FOURIER-JACOBI TYPE SPHERICAL FUNCTIONS ON $S_p(2,\mathbf{R})$; THE CASE OF $P_J$-PRINCIPAL SERIES AND DISCRETE SERIES (Automorphic Forms and Number Theory)

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FOURIER-JACOBI TYPE SPHERICAL FUNCTIONS ON $Sp(2, \mathbb{R})$;
THE CASE OF $P_J$-PRINCIPAL SERIES AND DISCRETE SERIES

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1. Introduction
In this note, we study a kind of generalized Whittaker models, or equally, of generalized spherical functions associated with automorphic forms on the real symplectic group of degree two. We call these spherical functions 'Fourier-Jacobi type', since these are closely connected with the coefficients of the 'Fourier-Jacobi expansions' of (holomorphic or non-holomorphic) automorphic forms. Also these can be considered as a non-holomorphic analogue of the local Whittaker-Shintani functions on $Sp(2, \mathbb{R})$ of Fourier-Jacobi type in the paper of Murase and Sugano [6].

2. Preliminaries

2.1. Groups and algebras. We denote by $\mathbb{Z}_{\geq m}$ the set of integers $n$ such that $n \geq m$. Moreover, we use the convention that unwritten components of a matrix are zero.

Let $G$ be the real symplectic group $Sp(2, \mathbb{R})$ of degree two given by

$$Sp(2, \mathbb{R}) = \left\{ g \in M_4(\mathbb{R}) \mid {}^t gJ_2 g = J_2 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \det g = 1 \right\}.$$ 

Let $\theta(g) = {}^t \bar{g}^{-1} (g \in G)$ be a Cartan involution of $G$ and $K$ be the set of fixed points of $\theta$. Then $K$ becomes a maximal compact subgroup of $G$ which is isomorphic to the unitary group $U(2)$.

Let $\mathfrak{g} = \{ X \in M_4(\mathbb{R}) \mid J_2 X + {}^t X J_2 = 0 \}$ be the Lie algebra of $G$. If we denote the differential of $\theta$ again by $\theta$, then we have $\theta(X) = -{}^t \bar{X} (X \in \mathfrak{g})$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and $-1$ eigenspaces of $\theta$ in $\mathfrak{g}$, respectively, and hence

$$\mathfrak{k} = \left\{ X \in \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A, B \in M_2(\mathbb{R}), {}^t A = -A, {}^t B = B \right\},$$

$$\mathfrak{p} = \left\{ X \in \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A, B \in M_2(\mathbb{R}), {}^t A = A, {}^t B = B \right\}. $$
Then we have a Cartan decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$. Of course, $\mathfrak{k}$ is the Lie algebra of $K$ which is isomorphic to the unitary algebra $u(2)$.

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_\mathbb{C} = \mathfrak{l} \otimes \mathbb{C}$ the complexification of $\mathfrak{l}$. Let $\mathfrak{h}$ be a compact Cartan subalgebra of $g$ given by

$$\mathfrak{h} = \left\{ H(\theta_1, \theta_2) = \begin{pmatrix} \theta_1 & & \\ -\theta_1 & \theta_2 \\ & -\theta_2 \end{pmatrix} \bigg| \theta_i \in \mathbb{R} \right\}.$$ 

Now we identify a linear form $\beta : \mathfrak{h}_\mathbb{C} \to \mathbb{C}$ with $(\beta_1, \beta_2) \in \mathbb{C}^2$ via $\beta = \beta_1 e_1 + \beta_2 e_2$, where $e_i(H(\theta_1, \theta_2)) = \sqrt{-1} \theta_i$. Then the set of roots $\Delta = \Delta(\mathfrak{h}_\mathbb{C}, \mathfrak{g}_\mathbb{C})$ of $(\mathfrak{h}_\mathbb{C}, \mathfrak{g}_\mathbb{C})$ is given by

$$\Delta = \{ \pm(2, 0), \pm(0, 2), \pm(1, 1), \pm(1, -1) \}.$$ 

Fix a positive root system $\Delta^+ = \{(2, 0), (0, 2), (1, 1), (1, -1)\}$, and put $\Delta_c^+$ and $\Delta_n^+$ the set of compact and non-compact positive roots, respectively. Then

$$\Delta_c^+ = \{(1, -1)\}, \quad \Delta_n^+ = \{(2, 0), (0, 2), (1, 1)\}.$$ 

If we denote the root space for $\beta \in \Delta$ by $\mathfrak{g}_\beta$, then we have a decomposition $\mathfrak{p}_\mathbb{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ with $\mathfrak{p}_+ = \sum_{\beta \in \Delta^+_c} \mathfrak{g}_\beta$ and $\mathfrak{p}_- = \sum_{\beta \in \Delta^+_n} \mathfrak{g}_\beta$.

