FOURIER-JACOBI TYPE SPHERICAL FUNCTIONS ON $Sp(2,\mathbb{R})$; 
THE CASE OF $P_J$-PRINCIPAL SERIES AND DISCRETE SERIES

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1. Introduction
In this note, we study a kind of generalized Whittaker models, or equally, of 
generalized spherical functions associated with automorphic forms on the real sym-
plectic group of degree two. We call these spherical functions 'Fourier-Jacobi type',
since these are closely connected with the coefficients of the 'Fourier-Jacobi expan-
sions' of (holomorphic or non-holomorphic) automorphic forms. Also these can be 
considered as a non-holomorphic analogue of the local Whittaker-Shintani functions 
on $Sp(2,\mathbb{R})$ of Fourier-Jacobi type in the paper of Murase and Sugano [6].

2. Preliminaries

2.1. Groups and algebras. We denote by $\mathbb{Z}_{\geq m}$ the set of integers $n$ such that 
n $\geq m$. Moreover, we use the convention that unwritten components of a matrix
are zero.

Let $G$ be the real symplectic group $Sp(2,\mathbb{R})$ of degree two given by

$$Sp(2,\mathbb{R}) = \left\{ g \in M_4(\mathbb{R}) \mid {}^t gJ_2 g = J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \det g = 1 \right\}.$$ 

Let $\theta(g) = {}^t g^{-1}$ ($g \in G$) be a Cartan involution of $G$ and $K$ be the set of fixed points 
of $\theta$. Then $K$ becomes a maximal compact subgroup of $G$ which is isomorphic to 
the unitary group $U(2)$.

Let $\mathfrak{g} = \{ X \in M_4(\mathbb{R}) \mid J_2 X + {}^t X J_2 = 0 \}$ be the Lie algebra of $G$. If we denote 
the differential of $\theta$ again by $\theta$, then we have $\theta(X) = -{}^t X$ ($X \in \mathfrak{g}$). Let $\mathfrak{k}$ and $\mathfrak{p}$ be 
the $+1$ and $-1$ eigenspaces of $\theta$ in $\mathfrak{g}$, respectively, and hence

$$\mathfrak{k} = \left\{ X \in \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A, B \in M_2(\mathbb{R}), {}^t A = -A, {}^t B = B \right\},$$

$$\mathfrak{p} = \left\{ X \in \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A, B \in M_2(\mathbb{R}), {}^t A = A, {}^t B = B \right\}.$$
Then we have a Cartan decomposition \( g = \mathfrak{k} \oplus \mathfrak{p} \). Of course, \( \mathfrak{k} \) is the Lie algebra of \( K \) which is isomorphic to the unitary algebra \( u(2) \).

For a Lie algebra \( \mathfrak{l} \), we denote by \( \mathfrak{l}_C = \mathfrak{l} \otimes \mathbb{R} \mathbb{C} \) the complexification of \( \mathfrak{l} \). Let \( \mathfrak{h} \) be a compact Cartan subalgebra of \( \mathfrak{g} \) given by

\[
\mathfrak{h} = \left\{ H(\theta_1, \theta_2) = \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_1 & -\theta_2 \end{pmatrix} \middle| \theta_i \in \mathbb{R} \right\}.
\]

Now we identify a linear form \( \beta : \mathfrak{h}_C \to \mathbb{C} \) with \((\beta_1, \beta_2) \in \mathbb{C}^2\) via \( \beta = \beta_1 e_1 + \beta_2 e_2 \), where \( e_i(H(\theta_1, \theta_2)) = \sqrt{-1}\theta_i \). Then the set of roots \( \Delta = \Delta(\mathfrak{h}_C, \mathfrak{g}_C) \) of \((\mathfrak{h}_C, \mathfrak{g}_C)\) is given by

\[
\Delta = \{ \pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1) \}.
\]

Fix a positive root system \( \Delta^+ = \{(2,0), (0,2), (1,1), (1,-1)\} \), and put \( \Delta^+_c \) and \( \Delta^+_n \) the set of compact and non-compact positive roots, respectively. Then

\[
\Delta^+_c = \{(1,-1)\}, \quad \Delta^+_n = \{(2,0), (0,2), (1,1)\}.
\]

If we denote the root space for \( \beta \in \Delta \) by \( \mathfrak{g}_\beta \), then we have a decomposition \( \mathfrak{p}_C = \mathfrak{p}_+ \oplus \mathfrak{p}_- \) with \( \mathfrak{p}_+ = \sum_{\beta \in \Delta^+_c} \mathfrak{g}_\beta \) and \( \mathfrak{p}_- = \sum_{\beta \in \Delta^+_n} \mathfrak{g}_\beta \).

