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The Complexity of Negation-Limited Inverters

Shao-Chin Sung[†] Keisuke Tanaka[‡]
(宋 少秋) (田中 圭介)

Summary

We mainly consider the complexity of *negation-limited inverters.* We show an upper bound $d+3\lceil\log(n+1)\rceil$ on depth for negation-limited inverter whose size is $O(ns+n^{2})$, where d and s is, respectively, the depth and the size of a monotone sorting network with optimal depth. If the optimal depth of monotone circuit for the *majority* function is not less than that for any other threshold function, this upper bound match the lower bound [11] and a superlinear lower bound $\Omega(n\log n)$ on size of depth-optimal negation-limited inverter on basis $\{\mathrm{AND},\mathrm{OR},\mathrm{NAND},\mathrm{NOR}\}$ is shown.

1 Introduction

1.1 Background and Motivation

We do not know much about the complexities of combinational circuits (i.e., circuits with AND, OR, and NEGATION gates) for explicitly defined functions. Even a superlinear lower bound for the size of combinational circuits is not known. The complexities of monotone circuits (i.e., combinational circuits *without* NEGATION gates) for many explicitly defined functions are well understood. For example, exponential lower bounds for the size of monotone circuits are known [2, 8, 9]. Exponential gaps between monotone and combinational circuit complexity are also shown [8, 12]. So, we cannot generally derive strong lower bounds for the combinational circuit complexity using those bounds for the monotone circuit complexity.

In this situation, there is no doubt that it is necessary to understand the effect of NEGATION gates in order to obtain strong lower bounds for combinational circuit complexity. This is a motivation for the study of *negation-limited circuit complexity*, the complexity for circuits in which the number of NEGATION gates are restricted.

Markov [7] gave the number of NEGATION gates which is necessary and sufficient to compute a system of boolean functions. Especially, he showed that $\lceil\log(n+1)\rceil$ NEGATION gates are sufficient to compute *any* system of boolean functions (all logarithms and exponents in this paper

† School of Information Science, Japan Advanced Institute of Science and Technology, Tatsunokuchi, Ishikawa 923-12, Japan. son@jaist.ac.jp
‡ Distributed Systems Architecture Laboratory, NTT Information and Communication Systems Laboratories, 1-1 Hikarinooka Yokosuka-shi, Kanagawa 239 Japan. keisuke@sucaba.isl.ntt.co.jp

are base two). In [5, 6], Fischer constructed a polynomial size circuit that contains only $\lceil\log(n+1)\rceil$ NEGATION gates, and inverts n input variables. Owing to this result, for *any* n-input boolean function f, if there exists a polynomial size combinational circuit for f, there also exists a polynomial size circuit with only $\lceil\log(n+1)\rceil$ NEGATION gates. Tanaka and Nishino [11], and Beals, Nishino, and Tanaka [3] improved the result of Fischer, and also gave lower bounds for the negation-limited circuit complexity of several functions. On the other hand, Santha and Wilson [10] studied the negation-limited complexity of constant depth circuits.

1.2 Definitions and Preliminaries

Let $\tilde{x} = \{x_1, \ldots, x_n\}$ be the set of inputs. Let $f = (f^1, \ldots, f^m)$ be an n-input m-output *boolean function* $\{0,1\}^n \to \{0,1\}^m$. A theorem of Markov [7] precisely determines the number r of NEGATION gates necessary and sufficient to compute f. A *chain* C in the boolean lattice $\{0,1\}^n$ is an increasing sequence $a^1 < \ldots < a^k \in \{0,1\}^n$. The *decrease* of f on C is the number of $i \leq k$ such that $f^j(a^{i-1}) > f^j(a^i)$ for some $1 \leq j \leq m$. We define $d(f)$ to be the maximum decrease of f on any chain C. Markov has shown that $\lceil\log(d(f)+1)\rceil$ NEGATION gates are necessary and sufficient to compute any f.

We denote by $C(f)$ the *circuit complexity* of f, i.e., the size (number of gates) of the smallest circuit of AND, OR, and NEGATION gates with inputs $x_1, \ldots, x_n$ and outputs $f^1(\tilde{x}), \ldots, f^m(\tilde{x})$. We call a circuit including at most r NEGATION gates an *r-circuit.* Especially, when $r = 0$, an r-circuit is said to be *monotone.* We denote by $C^r(f)$ the size of the smallest r-circuit computing f for $r \geq \lceil\log(d(f)+1)\rceil$.

Let $|\tilde{x}|$ denote the number of ones in $\tilde{x}$. We define the n-input *parity* function as a function which returns one iff $|\tilde{x}|$ is odd, and the n-input *k-th threshold* function as a function which returns one iff $|\tilde{x}| \geq k$ for $1 \leq k \leq n$, i.e., the *majority* function is the $\lceil n/2\rceil$-th threshold function.