Put $P_J$ the Jacobi maximal parabolic subgroup of $G$ with the Langlands decomposition $P_J = M_J A_J N_J$, where

$$M_J = \left\{ \begin{pmatrix} \varepsilon & b & \\ a & \varepsilon & c \\ & d & \end{pmatrix} \bigg| \varepsilon \in \{ \pm 1 \}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \right\} \simeq \{ \pm I \} \times SL(2, \mathbb{R}),$$

$$N_J = \left\{ n(x, y; z) = \begin{pmatrix} 1 & y & \\ & 1 & \\ & & 1 - y \end{pmatrix}, \begin{pmatrix} 1 & z & x \\ & 1 & \\ & & 1 \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\},$$

and $A_J = \{ \text{diag}(a, 1, a^{-1}, 1) \mid a > 0 \}$. Remark that the unipotent radical $N_J$ of $P_J$ is isomorphic to the 3-dimensional Heisenberg group $\mathcal{H}_1$. The Levi part $M_J A_J$ of $P_J$ acts on $N_J$ via the conjugate action, and $M_J$ gives the centralizer of the center $Z(N_J) = \{ n(0, 0; z) \mid z \in \mathbb{R} \} \simeq \mathbb{R}$ of $N_J$ in $M_J A_J$. Now we define the Jacobi group $R_J$ by the semidirect product $M_J^0 \ltimes N_J \simeq SL(2, \mathbb{R}) \ltimes \mathcal{H}_1$, where $M_J^0 \simeq SL(2, \mathbb{R})$ is the identity component of $M_J$.

2.2. Representations. First we investigate the irreducible unitary representations of the Jacobi group $R_J$. Since $Z(R_J) = Z(N_J) \simeq \mathbb{R}$, the central characters of elements in $\hat{R}_J$ and $\hat{N}_J$ are parametrized by the real numbers. Then we call an irreducible unitary representation of $R_J$ and $N_J$ of type $m$ if its central character is of the form $z \mapsto e^{2\pi \sqrt{-1} mz}$ with $m \in \mathbb{R}$. Let $\nu \in \hat{N}_J$ of type $m$. According to the
theorem of Stone-von Neumann (cf. Corwin-Greenleaf [1; pp.46-47, 51-52]), \( \nu \) is a character if \( m = 0 \) and \( \nu \) is infinite dimensional if \( m \neq 0 \). Moreover \( \nu \) of type \( m \neq 0 \) is uniquely determined by \( m \) up to unitary equivalence. Now we fix an irreducible unitary representation \( (\nu_m, \mathcal{U}_m) \) of \( N_J \) of type \( m \neq 0 \). From the theory of the Weil representation, \( (\nu_m, \mathcal{U}_m) \) can be extended to a continuous true projective unitary representation \( (\hat{\nu}_m, \mathcal{U}_m) \) of \( R_J \) by \( \hat{\nu}_m(\tilde{n}) = W_m(g)\nu_m(n) \) for \( \tilde{n} = g \cdot n \in M^\circ_J \ltimes \tilde{N}_J \) with the Weil representation \( W_m \) on \( M^\circ_J \). Here \( \hat{\nu}_m \) has a factor set \( \alpha \) which is a proper 2-cocycle.

**Lemma 2.1.** (Satake [7; Appendix I, Proposition 2]) Let \( \hat{\nu}_m \) \( (m \neq 0) \) as above. For every irreducible projective unitary representation \( \pi \) of \( M^\circ_J \) with factor set \( \alpha^{-1} \), put \( \rho(\tilde{n}) = \pi(g) \otimes \hat{\nu}_m(\tilde{n}) \) for \( \tilde{n} = g \cdot n \in M^\circ_J \ltimes \tilde{N}_J \). Then \( \rho \) is an irreducible unitary representation of \( R_J \). Conversely, all irreducible unitary representations of \( R_J \) of type \( m \neq 0 \) are obtained in this manner. Moreover \( \rho \) is square-integrable iff \( \pi \) is so.