Put \( P_J \) the Jacobi maximal parabolic subgroup of \( G \) with the Langlands decomposition \( P_J = M_J A_J N_J \), where

\[
M_J = \left\{ \begin{pmatrix} \varepsilon & a & b \\ \alpha & \varepsilon & d \\ \gamma & c & \varepsilon \end{pmatrix} \varepsilon \in \{\pm1\}, \begin{pmatrix} a & b \\ \alpha & d \\ c & \gamma \end{pmatrix} \in SL(2, \mathbb{R}) \right\} \simeq \{\pm I\} \times SL(2, \mathbb{R}),
\]

\[
N_J = \left\{ n(x, y; z) = \begin{pmatrix} 1 & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\},
\]

and \( A_J = \{ \text{diag}(a, 1, a^{-1}, 1) \mid a > 0 \} \). Remark that the unipotent radical \( N_J \) of \( P_J \) is isomorphic to the 3-dimensional Heisenberg group \( \mathcal{H}_1 \). The Levi part \( M_J A_J \) of \( P_J \) acts on \( N_J \) via the conjugate action, and \( M_J \) gives the centralizer of the center \( Z(N_J) = \{ n(0,0; z) \mid z \in \mathbb{R} \} \simeq \mathbb{R} \) of \( N_J \) in \( M_J A_J \). Now we define the Jacobi group \( R_J \) by the semidirect product \( M_J^o \ltimes N_J \simeq SL(2, \mathbb{R}) \ltimes \mathcal{H}_1 \), where \( M_J^o \simeq SL(2, \mathbb{R}) \) is the identity component of \( M_J \).

\[2.2.\text{Representations.}\] First we investigate the irreducible unitary representations of the Jacobi group \( R_J \). Since \( Z(R_J) = Z(N_J) \simeq \mathbb{R} \), the central characters of elements in \( R_J \) and \( N_J \) are parametrized by the real numbers. Then we call an irreducible unitary representation of \( R_J \) and \( N_J \) of type \( m \) if its central character is of the form \( z \mapsto e^{2\pi \sqrt{-1} m z} \) with \( m \in \mathbb{R} \). Let \( \nu \in N_J \) of type \( m \). According to the
theorem of Stone-von Neumann (cf. Corwin-Greenleaf [1; pp.46-47, 51-52]), $\nu$ is a character if $m = 0$ and $\nu$ is infinite dimensional if $m \neq 0$. Moreover $\nu$ of type $m \neq 0$ is uniquely determined by $m$ up to unitary equivalence. Now we fix an irreducible unitary representation $(\nu_m, U_m)$ of $N_J$ of type $m \neq 0$. From the theory of the Weil representation, $(\nu_m, U_m)$ can be extended to a continuous true projective unitary representation $(\hat{\nu}_m, \hat{U}_m)$ of $R_J$ by $\hat{\nu}_m(\hat{n}) = W_m(g)\nu_m(n)$ for $\hat{n} = g \cdot n \in \hat{M}_J \ltimes \hat{N}_J$ with the Weil representation $W_m$ on $M_J$. Here $\hat{\nu}_m$ has a factor set $\alpha$ which is a proper 2-cocycle.

**Lemma 2.1.** (Satake [7; Appendix I, Proposition 2]) Let $\hat{\nu}_m$ $(m \neq 0)$ as above. For every irreducible projective unitary representation $\pi$ of $\hat{M}_J$ with factor set $\alpha^{-1}$, put $\rho(\tilde{n}) = \pi(g) \otimes \hat{\nu}_m(\tilde{n})$ for $\tilde{n} = g \cdot n \in \hat{M}_J \ltimes \hat{N}_J$. Then $\rho$ is an irreducible unitary representation of $\hat{R}_J$. Conversely, all irreducible unitary representations of $\hat{R}_J$ of type $m \neq 0$ are obtained in this manner. Moreover $\rho$ is square-integrable iff $\pi$ is so.