1.3 Main Results

We first show an upper bound $d + 3\lceil\log(n+1)\rceil$ on depth for negation-limited inverter whose size is $O(ns + n^2)$, where d and s is, respectively, the depth and the size of a monotone sorting network with optimal depth. This upper bound matches the lower bound shown by Tanaka and Nishino [11] under the following natural assumption.

Assumption A *The optimal depth of monotone circuit for the majority function is not less than that for any other threshold function.*

Under this assumption, a superlinear lower bound $\Omega(n\log n)$ on size of depth-optimal negation-limited inverter on basis $\{\mathrm{AND}, \mathrm{OR}, \mathrm{NAND}, \mathrm{NOR}\}$ is shown.

2 Properties of Negation-Limited Inverters

Let $n+1$ be a power of two, and let $b(n) = \lceil\log(n+1)\rceil$. An inverter is an n-input n-output boolean function $I_n = (I_n^1, \ldots, I_n^n)$, where $I_n^i(\tilde{x}) = \neg x_i$ for all $1 \leq i \leq n$. A theorem of Markov implies that $b(n)$ NEGATION gates are necessary and sufficient to compute I_n. Tanaka and Nishino [11] showed the following properties (P1, P2, and P3) of NEGATION gates in any $b(n)$-circuit for I_n.

P1) There exists a path that goes through all $b(n)$ NEGATION gates.

P2) On any path between any two NEGATION gates or any NEGATION gate and any output, there are at least one AND gate and at least one OR gate.

From the properties P1 and P2, the lower bound on depth for negation-limited inverter is obtained.

Lemma 1 (Tanaka and Nishino [11]) *Any negation-limited inverter must have depth at least $d_{\mathcal{MAJ}} + 3b(n)$, where $d_{\mathcal{MAJ}}$ is the optimal depth of monotone circuit for the majority function.*

For $1 \le \alpha \le b(n)$, we denote the α-th NEGATION gate by N_α on the path in property P1. Let z_α and y_α, respectively, be the functions computed at the input and output of N_α, i.e., $z_\alpha = \neg y_\alpha$.

P3) $z_1 z_2 \dots z_r$ is the binary representation of $|\tilde{x}|$.

Observe that from property P3,

$$|\tilde{x}| + \sum_{\alpha=1}^{b(n)} y_\alpha(\tilde{x}) 2^{b(n)-\alpha} = n \ . \tag{1}$$

3 Upper Bound on Depth

Let s be the size of monotone sorting network with the optimal depth d. From the previous result [3, 5, 11], we have the following proposition.

Proposition 2 *$y_1(\tilde{x}), \dots, y_{b(n)}(\tilde{x})$ can be computed by a $b(n)$-circuit with size $s + O(n)$ and $d + 3b(n) - 2$. Especially, $y_\alpha(\tilde{x})$ can be computed in depth $d + 3\alpha - 2$ for $1 \le \alpha \le b(n)$.*

Let,

$$t_0^i = |\tilde{x}| - x_i \ , \ t_\alpha^i = t_0^i + \sum_{\gamma=1}^{\alpha} y_\gamma(\tilde{x}) 2^{b(n)-\gamma} \ ,$$

for $1 \le \alpha \le b(n)$, and let

$$T_{i,\alpha}^k(\tilde{x}) = \begin{cases} 1 & \text{if } t_\alpha^i \ge k \ , \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \le i, k \le n$ and $0 \le \alpha \le b(n)$. Each $T_{i,\alpha}^k(\tilde{x})$ is a monotone function over $\tilde{x} \cup \{y_1(\tilde{x}), \dots, y_\alpha(\tilde{x})\}$. Especially, $T_{i,0}^k(\tilde{x})$ is the k-th threshold function over $\tilde{x} - \{x_i\}$, and from (1)

$$I_n^i(\tilde{x}) = T_{i,b(n)}^n(\tilde{x}) = \begin{cases} 1 & \text{if } t_{b(n)}^i = n - x_i \ge n \ , \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3 *Let s be the size of monotone sorting network with the optimal depth d. There exists a negation-limited inverter with the depth $d + 3b(n)$ can be constructed in size $O(ns + n^2)$.*

Proof. In parallel, we construct n monotone sorting networks, $S_1, \ldots, S_n$. S_i sort the inputs in $\tilde{x} - \{x_i\}$ for $1 \le i \le n$. As an output of S_i, $T^k_{i,0}(\tilde{x})$ is computed in depth d for $1 \le k \le n$. From Proposition 2, $y_1(\tilde{x}), \ldots, y_{b(n)}(\tilde{x})$ are computed by a $b(n)$-circuit as shown in the previous result [3, 5, 11].