Let \( (\rho, \mathcal{F}_\rho) \) be an irreducible unitary representation of \( R_J \) of type \( m \neq 0 \). From the above lemma, we can regard \( (\rho, \mathcal{F}_\rho) \in \hat{R}_J \) as a tensor product representation \( (\pi_1 \otimes \hat{\nu}_m, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m) \). Here, if we write \( \overline{M}^\circ_J \) for the double cover of \( M^\circ_J \simeq SL(2, \mathbb{R}) \), \( (\hat{\nu}_m, \mathcal{U}_m) \) is a unitary representation of \( \overline{M}^\circ_J \ltimes N_J \) which is extended from \( (\nu_m, \mathcal{U}_m) \in \tilde{N}_J \) as above and \( (\pi_1, \mathcal{W}_{\pi_1}) \) is a unitary representation of \( \overline{M}^\circ_J \) which does not factor through \( M^\circ_J \). On the other hand, the unitary dual of \( \overline{M}^\circ_J \) is given as follows.

**Proposition 2.2.** (cf. Gelbert[2; Lemma 4.1, 4.2]) The following representations exhaust a set of representatives for the equivalence classes of irreducible unitary representations of \( SL(2, \mathbb{R}) \).

1. *(unitary principal series)* \( \mathcal{P}^\tau_s \), \( s \in \sqrt{-1}\mathbb{R}, \tau = 0, 1, \pm \frac{1}{2} \) except for the case \( (s, \tau) = (0, 1) \).
2. *(complementary series)* \( \mathcal{C}^\tau_s \), \( 0 < s < 1 \) for \( \tau = 0, 1 \) and \( 0 < s < \frac{1}{2} \) for \( \tau = \pm \frac{1}{2} \).
3. *(limit of)* discrete series \( D^\pm_k \), \( k \in \frac{1}{2}\mathbb{Z}_{\geq 2} \).
4. *(quotient representation)* \( D^\pm_{\frac{1}{2}}, D^+_{\frac{1}{2}} \).
5. *(the trivial representation of)* \( SL(2, \mathbb{R}) \).

In the above, the representations \( \mathcal{P}^\tau_s, \mathcal{C}^\tau_s \) for \( \tau = 0, 1, D^\pm_k \) for \( k \in \mathbb{Z}_{\geq 1} \) and (5) factor through \( SL(2, \mathbb{R}) \), and the otherwise not.

Hence we take as \( (\pi_1, \mathcal{W}_{\pi_1}) \) one of the irreducible unitary representations \( \mathcal{P}^\tau_s, \mathcal{C}^\tau_s \) with \( \tau = \pm \frac{1}{2} \) and \( D^\pm_k \) with \( k \in \frac{1}{2}\mathbb{Z}\backslash \mathbb{Z}, k \geq \frac{1}{2} \).

**Remark 2.3.** The Weil representation \( W_m \) considered as the representation of \( \overline{M}^\circ_J \) has the following irreducible decomposition;

\[
W_m = \begin{cases} 
D^+_{\frac{1}{2}} \oplus D^+_{\frac{3}{2}}, & \text{if } m > 0, \\
D^-_{\frac{1}{2}} \oplus D^-_{\frac{3}{2}}, & \text{if } m < 0.
\end{cases}
\]

Next, we treat the irreducible unitary representations of \( K \). Since \( \Delta^+_c \) is also a positive system of \( \Delta(\mathfrak{g}, \mathfrak{h}_C) \), then the set of the \( \Delta^+_c \)-dominant weights, and thus
$\hat{K}$, is parametrized by the set

$$\Lambda = \{ \lambda = (\lambda_1, \lambda_2) \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \}$$

(cf. Knapp [4; Theorem 4.28]). We denote by $(\tau_\lambda, V_\lambda)$ the element of $\hat{K}$ corresponding to $\lambda = (\lambda_1, \lambda_2) \in \Lambda$. Here $\dim V_\lambda = d_\lambda + 1$ with $d_\lambda = \lambda_1 - \lambda_2$.