Let $(\rho, F_\rho)$ be an irreducible unitary representation of $R_J$ of type $m \neq 0$. From the above lemma, we can regard $(\rho, F_\rho) \in \hat{R}_J$ as a tensor product representation $(\pi_1 \otimes \hat{\nu}_m, \mathcal{W}_{\pi_1} \otimes \hat{U}_m)$. Here, if we write $\overline{M}_J$ for the double cover of $M$ and $\mathcal{W}_{\pi_1}$ is a unitary representation of $\overline{M}_J \ltimes \hat{N}_J$ which is extended from $(\nu_m, U_m) \in \hat{N}_J$ as above and $(\pi_1, \mathcal{W}_{\pi_1})$ is a unitary representation of $\overline{M}_J$ which does not factor through $\hat{M}_J$. On the other hand, the unitary dual of $\overline{M}_J$ is given as follows.

**Proposition 2.2.** (cf. Gelbert[2; Lemma 4.1, 4.2]) The following representations exhaust a set of representatives for the equivalence classes of irreducible unitary representations of $SL(2, \mathbb{R})$.

1. (unitary principal series) $\mathcal{P}_s^\tau$, $s \in \sqrt{-1} \mathbb{R}$, $\tau = 0, 1, \pm \frac{1}{2}$ except for the case $(s, \tau) = (0, 1)$.
2. (complementary series) $\mathcal{C}_s^\tau$, $0 < s < 1$ for $\tau = 0, 1$ and $0 < s < \frac{1}{2}$ for $\tau = \pm \frac{1}{2}$.
3. ((limit of) discrete series) $\mathcal{D}_k^\pm$, $k \in \frac{1}{2} \mathbb{Z}_{\geq 2}$.
4. (quotient representation) $\mathcal{D}_k^\pm$, $\mathcal{D}_k^+$.
5. (The trivial representation of $SL(2, \mathbb{R})$.

In the above, the representations $\mathcal{P}_s^\tau, \mathcal{C}_s^\tau$ for $\tau = 0, 1$, $\mathcal{D}_k^\pm$ for $k \in \mathbb{Z}_{\geq 1}$ and (5) factor through $SL(2, \mathbb{R})$, and the otherwise not.

Hence we take as $(\pi_1, \mathcal{W}_{\pi_1})$ one of the irreducible unitary representations $\mathcal{P}_s^\tau, \mathcal{C}_s^\tau$ with $\tau = \pm \frac{1}{2}$ and $\mathcal{D}_k^\pm$ with $k \in \frac{1}{2} \mathbb{Z} \setminus \mathbb{Z}, k \geq \frac{1}{2}$.

**Remark 2.3.** The Weil representation $W_m$ considered as the representation of $\overline{M}_J$ has the following irreducible decomposition;

$$W_m = \begin{cases} \mathcal{D}_k^+ \oplus \mathcal{D}_k^-, & \text{if } m > 0, \\ \mathcal{D}_k^+ \oplus \mathcal{D}_k^-, & \text{if } m < 0. \end{cases}$$

Next, we treat the irreducible unitary representations of $K$. Since $\Delta^+_c$ is also a positive system of $\Delta(\mathfrak{k}_C, \mathfrak{h}_C)$, then the set of the $\Delta^+_c$-dominant weights, and thus
\( \hat{K} \), is parametrized by the set

\[
\Lambda = \{ \lambda = (\lambda_1, \lambda_2) \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \}
\]

(cf. Knapp[4; Theorem 4.28]). We denote by \((\tau_{\lambda}, V_{\lambda})\) the element of \( \hat{K} \) corresponding to \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \). Here \( \dim V_{\lambda} = d_\lambda + 1 \) with \( d_\lambda = \lambda_1 - \lambda_2 \).

Both of \( p_\pm \) become \( K \)-modules via the adjoint representation of \( K \), and we have isomorphisms \( p_+ \simeq V_{(2,0)} \) and \( p_- \simeq V_{(0,-2)} \). For a given irreducible \( K \)-module \( V_{\lambda} \) with the parameter \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \), the tensor product \( K \)-modules \( V_{\lambda} \otimes p_+ \) and \( V_{\lambda} \otimes p_- \) have the irreducible decompositions

\[
V_{\lambda} \otimes p_+ \cong \bigoplus_{\beta \in \Delta^+_n} V_{\lambda+\beta}, \quad V_{\lambda} \otimes p_- \cong \bigoplus_{\beta \in \Delta^-_n} V_{\lambda-\beta}.
\]

For each \( \beta \in \Delta^+_n \), let \( P^\beta : V_{\lambda} \otimes p_+ \to V_{\lambda+\beta} \) and \( P^{-\beta} : V_{\lambda} \otimes p_- \to V_{\lambda-\beta} \) be the projectors into the irreducible factors of \( V_{\lambda} \otimes p_\pm \).

