1}{2} \right)} \leq C_\epsilon' \left( \frac{z}{2} \right)^{\delta_{\nu}} \exp(z), \quad \text{for } z \in \mathbb{R} \text{ and } z \geq \epsilon.
\]

Here for \( \nu \in \mathbb{C} \) we denote by \( \delta_{\nu} \) the following number:

\[
\delta_{\nu} = \begin{cases} 
\nu, & \text{if } \Re(\nu) > 0, \\
-\nu, & \text{if } \Re(\nu) < 0.
\end{cases}
\]
We set for \( \nu \in \mathbb{R}, \ j = 0, 1, \)
\[
X_{j, \nu} = \begin{cases} 
\{ k \in \mathbb{N} \mid k \geq \lvert \nu \rvert + 1 \}, & \text{if } \nu \in \mathbb{Z} \text{ and } (-1)^j \nu < 0, \\
\mathbb{N}, & \text{otherwise},
\end{cases}
\]
\[
k_{j, \nu} = \min \{ k \in X_{j, \nu} \},
\]
\[
M_{j, \mu, \nu} = \sup_{l \in X_{j, \nu}} \frac{\lvert \frac{1}{2}(-\mu + (-1)^j \nu + 1) + l \rvert}{\lvert (-1)^j \nu + 1 + l \rvert},
\]
\[
M_{\mu, \nu} = \max_{j=0,1} M_{j, \mu, \nu}.
\]
We denote by \( c_{j, \mu, \nu} \) \( (j = 0, 1 ; \mu, \nu \in \mathbb{R}) \) the following constant:
\[
c_{j, \mu, \nu} = \begin{cases} 
\lvert \Gamma \left( \frac{1}{2}(-\mu - (-1)^j \nu + 1) \right) \rvert, & \text{if } \nu \in \mathbb{Z} \text{ and } (-1)^j \nu < 0, \\
\frac{\lvert \Gamma \left( \frac{1}{2}(-\mu + (-1)^j \nu + 1) \right) \rvert}{\lvert \Gamma((-1)^j \nu + 1) \rvert}, & \text{otherwise}.
\end{cases}
\]
For simplicity, we write \( c_{i,j} = c_{i,j,\mu,\nu}, M_i = M_{i,\mu,\nu}, M = M_{\mu,\nu} \) and \( k_j = k_{j,\nu} \). Then we obtain the following results:

**Corollary 2.1.** We assume that \( \kappa_1, \kappa_2, \mu, \nu \in \mathbb{R}, \kappa_2 \neq 0, \kappa_1 < 0 \) and \( a_1, a_2 > 0 \). Then we obtain the following results:

1. If \( \mu + \nu \) and \( -\mu - \nu \) are not contained in the set \( \{ x \in 2\mathbb{Z} + 1 \mid x \leq -1 \} \), then for any fixed \( \epsilon > 0 \), we obtain the following evaluations of \( \phi_i \) \( (1 \leq i \leq 4) \):