Both of $p_\pm$ become $K$-modules via the adjoint representation of $K$, and we have isomorphisms $p_+ \simeq V_{(2,0)}$ and $p_- \simeq V_{(0,-2)}$. For a given irreducible $K$-module $V_\lambda$ with the parameter $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, the tensor product $K$-modules $V_\lambda \otimes p_+$ and $V_\lambda \otimes p_-$ have the irreducible decompositions

$$V_\lambda \otimes p_+ \simeq \bigoplus_{\beta \in \Delta^+_n} V_{\lambda+\beta}, \quad V_\lambda \otimes p_- \simeq \bigoplus_{\beta \in \Delta^-_n} V_{\lambda-\beta}.$$  

For each $\beta \in \Delta^+_n$, let $P^\beta : V_\lambda \otimes p_+ \rightarrow V_{\lambda+\beta}$ and $P^-\beta : V_\lambda \otimes p_- \rightarrow V_{\lambda-\beta}$ be the projectors into the irreducible factors of $V_\lambda \otimes p_\pm$.

In this note, we consider the following two series of representations of $G$; one is the principal series induced from $P_J$, and the other is the discrete series. We explain these representations in the remaining of this section.

Let $\sigma = (\epsilon, D)$ be a representation of $M_J \simeq \{\pm I\} \times SL(2, \mathbb{R})$ with a character $\epsilon : \{\pm I\} \rightarrow \mathbb{C}^\times$ and a discrete series representation $D = D^{\pm}_n (n \in \mathbb{Z}_{\geq 2})$ of $SL(2, \mathbb{R})$, and take a quasi-character $\nu_z$ ($z \in \mathbb{C}$) of $A_J$ such that $\nu_z(diag(a, 1, a^{-1}, 1)) = a^z$. Then we can construct a induced representation $\text{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$ of $G$ from the Jacobi maximal parabolic subgroup $P_J = M_J A_J N_J$ by the usual manner (cf. Knapp [4; Chapter VII]), and call $\text{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$ the $P_J$-principal series representation of $G$. The following lemma is derived from the Frobenius reciprocity for induced representations.

**Lemma 2.4.** $\tau_\lambda \in \hat{K}$ with the parameter $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ such that $\lambda_1 < n$ (resp. $\lambda_2 > -n$) does not occur in the $K$-type of $\text{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$ for $D = D^+_n$ (resp. $D^-_n$). The 'comer' $K$-types $\tau_\lambda \in \hat{K}$ of $\text{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$ with the parameter $\lambda \in \Lambda$ given below occur with multiplicity one.

1. $\lambda = (n, n)$ for $\epsilon(\gamma) = (-1)^n$ and $D = D^+_n$,
2. $\lambda = (n, n-1)$ for $\epsilon(\gamma) = -(1)^n$ and $D = D^-_n$,
3. $\lambda = (-n, -n)$ for $\epsilon(\gamma) = (-1)^n$ and $D = D^-_n$,
4. $\lambda = (-n+1, -n)$ for $\epsilon(\gamma) = -(1)^n$ and $D = D^-_n$.

Here $\gamma = \text{diag}(-1, 1, -1, 1)$.

In order to parametrize the discrete series representations of $G$, we enumerate all the positive root systems compatible to $\Delta^+_c$:

1. $\Delta^+_I = \{(1, -1), (2, 0), (1, 1), (0, 2)\}$,
2. $\Delta^+_II = \{(1, -1), (2, 0), (1, 1), (0, -2)\}$,
3. $\Delta^+_III = \{(1, -1), (2, 0), (0, -2), (-1, -1)\}$,
4. $\Delta^+_IV = \{(1, -1), (0, -2), (-1, -1), (-2, 0)\}$. 


Let $J$ be a variable running over the set of indices I, II, III, IV, and let us denote the set of non-compact positive roots for the index $J$ by $\Delta^+_J = \Delta^+_J - \Delta_c^+$. Define a subset $\Xi_J$ of $\Delta^+_c$-dominant weights by

$$\Xi_J = \{\Lambda = (\Lambda_1, \Lambda_2), \Delta^+_c \text{- dominant weight } | \langle \Lambda, \beta \rangle > 0, \forall \beta \in \Delta^+_J \}.$$ 

The set $\bigcup_{J=1}^{IV} \Xi_J$ gives the Harish-Chandra parametrization of the discrete series representation of $G$. Let us write by $\pi_J$ the discrete series representation of $G$ with the Harish-Chandra parameter $\Lambda \in \bigcup_{J=1}^{IV} \Xi_J$. Then $\pi_{\Lambda}$ is called the holomorphic discrete series representation if $\Lambda \in \Xi_I$ and the anti-holomorphic one if $\Lambda \in \Xi_{IV}$. Moreover if $\Lambda \in \Xi_{II} \cup \Xi_{III}$, a discrete series representation $\pi_{\Lambda}$ is called large (in the sense of Vogan[8]).