Since

$$T^k_{i,\alpha}(\tilde{x}) = T^k_{i,\alpha-1}(\tilde{x}) \vee y_\alpha(\tilde{x}) \wedge T^{k-2^{b(n)-\alpha}}_{i,\alpha-1}(\tilde{x}) ,$$

$I^i_n(\tilde{x})$ can be computed from $y_\alpha(\tilde{x}), \ldots, y_{b(n)}(\tilde{x})$ and all $T^k_{i,\alpha-1}(\tilde{x})$'s for $n - 2^{b(n)-\alpha+1} < k \le n$. By using $2^{b(n)-\alpha-1}$ gates all $T^k_{i,\alpha}(\tilde{x})$ for $n - 2^{b(n)-\alpha} < k \le n$ can be obtained in depth $d + 3\alpha$, from $y_\alpha(\tilde{x})$ and $T^{k'}_{i,\alpha-1}(\tilde{x})$ for $n - 2^{b(n)-\alpha+1} < k' \le n$. Thus, $I^i_n(\tilde{x}) = T^n_{i,b(n)}(\tilde{x})$ for all $1 \le i \le n$ are computed in depth $d + 3b(n)$.

Finally, there are n monotone sorting networks such that each has size s, a subcircuit computing $y_1(\tilde{x}), \ldots, y_{b(n)}(\tilde{x})$ with size $s + O(n)$, and $2n - 2$ gates for computing $I^i_n(\tilde{x})$ from all $T^k_{i,0}(\tilde{x})$ and $y_1(\tilde{x}), \ldots, y_{b(n)}(\tilde{x})$ for each $1 \le i \le n$. Hence, the circuit has size $O(ns + n^2)$ and depth $d + 3b(n)$. ∎

Using the $\mathcal{AKS}$ monotone sorting networks (see [1]) in our construction, we have the following corollary.

Corollary 4 *Let $d_{\mathcal{AKS}}$ be the depth of the $\mathcal{AKS}$ monotone sorting network. A negation-limited inverter with depth $d_{\mathcal{AKS}} + 3b(n)$ can be constructed with size $O(n^2 \log n)$.* ∎

4 Under Assumption A

Here, we consider the negation-limited inverter under the following natural assumption.

Assumption A *The optimal depth of monotone circuit for the majority function is not less than that for any other threshold function.*

Under this assumption, the following corollary is obtained from Lemma 1 and Theorem 3.

Corollary 5 *Under Assumption A, the optimal depth of any negation-limited inverter is $d_{\mathcal{MAJ}} + 3b(n)$.* ∎

Next, we consider the size of depth-optimal negation-limited inverter under Assumption A. We change, in this section, our circuit model from the previous works. Namely, in the new model we ignore the size and depth of the NEGATION gates. This is a natural formulation, since the new model can be considered as the model of circuits with basis $\{\mathrm{AND}, \mathrm{OR}, \mathrm{NAND}, \mathrm{NOR}\}$, where we can interpret circuits with a limited number of NAND and NOR gates as standard negation-limited circuits. An $\Omega(n \log n)$ lower bound is shown for any depth-optimal negation-limited inverter in this model.

In a depth-optimal negation-limited inverter in this model, N_1 is in depth $d_{\mathcal{MAJ}}$, that is, N_α in depth $d_{\mathcal{MAJ}} + 2\alpha - 2$ for all $1 \le \alpha \le b(n)$ and the circuit has depth $d_{\mathcal{MAJ}} + 2b(n)$. We find n disjoint paths with no NEGATION gate from n gates in depth $d_{\mathcal{MAJ}} + 1$ or $d_{\mathcal{MAJ}} + 2$ to the n output gates. Each of these paths has $2b(n) - 1$ gates on it.

We say two assignments π and π' that $\pi \geq \pi'$ iff $x_i|_\pi \geq x_i|_{\pi'}$ for all $1 \leq i \leq n$. For any functions $g(\tilde{x})$ and $g'(\tilde{x})$ and a set of assignments A, we say that $g(\tilde{x}) = g'(\tilde{x})$ *over* A iff $g(\pi) = g'(\pi)$ for all $\pi \in A$. Let $A_{\alpha,\beta} \subseteq \{0,1\}^n$ be a set of assignment that for all $\pi \in A_{\alpha,\beta}$, $|\pi| \bmod 2^{b(n)-\alpha} \equiv \beta$, for all $1 \leq \alpha \leq b(n)$ and $0 \leq \beta < 2^{b(n)-\alpha}$.

Lemma 6 *Suppose there exists a gate G which computes a function $g(\tilde{x})$, that $g(\tilde{x}) = I_n^i(\tilde{x})$ over $A_{\alpha,\beta}$. For $2 \leq \alpha \leq b(n)$ and $0 \leq \beta \leq 2^{b(n)-\alpha}$,*

1. *There are at least one path from N_α to G with no NEGATION gate except N_α.*
2. *On each of such paths, there are at least one AND gate and at least one OR gate.*
3. *If G is in depth $d_{\mathcal{MAJ}} + 2\alpha$, then there are exactly one such path.*

Proof.