In this note, we consider the following two series of representations of \( G \); one is the principal series induced from \( P_J \), and the other is the discrete series. We explain these representations in the remaining of this section.

Let \( \sigma = (\epsilon, D) \) be a representation of \( M_J \simeq \{ \pm I \} \times SL(2, \mathbb{R}) \) with a character \( \epsilon : \{ \pm I \} \to \mathbb{C}^\times \) and a discrete series representation \( D = D^\pm_n \ (n \in \mathbb{Z}_{\geq 2}) \) of \( SL(2, \mathbb{R}) \), and take a quasi-character \( \nu_z \ (z \in \mathbb{C}) \) of \( A_J \) such that \( \nu_z(\text{diag}(a, 1, a^{-1}, 1)) = a^z \). Then we can construct a induced representation \( \text{Ind}_{P_J}^{G}(\sigma \otimes \nu_z \otimes 1_{N_J}) \) of \( G \) from the Jacobi maximal parabolic subgroup \( P_J = M_J A_J N_J \) by the usual manner (cf. Knapp[4; Chapter VII]), and call \( \text{Ind}_{P_J}^{G}(\sigma \otimes \nu_z \otimes 1_{N_J}) \) the \( P_I \)-principal series representation of \( G \). The following lemma is derived from the Frobenius reciprocity for induced representations.

**Lemma 2.4.** \( \tau_{\lambda} \in \hat{K} \) with the parameter \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \) such that \( \lambda_1 < n \) (resp. \( \lambda_2 > -n \)) does not occur in the \( K \)-type of \( \text{Ind}_{P_J}^{G}(\sigma \otimes \nu_z \otimes 1_{N_J}) \) for \( D = D^+_n \) (resp. \( D^-_n \)). The 'corner' \( K \)-types \( \tau_{\lambda} \in \hat{K} \) of \( \text{Ind}_{P_J}^{G}(\sigma \otimes \nu_z \otimes 1_{N_J}) \) with the parameter \( \lambda \in \Lambda \) given below occur with multiplicity one.

1. \( \lambda = (n, n) \) for \( \epsilon(\gamma) = (-1)^n \) and \( D = D^+_n \),
2. \( \lambda = (n, n-1) \) for \( \epsilon(\gamma) = (-1)^{n-1} \) and \( D = D^-_n \),
3. \( \lambda = (-n, n) \) for \( \epsilon(\gamma) = (-1)^{-n} \) and \( D = D^+_n \),
4. \( \lambda = (-n, -n) \) for \( \epsilon(\gamma) = (-1)^{-n} \) and \( D = D^-_n \).

Here \( \gamma = \text{diag}(-1, 1, -1, 1) \).

In order to parametrize the discrete series representations of \( G \), we enumerate all the positive root systems compatible to \( \Delta^c_+ \):

1. \( \Delta^+_I = \{(1, -1), (2, 0), (1, 1), (0, 2)\} \),
2. \( \Delta^+_II = \{(1, -1), (2, 0), (1, 1), (0, -2)\} \),
3. \( \Delta^+_III = \{(1, -1), (2, 0), (0, -2), (-1, -1)\} \),
4. \( \Delta^+_IV = \{(1, -1), (0, -2), (-1, -1), (-2, 0)\} \).
Let $J$ be a variable running over the set of indices I, II, III, IV, and let us denote the set of non-compact positive roots for the index $J$ by $\Delta_{J,n}^{+} = \Delta_{J}^{+} - \Delta_{c}^{+}$. Define a subset $\Xi_{J}$ of $\Delta_{c}^{+}$-dominant weights by

$$\Xi_{J} = \{ \Lambda = (\Lambda_1, \Lambda_2), \Delta_{c}^{+} - \text{dominant weight} \mid \langle \Lambda, \beta \rangle > 0, \forall \beta \in \Delta_{J,n}^{+} \}.$$ 

The set $\bigcup_{J=1}^{IV} \Xi_{J}$ gives the Harish-Chandra parametrization of the discrete series representation of $G$. Let us write by $\pi_{\Lambda}$ the discrete series representation of $G$ with the Harish-Chandra parameter $\Lambda \in \bigcup_{J=1}^{IV} \Xi_{J}$. Then $\pi_{\Lambda}$ is called the holomorphic discrete series representation if $\Lambda \in \Xi_{I}$ and the anti-holomorphic one if $\Lambda \in \Xi_{IV}$. Moreover if $\Lambda \in \Xi_{II} \cup \Xi_{III}$, a discrete series representation $\pi_{\Lambda}$ is called large (in the sense of Vogan[8]).