The Blattner formula gives the description of the $K$-types of $\pi_{\Lambda}$. In particular, the minimal $K$-type $(\tau_{\lambda}, V_{\lambda})$ of $\pi_{\Lambda}$ is given by the formula $\lambda = \Lambda - \rho_c + \rho_n$, where $\rho_c$ (resp. $\rho_n$) is the half sum of compact (resp. non-compact) positive roots in $\Delta^+_J$.

We call such $\lambda$ the Blattner parameter of $\pi_{\Lambda}$.

3. Fourier-Jacobi type spherical functions

3.1. Radial parts. Let $(\rho, F_{\rho})$ be an irreducible unitary representation of $R_J$ and let $(\tau, V_{\tau})$ be a finite dimensional $K$-module. We denote by $C^\infty_{\rho, \tau}(R_J \backslash G/K)$ the space of smooth functions $F : G \to F_{\rho} \otimes V_{\tau}$ with the property

$$F(r g k) = (\rho(r) \otimes \tau(k)^{-1}) F(g), \quad (r, g, k) \in R_J \times G \times K.$$ 

On the other hand, let $C^\infty(A_J; \rho, \tau)$ be the space of smooth functions $\varphi : A_J \to F_{\rho} \otimes V_{\tau}$ satisfying

$$(\rho(m) \otimes \tau(m)) \varphi(a) = \varphi(a), \quad m \in R_J \cap K = M^J_J \cap K, \quad a \in A_J.$$ 

Because of an Iwasawa decomposition of $G$, we have $G = R_J A_J K$. Also we remark that all elements in $M^J_J \cap K$ are commutative with $a \in A_J$. Then the restriction to $A_J$ gives a linear map from $C^\infty_{\rho, \tau}(R_J \backslash G/K)$ to $C^\infty(A_J; \rho, \tau)$, which is injective. For each $f \in C^\infty_{\rho, \tau}(R_J \backslash G/K)$, we call $f|_{A_J} \in C^\infty(A_J; \rho, \tau)$ the radial part of $f$, where $|_{A_J}$ means the restriction to $A_J$.

Let $(\tau', V_{\tau'})$ be also a finite dimensional $K$-module. For each $C$-linear map $u : C^\infty_{\rho, \tau}(R_J \backslash G/K) \to C^\infty_{\rho', \tau}(R_J \backslash G/K)$, we have a unique $C$-linear map $R(u) : C^\infty(A_J; \rho, \tau) \to C^\infty(A_J; \rho, \tau')$ with the property $(u f)|_{A_J} = R(u)(f|_{A_J})$ for $f \in C^\infty_{\rho, \tau}(R_J \backslash G/K)$. We call $R(u)$ the radial part of $u$.

3.2. Fourier-Jacobi type spherical functions. Let $(\rho, F_{\rho})$ be as above and consider a $C^\infty$-induced representation $C^\infty \text{Ind}^G_{R_J}(\rho)$ with the representation space

$$C^\infty_{\rho}(R_J \backslash G) = \{F : G \to F_{\rho}, \quad C^\infty | F(r g) = \rho(r)F(g), \quad (r, g) \in R_J \times G\}$$ 

on which $G$ acts by the right translation. Then $C^\infty_{\rho}(R_J \backslash G)$ becomes a smooth $G$-module and a $(\mathfrak{g}_C, K)$-module naturally. Moreover let $(\tau, V_{\tau}) \in \hat{K}$ and take an
irreducible Harish-Chandra module $\pi$ of $G$ with the $K$-type $\tau^*$, where $\tau^*$ is the contragredient representation of $\tau$. Now we consider the intertwining space

$$I_{\rho, \pi} := \text{Hom}_{(g_C, K)}(\pi, C^\infty \text{Ind}_{R_J}^G(\rho))$$

between $(g_C, K)$-modules and its restriction to the $K$-type $\tau^*$ of $\pi$.