\[
| \phi_1(a_1, a_2) | \leq \frac{2 \pi c_0 C_0'}{4^\mu} M_{-k_0}^{-\kappa_1|\kappa_2|} \left( \frac{a_1^2}{4} \right)^{\frac{1}{2}(-\mu+\nu)} \exp \left( -M_{0} \frac{\kappa_1|\kappa_2|}{4} a_1^2 + \sqrt{\kappa_1} \frac{a_1}{a_2} \right),
\]
\[
| \phi_2(a_1, a_2) | \leq \frac{2 \pi c_0 C_0'}{4^\mu} M_{-k_0}^{-\kappa_1|\kappa_2|} \left( \frac{a_1^2}{4} \right)^{\frac{1}{2}(-\mu+\nu)} \exp \left( -M_{0} \frac{\kappa_1|\kappa_2|}{4} a_1^2 - \sqrt{\kappa_1} \frac{a_1}{a_2} \right),
\]
\[
| \phi_3(a_1, a_2) | \leq \frac{\pi^2 (c_0 + c_1) C_0'}{4^\mu} \max_{j=0,1} \left( \frac{\kappa_1|\kappa_2|}{4} a_1^2 \right)^{\frac{1}{2}(-\mu+(-1)^j \nu)} M_{-k_j} M_{-k_j}^{-j}
\]
\[
\times \exp \left( -M \frac{\kappa_1|\kappa_2|}{4} a_1^2 + \sqrt{\kappa_1} \frac{a_1}{a_2} \right),
\]
\[
| \phi_4(a_1, a_2) | \leq \frac{\pi (c_0 + c_1) C_0'}{4^\mu} \max_{j=0,1} \left( \frac{\kappa_1|\kappa_2|}{4} a_1^2 \right)^{\frac{1}{2}(-\mu+(-1)^j \nu)} M_{-k_j}^{-j}
\]
\[
\times \exp \left( -M \frac{\kappa_1|\kappa_2|}{4} a_1^2 - \sqrt{\kappa_1} \frac{a_1}{a_2} \right),
\]
for \( \frac{a_1}{a_2} \geq \epsilon, a_2 > 0. \)
(2) \( \phi_2 \) (resp. \( \phi_4 \)) is positive real valued for any \( \nu > 0 \) (resp. \( \nu \in \mathbb{R} \)). Moreover we assume that \( \kappa_2 < 0 \). Then, for any fixed \( a_1 > 0 \), \( \phi_3(a_1, a_2) \) and \( \phi_4(a_1, a_2) \) are rapidly decreasing as \( a_2 \to +\infty \).

3. WHITTAKER FUNCTIONS FOR THE REPRESENTATIONS OF THE DISCRETE SERIES - THE CASE OF \( Sp(2; \mathbb{R}) \) -

3.1. Structure of Lie group and Lie algebra. Let \( G \) be the symplectic group \( Sp(2; \mathbb{R}) \) realized as

\[ G = \{ g \in SL_4(\mathbb{R}) \mid {}^t g J g = J \}, \quad \text{with} \quad J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \in M_4(\mathbb{R}), \]

where \( {}^t g \) denotes the transpose of a matrix \( g \) and \( 1_2 \) denotes a unit matrix of size 2.

Let \( O(4) \) be the orthogonal group of degree 2. Take a maximal compact subgroup \( K = G \cap O(4) \). We denote by \( \mathfrak{g}, \mathfrak{t} \) the Lie algebra of \( G, K \), respectively. Let \( \theta(X) = -{}^t X \) be a Cartan involution and \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) is the Cartan decomposition of \( \mathfrak{g} \).

We set \( a = \mathbb{R}H_1 + \mathbb{R}H_2 \) with \( H_1 = \text{diag}(1, 0, -1, 0), H_2 = \text{diag}(0, 1, 0, -1) \). Then \( a \) is a maximally Cartan subalgebra of \( \mathfrak{g} \) and the restricted root system \( \Delta = \Delta(g; a) \) is expressed as \( \Delta = \Delta(g; a) = \{ \pm \lambda_1, \pm \lambda_2, \pm 2 \lambda_1, \pm 2 \lambda_2 \} \), where \( \lambda_j \) is the dual of \( H_j \). We choose a positive root system \( \Delta^+ \) as \( \Delta^+ = \{ \lambda_1, \pm \lambda_2, 2 \lambda_1, 2 \lambda_2 \} \). We also denote the corresponding nilpotent subalgebra by \( n = \sum_{\beta \in \Delta^+} \mathfrak{g}_\beta \). Here \( \mathfrak{g}_\beta \) is the root subspace of \( \mathfrak{g} \) corresponding to \( \beta \in \Delta^+ \). Then one obtains an Iwasawa decomposition of \( \mathfrak{g} \) and \( G; \mathfrak{g} = n + a + t, G = NAK \) with \( A = \exp a, N = \exp n \).