The Blattner formula gives the description of the $K$-types of $\pi_{\Lambda}$. In particular, the minimal $K$-type $(\tau_{\Lambda}, V_{\Lambda})$ of $\pi_{\Lambda}$ is given by the formula $\lambda = \Lambda - \rho_{c} + \rho_{n}$, where $\rho_{c}$ (resp. $\rho_{n}$) is the half sum of compact (resp. non-compact) positive roots in $\Delta_{J}^{+}$. We call such $\lambda$ the Blattner parameter of $\pi_{\Lambda}$.

3. Fourier-Jacobi type spherical functions

3.1. Radial parts. Let $(\rho, \mathcal{F}_{\rho})$ be an irreducible unitary representation of $R_{J}$ and let $(\tau, V_{\tau})$ be a finite dimensional $K$-module. We denote by $C_{\rho,\tau}^{\infty}(R_{J}\backslash G/K)$ the space of smooth functions $F: G \rightarrow \mathcal{F}_{\rho} \otimes V_{\tau}$ with the property

$$F(rgk) = (\rho(r) \otimes \tau(k)^{-1})F(g), \quad (r, g, k) \in R_{J} \times G \times K.$$ 

On the other hand, let $C^{\infty}(A_{J}; \rho, \tau)$ be the space of smooth functions $\varphi: A_{J} \rightarrow \mathcal{F}_{\rho} \otimes V_{\tau}$ satisfying

$$(\rho(m) \otimes \tau(m))\varphi(a) = \varphi(a), \quad m \in R_{J} \cap K = M_{J}^{o} \cap K, \quad a \in A_{J}.$$ 

Because of an Iwasawa decomposition of $G$, we have $G = R_{J} A_{J} K$. Also we remark that all elements in $M_{J}^{o} \cap K$ are commutative with $a \in A_{J}$. Then the restriction to $A_{J}$ gives a linear map from $C_{\rho,\tau}^{\infty}(R_{J}\backslash G/K)$ to $C^{\infty}(A_{J}; \rho, \tau)$, which is injective. For each $f \in C_{\rho,\tau}^{\infty}(R_{J}\backslash G/K)$, we call $f|_{A_{J}} \in C^{\infty}(A_{J}; \rho, \tau)$ the radial part of $f$, where $|_{A_{J}}$ means the restriction to $A_{J}$.

Let $(\tau', V_{\tau'})$ be also a finite dimensional $K$-module. For each $C$-linear map $u: C_{\rho,\tau}^{\infty}(R_{J}\backslash G/K) \rightarrow C_{\rho',\tau'}^{\infty}(R_{J}\backslash G/K)$, we have a unique $C$-linear map $\mathcal{R}(u): C^{\infty}(A_{J}; \rho, \tau) \rightarrow C^{\infty}(A_{J}; \rho, \tau')$ with the property $(uf)|_{A_{J}} = \mathcal{R}(u)(f|_{A_{J}})$ for $f \in C_{\rho,\tau}^{\infty}(R_{J}\backslash G/K)$. We call $\mathcal{R}(u)$ the radial part of $u$.

3.2. Fourier-Jacobi type spherical functions. Let $(\rho, \mathcal{F}_{\rho})$ be as above and consider a $C^{\infty}$-induced representation $C^{\infty}\text{Ind}_{R_{J}}^{G}(\rho)$ with the representation space

$$C_{\rho}^{\infty}(R_{J}\backslash G) = \{ F: G \rightarrow \mathcal{F}_{\rho}, \quad C^{\infty} \mid F(rg) = \rho(r)F(g), \quad (r, g) \in R_{J} \times G \}$$ 

on which $G$ acts by the right translation. Then $C_{\rho}^{\infty}(R_{J}\backslash G)$ becomes a smooth $G$-module and a $(\mathfrak{g}_{C}, K)$-module naturally. Moreover let $(\tau, V_{\tau}) \in \hat{K}$ and take an
irreducible Harish-Chandra module $\pi$ of $G$ with the $K$-type $\tau^*$, where $\tau^*$ is the contragredient representation of $\tau$. Now we consider the intertwining space

$$I_{\rho,\pi} := \text{Hom}_{(g_C, K)}(\pi, C^\infty \text{Ind}^G_{R_J} (\rho))$$

between $(g_C, K)$-modules and its restriction to the $K$-type $\tau^*$ of $\pi$.