Let $i : \tau^* \to \pi|_K$ be a $K$-equivariant map and let $i^*$ be the pullback via $i$. Then the map

$$I_{\rho, \pi} \xrightarrow{i^*} \text{Hom}_K(\tau^*, C^\infty_R(\tau^*|_{R_J\backslash G})) \cong C^\infty_{\rho, \tau}(\tau^*|_{R_J\backslash G/K})$$

gives the restriction of $T \in I_{\rho, \pi}$ to the $K$-type $\tau^*$ and we denote the image of $T$ in $C^\infty_{\rho, \tau}(\tau^*|_{R_J\backslash G/K})$ by $T_i$. Now the space $J_{\rho, \pi}(\tau)$ of the algebraic Fourier-Jacobi type spherical functions of type $(\rho, \pi; \tau)$ on $G$ is defined by

$$J_{\rho, \pi}(\tau) := \bigcup_{i \in \text{Hom}_K(\tau^*, \pi|_K)} \{T_i | T \in I_{\rho, \pi}\}.$$

Moreover put

$$J^\circ_{\rho, \pi}(\tau) = \{f \in J_{\rho, \pi}(\tau) | f|_{A_J}(\text{diag}(a, 1, a^{-1}, 1)) \text{ is of moderate growth as } a \to \infty\}.$$

We call $f \in J^\circ_{\rho, \pi}(\tau)$ a Fourier-Jacobi type spherical functions of type $(\rho, \pi; \tau)$.

In this note, we investigate the space $J^\circ_{\rho, \pi}(\tau)$ for the following triplet $(\rho, \pi; \tau)$: As $\pi \in \hat{G}$ and $\tau^* \in \hat{K}$, we take either the $P_J$-principal series representation and the corner $K$-type or the discrete series representation and the minimal $K$-type, and also as $\rho \in \hat{R}_J$ the one with the non-trivial central character, i.e. of type $m \neq 0$.

4. Differential equations

4.1 Differential operators. In this subsection, we introduce some differential operators acting on $C^\infty_{\rho, \tau}(\tau^*|_{R_J\backslash G/K})$.

Take an orthonormal basis $\{X_i\}$ of $p$ with respect to the Killing form of $g$. Now we define a first order gradient type differential operator

$$\nabla_{\rho, \tau} : C^\infty_{\rho, \tau}(R_J\backslash G/K) \to C^\infty_{\rho, \tau \oplus \text{Ad}_{p_C}}(R_J\backslash G/K)$$

by

$$\nabla_{\rho, \tau} f = \sum_i R_{X_i} f \otimes X_i, \quad f \in C^\infty_{\rho, \tau}(R_J\backslash G/K),$$

where

$$R_{X} f(g) = \frac{d}{dt} f(g \cdot \exp(tX))|_{t=0}, \quad X \in g_C, \ g \in G.$$

This differential operator $\nabla_{\rho, \tau}$ is called the Schmid operator. Then $\nabla_{\rho, \tau}$ can be decomposed as $\nabla_{\rho, \tau}^+ \oplus \nabla_{\rho, \tau}^-$ with $\nabla_{\rho, \tau}^\pm : C^\infty_{\rho, \tau}(R_J\backslash G/K) \to C^\infty_{\rho, \tau \oplus \text{Ad}_{p_{\pm}}}(R_J\backslash G/K)$ corresponding to the decomposition $p_C = p_+ \oplus p_-$. For each $\beta \in \Delta^+_n$, the shift operator $\nabla_{\rho, \tau, \lambda}^\beta : C^\infty_{\rho, \tau, \lambda}(R_J\backslash G/K) \to C^\infty_{\rho, \tau, \lambda \pm \beta}(R_J\backslash G/K)$ is defined as the composition of
$\nabla_{\rho,\tau_{\lambda}}^{\pm\beta}$ with the projector $P_{\rho,\tau_{\lambda}}^{\pm\beta}$ from $V_{\tau_{\lambda}} \otimes p_{\rho,\tau_{\lambda}}^{\pm\beta}$ into the irreducible component $V_{\tau_{\lambda}^{\pm\beta}}$; $\nabla_{\rho,\tau_{\lambda}}^{\pm\beta} = (1_{F_{\rho}} \otimes P^{\pm\beta}) \nabla_{\rho,\tau_{\lambda}}^{\pm}.

On the other hand, the Casimir element $\Omega$ is defined by $\Omega = \sum X_{i} - \sum Y_{j}$, where $\{Y_{j}\}$ is an orthonormal basis of $\mathfrak{f}$ with respect to the Killing form of $\mathfrak{g}$. It is well known that $\Omega$ is in the center $Z(\mathfrak{g}_{C})$ of the universal enveloping algebra of $\mathfrak{g}_{C}$.