3.2. Representation of the maximal compact subgroup. Firstly, we review the parametrization of the finite-dimensional irreducible representations of \( SL_2(\mathbb{C}) \).

Let \( \{ f_1, f_2 \} \) be the standard basis of the vector space \( V = V_1 = \mathbb{C} \oplus \mathbb{C} \). Then \( GL_2(\mathbb{C}) \) acts on \( V \) by matrix multiplication. We denote the symmetric tensor space of 2 dimension by \( V_d = S^d(V) \). Here \( V_0 = \mathbb{C} \). We consider \( V_d \) as a \( SL_2(\mathbb{C}) \)-module by

\[ \text{sym}^d(g)(v_1 \otimes v_2 \otimes \cdots \otimes v_d) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_d. \]

It is well known that all the finite-dimensional irreducible (polynomial) representations of \( SL_2(\mathbb{C}) \) can be obtained in this way. By Weyl’s unitary trick, all irreducible unitary representations of \( SU(2) \) are obtained by restriction of \( \text{sym}^d \) \( (d \geq 0) \).

The maximal compact subgroup \( K \) is isomorphic to the unitary group \( U(2) \) of degree 2 by

\[ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \to A + \sqrt{-1}B, \quad \text{for} \quad \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K. \]

For \( d, m \in \mathbb{Z}, d \geq 0 \), we define a holomorphic representation \( (\sigma_{d,m}, V_d) \) of \( GL_2(\mathbb{C}) \) by \( \sigma_{d,m}(g) = \text{sym}^d(g) \otimes \det(g)^m \). Then we know \( U(2) = \{ \sigma_{d,m}|_{U(2)} \mid d, m \in \mathbb{Z}, d \geq 0 \} \).

We set \( \lambda = (\lambda_1, \lambda_2) = (m + d, m) \) and \( \tau_\lambda = \sigma_{d,m}|_{U(2)} \). By the isomorphism between
we obtain 

\[
\hat{K} = \{ (\tau_{\lambda}, V_{\lambda}) \mid \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \}.
\]

3.3. Characters of the unipotent radical. The commutator subgroup \([N, N]\) of \(N\) is given by

\[
[N, N] = \left\{ (n_1, n_2) \in \mathbb{R} \mid n_1, n_2 \in \mathbb{R} \right\}
\]

Hence a unitary character \(\eta\) of \(N\) is written for some constant \(\eta_0, \eta_3 \in \mathbb{R}\) as

\[
\exp\{\sqrt{-1}(\eta_0 n_0 + \eta_3 n_3)\} \in \mathbb{C}^\times.
\]

A unitary character \(\eta\) of \(N\) is said to be non-degenerate if \(\eta_0 \eta_3 \neq 0\).

3.4. Parametrization of the discrete series. Let us now parametrize the discrete series of \(Sp(2; \mathbb{R})\). Take a compact Cartan subalgebra \(\mathfrak{h}\) defined by \(\mathfrak{h} = \mathbb{R}h_1 \oplus \mathbb{R}h_2\) with \(h_1 = X_{13} - X_{31}, h_2 = X_{24} - X_{42}\), where the \(X_{ij}\)'s are elementary matrices given by \(X_{ij} = (\delta_{ip}\delta_{j}q)_{1 \leq p, q \leq 4}\), with Kronecker's delta \(\delta_{i,p}\), and let \(\mathfrak{h}_\mathbb{C}\) be its complexification. Then the root system is expressed as

\[
\tilde{\Delta} = \Delta(\mathfrak{g}; \mathfrak{h}) = \{(2,0), (0,2), (1,1), (1, -1)\},
\]

where by \(\beta = (r, s)\), we mean \(r = \beta(-\sqrt{-1}h_1), s = \beta(-\sqrt{-1}h_2)\). Let

\[
\tilde{\Delta}^+ = \{(2,0), (0,2), (1,1)(1, -1)\}.
\]