Let $i : \tau^* \to \pi|_K$ be a $K$-equivariant map and let $i^*$ be the pullback via $i$. Then the map

$$I_{\rho,\pi} \overset{i^*}{\rightarrow} \text{Hom}_K(\tau^*, C^\infty_{\rho}(R_J \backslash G)) \simeq C^\infty_{\rho,\tau}(R_J \backslash G/K)$$

gives the restriction of $T \in I_{\rho,\pi}$ to the $K$-type $\tau^*$ and we denote the image of $T$ in $C^\infty_{\rho,\tau}(R_J \backslash G/K)$ by $T_i$. Now the space $J_{\rho,\pi}(\tau)$ of the algebraic Fourier-Jacobi type spherical functions of type $(\rho, \pi; \tau)$ on $G$ is defined by

$$J_{\rho,\pi}(\tau) := \bigcup_{i \in \text{Hom}_K(\tau^*, \pi|_K)} \{T_i \mid T \in I_{\rho,\pi}\}.$$

Moreover put

$$J_{\rho,\pi}^\circ(\tau) = \{f \in J_{\rho,\pi}(\tau) \mid f|_{A_J}(\text{diag}(a, 1, a^{-1}, 1)) \text{ is of moderate growth as } a \to \infty\}.$$

We call $f \in J_{\rho,\pi}^\circ(\tau)$ a Fourier-Jacobi type spherical functions of type $(\rho, \pi; \tau)$.

In this note, we investigate the space $J_{\rho,\pi}^\circ(\tau)$ for the following triplet $(\rho, \pi; \tau)$: As $\pi \in \hat{G}$ and $\tau^* \in \hat{K}$, we take either the $P_J$-principal series representation and the corner $K$-type or the discrete series representation and the minimal $K$-type, and also as $\rho \in \hat{R}_J$ the one with the non-trivial central character, i.e. of type $m \neq 0$.

4. Differential equations

4.1. Differential operators. In this subsection, we introduce some differential operators acting on $C^\infty_{\rho,\tau}(R_J \backslash G/K)$.

Take an orthonormal basis $\{X_i\}$ of $p$ with respect to the Killing form of $g$. Now we define a first order gradient type differential operator

$$\nabla_{\rho,\tau} : C^\infty_{\rho,\tau}(R_J \backslash G/K) \to C^\infty_{\rho,\tau \otimes \text{Ad}_{g_C}}(R_J \backslash G/K)$$

by

$$\nabla_{\rho,\tau} f = \sum_i R_{X_i} f \otimes X_i, \quad f \in C^\infty_{\rho,\tau}(R_J \backslash G/K),$$

where

$$R_X f(g) = \frac{d}{dt} f(g \cdot \exp(tX)) \bigg|_{t=0}, \quad X \in g_C, \ g \in G.$$  

This differential operator $\nabla_{\rho,\tau}$ is called the Schmid operator. Then $\nabla_{\rho,\tau}$ can be decomposed as $\nabla_{\rho,\tau}^+ \oplus \nabla_{\rho,\tau}^-$ with $\nabla_{\rho,\tau}^+ : C^\infty_{\rho,\tau}(R_J \backslash G/K) \to C^\infty_{\rho,\tau \otimes \text{Ad}_{g_C}}(R_J \backslash G/K)$ corresponding to the decomposition $p_C = p_+ \oplus p_-$ for each $\beta \in \Delta_n^+$, the shift operator $\nabla_{\rho,\tau,\lambda}^\pm : C^\infty_{\rho,\tau}(R_J \backslash G/K) \to C^\infty_{\rho,\tau \pm \beta}(R_J \backslash G/K)$ is defined as the composition of
with the projector $P^{\pm \rho}_{\tau, \lambda}$ from $V_{\tau, \lambda} \otimes \mathfrak{p}_{\pm}$ into the irreducible component $V_{\tau, \lambda \pm \rho}$; 

\[ \nabla^{\pm}_{\rho_{\tau, \lambda}} = (1_{\mathfrak{g}} \otimes P^{\pm \rho}_{\tau, \lambda}) \nabla^{\pm}_{\rho_{\tau, \lambda}}. \]

On the other hand, the Casimir element $\Omega$ is defined by $\Omega = \sum X_i - \sum Y_j$, where \( \{Y_j\} \) is an orthonormal basis of $\mathfrak{f}$ with respect to the Killing form of $\mathfrak{g}$. It is well known that $\Omega$ is in the center $Z(\mathfrak{g}_C)$ of the universal enveloping algebra of $\mathfrak{g}_C$.