4.2. Differential equations. In this subsection, we consider the system of differential equations satisfied by the Fourier-Jacobi type spherical functions.

First we discuss the case of the $P_{J}$-principal series representation $\pi \in \hat{G}$ and the corner $K$-type $\tau^{*}$. It is well known that the Casimir element $\Omega \in Z(\mathfrak{g}_{C})$ acts on $\pi$, hence on $J_{\rho,\pi}(\tau)$, as the scalar operator $\chi_{\Omega}$ (cf. Knapp[4; Corollary 8.14]). Let $\pi = \text{Ind}_{P_{J}}^{G}(\sigma \otimes \nu_{z} \otimes 1_{N_{J}})$ with data $\sigma = (\epsilon, D_{n}^{+}), \epsilon(\gamma) = (-1)^{n}$, and $\tau^{*} = \tau_{\lambda}^{*}$ be the corner $K$-type of $\pi$, i.e. $\lambda = (-n, -n)$. Since $\tau_{\lambda}^{*} = \tau_{(n-2,n-2)} \in \hat{K}$ does not occur in the $K$-types of $\pi$ from Lemma 2.4, an element in $J_{\rho,\pi}(\tau)$ is annihilated by the action of the composition of the shift operators

$$\nabla_{\rho,\tau_{\lambda}^{*}+(2,2)}^{(0,2)} \circ \nabla_{\rho,\tau_{\lambda}^{*}}^{(2,0)} : C_{\rho,\tau_{\lambda}^{*}}(R_{J}\backslash G/K) \rightarrow C_{\rho,\tau_{\lambda}^{*}+(2,2)}(R_{J}\backslash G/K).$$

Hence we have a system of differential equations satisfied by $f$ in $J_{\rho,\pi}(\tau)$;

$$\begin{cases}
\Omega f = \chi_{\Omega}f, \\
\nabla_{\rho,\tau_{\lambda}^{*}+(2,0)}^{(0,2)} \circ \nabla_{\rho,\tau_{\lambda}^{*}}^{(2,0)} f = 0.
\end{cases}$$

(4.1)

Let $\pi = \text{Ind}_{P_{J}}^{G}(\sigma \otimes \nu_{z} \otimes 1_{N_{J}})$ with data $\sigma = (\epsilon, D_{n}^{+}), \epsilon(\gamma) = (-1)^{n}$, and $\tau^{*} = \tau_{\lambda}^{*}$ be the corner $K$-type of $\pi$, i.e. $\lambda = (-n+1, -n)$. Since $\tau_{\lambda}^{*} = \tau_{(n-2,n-1)} \in \hat{K}$ does not occur in the $K$-types of $\pi$ from Lemma 2.4, therefore an element in $J_{\rho,\pi}(\tau)$ vanishes by the action of the shift operator

$$\nabla_{\rho,\tau_{\lambda}^{*}+(1,1)}^{(1,1)} : C_{\rho,\tau_{\lambda}^{*}}(R_{J}\backslash G/K) \rightarrow C_{\rho,\tau_{\lambda}^{*}+(1,1)}(R_{J}\backslash G/K).$$

Hence we have a system of differential equations satisfied by $f$ in $J_{\rho,\pi}(\tau)$;

$$\begin{cases}
\Omega f = \chi_{\Omega}f, \\
\nabla_{\rho,\tau_{\lambda}^{*}+(1,1)}^{(1,1)} f = 0.
\end{cases}$$

(4.2)

For the case with the data $\sigma = (\epsilon, D_{n}^{+})$, we have similar systems of equations from the Casimir operator and the shift operators.

Let $\pi = \pi_{\Lambda}$ be a discrete series representation of $G$ with the Harish-Chandra parameter $\Lambda \in \Xi_{J}$ and $\tau^{*} = \tau_{\Lambda}^{*} \in \hat{K}$ be the minimal $K$-type of $\pi$. Now we refer the following proposition which enables us to identify the intertwining space $I_{\rho,\pi}$ with a solution space of differential equations for any $\rho \in \hat{R}_{J}$.