We write the set of compact positive roots by \(\tilde{\Delta}^+_c = \{(1, -1)\}\). Then there are 4 sets of positive roots \(\tilde{\Delta}^+_J (J = I, II, III, IV)\) of \((\mathfrak{g}, \mathfrak{h})\) containing \(\tilde{\Delta}^+_c\) as follows:

\[
\tilde{\Delta}^+_I = \{(2,0), (1,1), (0,2), (1, -1)\}, \quad \tilde{\Delta}^+_II = \{(1,1), (2,0), (1, -1), (0, -2)\},
\]

\[
\tilde{\Delta}^+_III = \{(2,0), (1, -1), (0, -2), (-1, -1)\}, \quad \tilde{\Delta}^+_IV = \{(1, -1), (0, -2), (-1, -1), (-2,0)\}.
\]

We put \(\delta_G, J = 2^{-1} \sum_{\beta \in \tilde{\Delta}^+_J} \beta\) (resp. \(\delta_K = 2^{-1} \sum_{\beta \in \tilde{\Delta}^+_+} \beta\)), the half sum of positive roots (resp. the half sum of compact positive roots). By definition, the space of Harish-Chandra parameters \(\Xi^+_c\) is given by

\[
\Xi^+_c = \{ \Lambda \in \mathfrak{h}^*_\mathbb{C} \mid \Lambda + \delta_G, I \text{ is analytically integral and} \}
\]

\[
\Lambda \text{ is regular and } \tilde{\Delta}^+_+\text{-dominant}\}.
\]

For each \(J = I, II, III, IV\), we set \(\Xi_J = \{ \Lambda \in \Xi^+_c \mid \langle \Lambda, \alpha \rangle > 0 (\alpha \in \tilde{\Delta}^+_J)\}\). Then \(\Xi^+_c\) is written as a disjoint union \(\Xi^+_c = \bigsqcup_{J=I}^IV J \Xi_J\).
It is well-known that there exists a bijection from $\Xi^+_c$ to the set of equivalence classes of discrete series representations of $G$. Let $\pi_\Lambda$ be the discrete series representation associated to $\Lambda$ in $\Xi^+_c$, then $\tau_\Lambda (\lambda = \Lambda + \delta_{G,J} - 2\delta_K)$ is the unique minimal $K$-type of $\pi_\Lambda$. We note that for each $\Lambda$ in $\Xi^+_c$, $\lambda = \Lambda + \delta_{G,J} - 2\delta_K$ is called the Blattner parameter. An easy computation implies

$$\Xi^+_c = \{ (\Lambda_1, \Lambda_2) \in \mathbb{Z} \oplus \mathbb{Z} \mid \Lambda_1 \neq 0, \Lambda_2 \neq 0, \Lambda_2 < \Lambda_1, \Lambda_1 + \Lambda_2 \neq 0 \}.$$ 

We note that $\Xi_I$ (resp. $\Xi_M$) corresponds to the holomorphic (resp. anti-holomorphic) discrete series, and $\Xi_{II}$ and $\Xi_{III}$ corresponds to the large discrete series in the sense of Vogan, [V].

3.5. Characterization of the minimal $K$-type of a discrete series representation. Let $\eta$ be a unitary character of $N$. Then we set

$$C^\infty_\eta (N \setminus G) = \{ \phi : G \to \mathbb{C}, \text{ $C^\infty$-class} \mid \phi(n g) = \eta(n) \phi(g), \ (n,g) \in N \times G \}.$$ 

By the right regular action of $G$, $C^\infty_\eta (N \setminus G)$ has a structure of smooth $G$-module. For any finite dimensional $K$-module $(\tau, V)$, we set

$$C^\infty_{\eta, \tau} (N \setminus G/K) = \{ F : G \to V, \text{ $C^\infty$-class} \mid F(ngk^{-1}) = \eta(n)\tau(k)F(g), \ (n,g,k) \in N \times G \times K \}.$$ 