4.2. Differential equations. In this subsection, we consider the system of differential equations satisfied by the Fourier-Jacobi type spherical functions.

First we discuss the case of the $P_J$-principal series representation $\pi \in \widehat{G}$ and the corner $K$-type $\tau^*$. It is well known that the Casimir element $\Omega \in Z(\mathfrak{g}_C)$ acts on $\pi$, hence on $J_{\rho, \pi}(\tau)$, as the scalar operator $\chi_{\Omega}$ (cf. Knapp\cite{4}; Corollary 8.14]). Let $\pi = \text{Ind}_{\mathbb{P}J}^{\mathbb{P}J}(\sigma \otimes \nu \otimes 1_{N_J})$ with data $\sigma = (\varepsilon, D_n^+)$, $\varepsilon(\gamma) = (-1)^n$, and $\tau^* = \tau^{\star}_{\lambda}$ be the corner $K$-type of $\pi$, i.e. $\lambda = (-n, -n)$.

Since $\tau^{\star}_{\lambda+1, 1} = \tau_{(n-1, n-1)} \in \hat{K}$ does not occur in the $K$-types of $\pi$ from Lemma 2.4, an element in $J_{\rho, \pi}(\tau)$ is annihilated by the action of the composition of the shift operators

\[ \nabla^{(0,2)}_{\rho, \tau, \lambda+2, 0} \circ \nabla^{(2,0)}_{\rho, \tau, \lambda} : C^\infty_{\rho, \tau, \lambda}(R_J \backslash G/K) \to C^\infty_{\rho, \tau, \lambda+2, 0}(R_J \backslash G/K). \]

Hence we have a system of differential equations satisfied by $f$ in $J_{\rho, \pi}(\tau)$;

\begin{align}
\begin{cases}
\Omega f = \chi_{\Omega} f, \\
\nabla^{(0,2)}_{\rho, \tau, \lambda+2, 0} \circ \nabla^{(2,0)}_{\rho, \tau, \lambda} f = 0.
\end{cases}
\end{align}

(4.1)

Let $\pi = \text{Ind}_{\mathbb{P}J}^{\mathbb{P}J}(\sigma \otimes \nu \otimes 1_{N_J})$ with data $\sigma = (\varepsilon, D_n^+)$, $\varepsilon(\gamma) = (-1)^n$, and $\tau^* = \tau^{\star}_{\lambda}$ be the corner $K$-type of $\pi$, i.e. $\lambda = (-n+1, -n)$. Since $\tau^{\star}_{\lambda+1, 1} = \tau_{(n-2, n-2)} \in \hat{K}$ does not occur in the $K$-types of $\pi$ from Lemma 2.4, therefore an element in $J_{\rho, \pi}(\tau)$ vanishes by the action of the shift operator

\[ \nabla^{(1,1)}_{\rho, \tau, \lambda+1, 1} : C^\infty_{\rho, \tau, \lambda}(R_J \backslash G/K) \to C^\infty_{\rho, \tau, \lambda+1, 1}(R_J \backslash G/K). \]

Hence we have a system of differential equations satisfied by $f$ in $J_{\rho, \pi}(\tau)$;

\begin{align}
\begin{cases}
\Omega f = \chi_{\Omega} f, \\
\nabla^{(1,1)}_{\rho, \tau, \lambda+1, 1} f = 0.
\end{cases}
\end{align}

(4.2)

For the case with the data $\sigma = (\varepsilon, D_{n-})$, we have similar systems of equations from the Casimir operator and the shift operators.

Let $\pi = \pi_{\Lambda}$ be a discrete series representation of $G$ with the Harish-Chandra parameter $\Lambda \in \Xi_J$ and $\tau^* = \tau^{\star}_{\lambda} \in \hat{K}$ be the minimal $K$-type of $\pi$. Now we refer the following proposition which enables us to identify the intertwining space $I_{\rho, \pi}$ with a solution space of differential equations for any $\rho \in \mathcal{R}_J$.