**Proposition 4.1.** (Yamashita [9; Theorem 2.4]) Let $\pi = \pi_{\Lambda} \in \hat{G}$ and $\tau^{*} = \tau_{\Lambda}^{*} \in \hat{K}$ be as above. Then we have a linear isomorphism

$$I_{\rho,\pi} \simeq \bigcap_{\beta \in \Delta_{J}^{*}, \rho,\tau_{\Lambda}^{*}} \ker(\nabla_{\rho,\tau_{\Lambda}^{*}}^{-\beta}) \subset C_{\rho,\tau_{\Lambda}^{*}}(R_{J}\backslash G/K).$$
for any $\rho \in \hat{R}_J$. In particular,

$$J_{\rho,\pi}(\tau) = \{ F \in C^\infty(R_J \backslash G/K) \mid \nabla_{\rho,\tau}^{-\beta} F = 0, \quad \forall \beta \in \Delta^*_{J,n} \}. $$

Here the index $J^*$ means IV, III, II and I for $J = I$, II, III and IV, respectively.

5. Result

Solving the systems of the differential equations given by (4.1), (4.2) and Proposition 4.1, we obtain the following theorem.

**Theorem 5.1.** Let $\pi$ be a $P_J$-principal series representation (resp. a discrete series representation) of $G = \text{Sp}(2, \mathbb{R})$ and $\tau^*$ be the 'corner' $K$-type (resp. the minimal $K$-type) of $\pi$. For each irreducible unitary representation $\rho$ of $R_J$ of type $m \neq 0$, we have

$$\dim J^0_{\rho,\pi}(\tau) \leq 1.$$ 

Moreover the radial parts of the functions in $J^0_{\rho,\pi}(\tau)$ are expressed by the Meijer's $G$-function $G^3,0_{2,3}(x \mid a_{1},a_{2} \mid b_{1},b_{2},b_{3})$ or more degenerate similar functions.

Here the Meijer's $G$-function $G^3,0_{2,3}(x) = G^3,0_{2,3}(x \mid a_{1},a_{2} \mid b_{1},b_{2},b_{3})$ with the complex parameters $a_{i}, b_{j} (1 \leq i \leq 2, 1 \leq j \leq 3)$ is the many-valued function defined by the integral

$$G^3,0_{2,3}(x) = C^3,0_{2,3}(x \mid a_{1},a_{2} \mid b_{1},b_{2},b_{3}) = \frac{1}{2\pi \sqrt{-1}} \int_{L} \prod_{j=1}^{3} \Gamma(b_{j}-t)(1 \leq j \leq 3) \prod_{i=1}^{2} \Gamma(a_{i}-t) x^{t} dt$$

of Mellin-Barnes type, where the contour $L$ is a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma(b_{j}-t)$ $(1 \leq j \leq 3)$ once in the negative direction. It is known that, up to constant multiple, $G^3,0_{2,3}(x)$ is the unique solution of the linear differential equation of 3-rd order

$$\left\{ x^{3} \frac{d^{3}}{dx^{3}} + \alpha_{2}(x) x^{2} \frac{d^{2}}{dx^{2}} + \alpha_{1}(x) x \frac{d}{dx} + \alpha_{0}(x) \right\} y = 0$$

with

$$\alpha_{2}(x) = 3 - b_{1} - b_{2} - b_{3} + x, $$

$$\alpha_{1}(x) = (1 - b_{1})(1 - b_{2})(1 - b_{3}) + b_{1}b_{2}b_{3} + (3 - a_{1} - a_{2})x,$$

$$\alpha_{0}(x) = -b_{1}b_{2}b_{3} + (1 - a_{1})(1 - a_{2})x,$$

which decays exponentially as $|x| \to \infty$ in $-\frac{3}{2} \pi < \arg x < \frac{1}{2} \pi$ (See the Meijer's original paper [5] for details).

**Remark 5.2.** Let $\pi$ be a holomorphic discrete series representation of $G$ and $\tau^*$ be the minimal $K$-type of $\pi$. Moreover, put $\rho = \pi_{1} \otimes \tilde{\nu}_{m} \in \hat{R}_J$ as in §2. For each $m \neq 0$, there is at most finitely many $\rho$ such that $\dim J^0_{\rho,\pi}(\tau) = 1$, and then the $\pi_{1}$-factors of such $\rho$'s are the holomorphic discrete series representations of $\widetilde{SL}(2, \mathbb{R})$. Moreover, the radial parts of the functions in $J^0_{\rho,\pi}(\tau)$ are expressed by the function of the form $x^{p}e^{qx}$ for some constant $p$, $q$. 
REFERENCES


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