Let $(\pi_\Lambda, H)$ be the discrete series representation of $G$ with Harish-Chandra parameter $\Lambda$ in $\Xi_J$, $(J = I, II, III, IV)$, and denote its associated $(g_\mathbb{C}, K)$-module by the same symbol. For $W$ in $Hom_{(g_\mathbb{C}, K)}(\pi_\Lambda, C^\infty_\eta (N \setminus G))$, we define $F_W$ in $C^\infty_{\eta, \tau_\Lambda} (N \setminus G/K)$ by

$$W(v^*)(g) = \langle v^*, F_W(g) \rangle, \quad (v^* \in V_\Lambda^*, g \in G).$$ 

Here $(\tau_\Lambda, V_\Lambda)$ denotes the minimal $K$-type of $\pi_\Lambda$ and $(\ast, \ast)$ denotes the canonical pairing on $V_\Lambda^* \times V_\Lambda$.

Now let us recall the definition of the Schmid operator. Let $g = t \oplus p$ be a Cartan decomposition of $g$ and $Ad = Ad_{p_\mathbb{C}}$ be the adjoint representation of $K$ on $p_\mathbb{C}$. Then we can define a differential operator $\nabla_{\eta, \lambda}$ from $C^\infty_{\eta, \tau_\Lambda} (N \setminus G/K)$ to $C^\infty_{\eta, \tau_\Lambda \otimes Ad} (N \setminus G/K)$ as

$$\nabla_{\eta, \lambda}F = \sum_i R_X F(\cdot) \otimes X_i.$$ 

Here the set $\{X_i\}_i$ is any fixed orthonormal basis of $p$ with respect to the Killing form on $g$ and $R_X F$ denotes the right differential of the function $F$ by $X$ in $g$ i.e. $R_X F(g) = \left. \frac{d}{dt} F(g \cdot \exp t X) \right|_{t=0}$. This operator $\nabla_{\eta, \lambda}$ is called the Schmid operator.

Let $(\tau^\Lambda, V^\Lambda)$ be the sum of irreducible $K$-submodules of $V_\Lambda \otimes p_\mathbb{C}$ with highest weight of the form $\lambda - \beta$ ($\beta \in \Delta_{\mathbb{C}}^+$, $J = I, II, III, IV$). Let $P_\Lambda$ be the projection from $V_\Lambda \otimes p_\mathbb{C}$ to $V^\Lambda$. We define a differential operator from $C^\infty_{\eta, \tau_\Lambda} (N \setminus G/K)$ to $C^\infty_{\eta, \tau_\Lambda} (N \setminus G/K)$ by $D_{\eta, \lambda} F(g) = P_\Lambda(\nabla_{\eta, \lambda} F(g))$ for $F \in C^\infty_{\eta, \tau_\Lambda} (N \setminus G/K)$, $g \in G$. We have the following:
Proposition 3.1 ([Y1] H. Yamashita, Proposition (2.1)). Let $\pi_\Lambda$ be a representation of discrete series with Harish-Chandra parameter $\Lambda \in \Xi_J$ of $Sp(2; \mathbb{R})$. Set $\lambda = \Lambda + \delta_G - 2\delta_K$. Then the linear map

$$W \in \text{Hom}_{\mathfrak{g}_{c,K}}(\pi_\Lambda^*, C_\infty^\om (N \backslash G)) \rightarrow F_W \in \text{Ker}(D_{\eta, \lambda})$$

is injective, and if $\Lambda$ is far from the walls of the Weyl chambers, it is bijective.