Proposition 4.1. (Yamashita\cite{9}; Theorem 2.4]) Let $\pi = \pi_{\Lambda} \in \widehat{G}$ and $\tau^* = \tau^{\star}_{\lambda} \in \hat{K}$ be as above. Then we have a linear isomorphism

\[ I_{\rho, \pi} \simeq \bigcap_{\beta \in \Delta_{J^*, n}} \ker(\nabla^{-\beta}_{\rho, \tau}) \subset C^\infty_{\rho, \tau}(R_J \backslash G/K) \]
for any \( \rho \in \hat{R}_J \). In particular,
\[
J_{\rho, \pi}(\tau) = \{ F \in C^\infty_{\rho, \tau}(R_J \backslash G/K) \mid \nabla^{-\beta}_F F = 0, \ \forall \beta \in \Delta^+_J \cdot n \}.
\]
Here the index \( J^* \) means IV, III, II and I for \( J = \text{I}, \text{II}, \text{III} \) and \( \text{IV} \), respectively.

5. Result
Solving the systems of the differential equations given by (4.1), (4.2) and Proposition 4.1, we obtain the following theorem.

Theorem 5.1. Let \( \pi \) be a \( P_J \)-principal series representation (resp. a discrete series representation) of \( G = \text{Sp}(2, \mathbb{R}) \) and \( \tau^* \) be the 'corner' \( K \)-type (resp. the minimal \( K \)-type) of \( \pi \). For each irreducible unitary representation \( \rho \) of \( R_J \) of type \( m \neq 0 \), we have
\[
\dim J^0_{\rho, \pi}(\tau) \leq 1.
\]
Moreover the radial parts of the functions in \( J^0_{\rho, \pi}(\tau) \) are expressed by the Meijer's \( G \)-function \( G^{3,0}_{2,3}(x \mid b_1, b_2, b_3) \) or more degenerate similar functions.

Here the Meijer's \( G \)-function \( G^{3,0}_{2,3}(x) = G^{3,0}_{2,3}(x \mid b_1, b_2, b_3) \) with the complex parameters \( a_i, b_j (1 \leq i \leq 2, 1 \leq j \leq 3) \) is the many-valued function defined by the integral
\[
G^{3,0}_{2,3}(x) = G^{3,0}_{2,3}(x \mid b_1, b_2, b_3) = \frac{1}{2\pi \sqrt{-1}} \int_L \frac{\prod_{j=1}^{3}(b_j - t)}{\prod_{i=1}^{2}(a_i - t)} x^{t} dt
\]
of Mellin-Barnes type, where the contour \( L \) is a loop starting and ending at \( +\infty \) and encircling all poles of \( \Gamma(b_j - t) \) \( (1 \leq j \leq 3) \) once in the negative direction. It is known that, up to constant multiple, \( G^{3,0}_{2,3}(x) \) is the unique solution of the linear differential equation of 3-rd order
\[
\left\{ x^3 \frac{d^3}{dx^3} + \alpha_2(x) x^2 \frac{d^2}{dx^2} + \alpha_1(x) x \frac{d}{dx} + \alpha_0(x) \right\} y = 0
\]
with
\[
\begin{align*}
\alpha_2(x) &= 3 - b_1 - b_2 - b_3 + x, \\
\alpha_1(x) &= (1 - b_1)(1 - b_2)(1 - b_3) + b_1 b_2 b_3 + (3 - a_1 - a_2)x, \\
\alpha_0(x) &= -b_1 b_2 b_3 + (1 - a_1)(1 - a_2)x,
\end{align*}
\]
which decays exponentially as \( |x| \to \infty \) in \(-\frac{3}{2}\pi < \arg x < \frac{1}{2}\pi \) (See the Meijer's original paper [5] for details).

Remark 5.2. Let \( \pi \) be a holomorphic discrete series representation of \( G \) and \( \tau^* \) be the minimal \( K \)-type of \( \pi \). Moreover, put \( \rho = \pi_1 \otimes \tilde{\nu}_m \in \hat{R}_J \) as in \( \S 2 \). For each \( m \neq 0 \), there is at most finitely many \( \rho \) such that \( \dim J^0_{\rho, \pi}(\tau) = 1 \), and then the \( \pi_1 \)-factors of such \( \rho \)'s are the holomorphic discrete series representations of \( \widetilde{SL}(2, \mathbb{R}) \). Moreover, the radial parts of the functions in \( J^0_{\rho, \pi}(\tau) \) are expressed by the function of the form \( x^pe^{q\tau} \) for some constant \( p, q \).
REFERENCES


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