3.6. A basis on the Whittaker space on $Sp(2; \mathbb{R})$. By the result of Kostant [Ko], and Vogan [V], if $\eta$ is non-degenerate, we obtain

$$\dim \text{Hom}_{(\mathfrak{g}_{c,K})}(\pi_\Lambda, C_\infty^\om (N \backslash G)) = \begin{cases} 4, & \text{if } \Lambda \in \Xi_\II \cup \Xi_\III, \\ 0, & \text{if } \Lambda \in \Xi_I \cup \Xi_M. \end{cases}$$

Oda proved the following:

Theorem 3.1 ([O] Oda). Let us assume that $\eta$ is non-degenerate and $\Lambda \in \Xi_\II$. We choose the basis $V_\lambda = \{v_k | 0 \leq k \leq d\}_\mathbb{C}$ defined in §4.2. Here $d = \lambda_1 - \lambda_2$.

(1) $F \in \text{Ker} D_{\eta, \lambda}$ if and only if $F$ satisfies the following conditions:

$$(\partial_1 - k)h_{d-k} + \sqrt{-1} \eta_0 h_{d-k-1} = 0, \quad k = 0, 1, \ldots, d-1,$$

(3.1) $$\{\partial_1 \partial_2 + (a_1/a_2)^2 \eta_0^2\}h_d = 0,$$

(3.2) $$\{(\partial_1 + \partial_2)^2 + 2(\lambda_2 - 1)(\partial_1 + \partial_2) - 2\lambda_2 + 1 + 4\eta_3 a_2 \partial_2\}h_d = 0.$$

Here $\partial_i = \frac{\partial}{\partial a_i}$, $i = 1, 2$ and $\{h_k | 0 \leq k \leq d\}$ is determined by

$$F|_\Lambda(a) = \sum_{k=0} c_k(a) v_k,$$

$$c_k(a) = a_1^{\lambda_2+1} a_2^{\lambda_1} \left( \frac{a_1}{a_2} \right)^k \exp(\eta_3 a_2^2) h_k(a), \quad (a \in A; \, k = 0, 1, \ldots, d).$$

(2) If $\eta_3 < 0$, $\text{Ker} D_{\eta, \lambda}$ contains the function $F$ such that $h_d$ has an integral representation

$$h_d(a) = \int_0^\infty t^{-\lambda_2+\frac{3}{2}} W_{\lambda_2}(t) \exp \left( \frac{t^2}{32\eta_3 a_2^2} + \frac{8\eta_3^2 \eta_2 a_1^2}{t^2} \right) \frac{dt}{t}.$$

By Theorem 3.1, Oda showed that if $\Lambda \in \Xi_\II \cup \Xi_\III$ and $\eta$ is non-degenerate,

$$\text{Hom}_{(\mathfrak{g}_{c,K})}(\pi_\Lambda^*, A_\eta(N \backslash G)) \cong \begin{cases} \mathbb{C}, & \eta_3 < 0, \\ 0, & \eta_3 > 0. \end{cases}$$

Here we put

$$A_\eta(N \backslash G) = \{F \in C_\infty^\om (N \backslash G) \mid K\text{-finite and for any } X \in U(\mathfrak{g}_c) \text{ there exists a constant } C_X > 0 \text{ such that } |F(g)| \leq C_X \text{tr}(tgg), \, g \in G \}$$

and $U(\mathfrak{g}_c)$ denotes the universal enveloping algebra of $\mathfrak{g}_c$. 

The system of equations (3.1), (3.2) is coincide with the system (1.1), (1.2) with the parameters \( \kappa_1 = \eta^2_0, \kappa_2 = -2\sqrt{-1}\eta_3, \mu = \lambda_2-1, \nu = -\lambda_2 \). And these parameters satisfies the assumptions in the Corollary(2.1). So let us denote by \( \phi_i(\kappa_1, \kappa_2, \mu, \nu; a_1, a_2) \) for the function \( \phi_i(a_1, a_2) \) (1 \leq i \leq 4) given for \( \kappa_1, \kappa_2, \mu, \nu \in \mathbb{C} \) in §3. We set

\[
h^{(i)}_d(a_1, a_2) = \phi_i(\eta^2_0, -2\sqrt{-1}\eta_3, \lambda_2-1, -\lambda_2; a_1, a_2), \quad \text{for } 1 \leq i \leq 4, \; a_1, a_2 > 0,
\]

and determine \( h^{(i)}_k \) by the relations

\[
(\partial_1 - k)h^{(i)}_{d-k} + \sqrt{-1}\eta_0 h^{(i)}_{d-k-1} = 0, \quad \text{for } 0 \leq k \leq d-1, \; 1 \leq i \leq 4.
\]

We define the function \( F^{(i)} \in C_\eta^\infty(N \backslash G/K) \) by

\[
F^{(i)}|_A(a) = \sum_{0 \leq k \leq d} c_k(a) v_k, \quad \text{with } c_k(a) = a_1^{\lambda_2+1}a_2^1 \left( \frac{a_1}{a_2} \right)^k \exp(\eta_3a_2^2) h_k(a),
\]

for \( a \in A, \; 0 \leq k \leq d, \; 1 \leq i \leq 4 \).

We set for \( t \in \mathbb{C}, \; |\arg t| < \pi \),

\[
k_{i,\nu}(t) = \begin{cases} K_\nu(\sqrt{t}/2), & \text{if } i = 1, 2, \\ I_\nu(\sqrt{t}/2), & \text{if } i = 3, 4, \end{cases}
\]

Then we obtain the following result:

**Theorem 3.2.** Let us assume that \( \eta \) is non-degenerate and \( \Lambda \in \Xi_\eta \). Then we obtain the following results:

1. \( \text{Ker} D_{\eta,\lambda} \) has the basis \( \{F^{(i)}|1 \leq i \leq 4\} \) and \( h^{(i)}_d(1 \leq i \leq 4) \) have the following integral expressions:

\[
h^{(i)}_d(a) = \int_{C_i} t^{\frac{1}{2}(1-\lambda_2)} k_{i,\nu}(t) \exp \left( \frac{t}{32\eta_3a_2^2} + \frac{8\eta^2_0\eta_3a_1^2}{t} \right) \frac{dt}{t}.
\]

Here we denote by \( C_i \) (1 \leq i \leq 4) the following contour:

\[
\int_{C_i} dt = \begin{cases} \int_C dt, & \text{if } i = 1, 3, \\ \int_0^\infty dt, & \text{if } i = 2, 4, \end{cases}
\]

where \( \int_C dt \) is the contour integral on \( C \) given in Theorem (1.1)-(2) and \( \int_0^\infty dt \) is the usual integral on \((0, \infty) \subset \mathbb{R}\).

2. For any fixed constant \( R_1, R_2 > 0 \), we denote by \( D_{R_1,R_2} \) the domain

\[
D_{R_1,R_2} = \{(a_1, a_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \mid a_1a_2 \leq R_1 \text{ and } a_1 \leq R_2 \}.
\]
Then there exist constants $C^{(i)} = C^{(i)}_{R_1,R_2} (1 \leq i \leq 4)$ and $C^{(i)}_k = C^{(i)}_{R_1,R_2,k} (0 \leq k \leq d; \ i = 1,2)$ such that

$$
|c^{(i)}_{d}(a_1, a_2)| \leq C^{(i)} a_1^{1+1} a_2^{1+|\eta_0|} \frac{a_1}{a_2} + m_i \eta_3 a_2^2 
$$

$$
|c^{(i)}_{k}(a_1, a_2)| \leq C^{(i)}_k a_1^{1+1} a_2^{1+|\eta_0|} \frac{a_1}{a_2} + m_i \eta_3 a_2^2 
$$

for $(a_1, a_2) \in D_{R_1,R_2}$.

Here we set for $1 \leq i \leq 4$

$$
m_i = \begin{cases} 
-1, & \text{if } i = 1, 2, \\
1, & \text{if } i = 3, 4 
\end{cases}
$$

REFERENCES