1. Suppose there is no such path from N_α to G with no NEGATION gate except N_α. Then, g is a monotone function on $x_1, \ldots, x_n$ and $y_\gamma(\tilde{x})$ for all $\gamma \neq \alpha$. There exists an assignment $\pi \in A_{\alpha,\beta}$ that $|\pi| = \beta < n$ and $x_i|_\pi = 0$. Then, we have $y_\alpha(\pi) = 1$ and $g(\pi) = I_n^i(\pi) = 1$. For such an assignment π, another assignment $\pi' \in A_{\alpha,\beta}$ that $\pi' \geq \pi$, $|\pi'| = |\pi| + 2^{b(n)-\alpha} > 0$ and $x_i|_{\pi'} = 1$. We have $y_\alpha(\pi') = 0$ and $y_\gamma(\pi) = y_\gamma(\pi')$ for all $\gamma \neq \alpha$. By the monotonicity of g, we have $g(\pi') \geq g(\pi) = 1$. However, $1 = g(\pi') \neq I_n^i(\pi') = 0$.

2. Suppose on one of such paths, all gates (including G) except N_α are AND gates. There exists an assignment $\pi'' \in A_{\alpha,\beta}$, that $|\pi''| = \beta + 2^{b(n)-\alpha}$ and $x_i|_{\pi''} = 0$, since $\beta + 2^{b(n)-\alpha} < 2^{b(n)} - 1 = n$. Then, we have $y_\alpha(\pi'') = 0$. However, since all gates including except N_α are AND gates on such a path, $y_\alpha(\pi'') = 0$ implies that $g(\pi'') = 0$. Again, $0 = g(\pi'') \neq I_n^i(\pi'') = 1$.

   For case that on such a path all gates except N_α are OR gates can be prove similarly.

3. Since N_α is in depth $d_{\mathcal{MAJ}} + 2\alpha - 2$ and G in depth $d_{\mathcal{MAJ}} + 2\alpha$, on such a path from N_α to G, there are one NEGATION gate, N_α, and including G exactly one AND gate and one OR gate. Thus, there are at most 2 such paths from N_α to G. Suppose there are 2 such paths from N_α to G. That is, both predecessors of G have a common predecessor, N_α. If G is an AND gate (OR gate), then both predecessors of G are OR gates (AND gates). There exist an assignment $\pi \in A_{\alpha,\beta}$ that $|\pi| = \beta + 2^{b(n)-\alpha} > 0$ and $x_i|_\pi = 1$. Since $y_\alpha(\pi) = 1$, we have $1 = g(\pi) \neq I_n^i(\pi) = 0$. ∎

Let G be a gate in depth $d_{\mathcal{MAJ}} + 2\alpha$ which computes function g with $g(\tilde{x}) = I_n^i(\tilde{x})$ over $A_{\alpha,\beta}$ ($2 \leq \alpha \leq b(n)$ and $0 \leq \beta < 2^{b(n)-\alpha}$). From Lemma 6, there exists exactly one path from N_α to G that consists of N_α, and including G exactly one AND gate and one OR gate. Let G' be the gate between N_α and G on such a path. Thus, G' is in depth $d_{\mathcal{MAJ}} + 2\alpha - 1$, since N_α is in depth $d_{\mathcal{MAJ}} + 2\alpha - 2$ and G is in depth $d_{\mathcal{MAJ}} + 2\alpha$. Let H_1 be the predecessor of G other than G', and H_2 be the predecessor of G' other than N_α. We denote the function that H_i computes by $h_i(\tilde{x})$ for $1 \in \{1,2\}$. Note that $h_1(\tilde{x})$ and $h_2(\tilde{x})$ are monotone functions over $\tilde{x} \cup \{y_\gamma(\tilde{x}) \,|\, \text{for } \gamma < \alpha\}$, since there is no path from N_α to H_i for $i \in \{1,2\}$.

Lemma 7 *Suppose G is an OR gate and G' is an AND gate. That is, $g(\tilde{x}) = h_1(\tilde{x}) \vee (h_2(\tilde{x}) \wedge y_\alpha(\tilde{x}))$. Then, $h_1(\tilde{x}) \wedge y_\alpha(\tilde{x}) = 0$ over $A_{\alpha,\beta}$.*

Proof. Suppose there exist an assignment $\pi \in A_{\alpha,\beta}$ such that $h_1(\pi) = y_\alpha(\pi) = 1$. It implies that $g(\pi) = I_n^i(\pi) = 1$ and $x_i|_\pi = 0$. Note that $y_\alpha(\pi) = 1$ implies that $|\pi| \leq n - 2^{b(n)-\alpha}$. There exists an assignment $\pi' \in A_{\alpha,\beta}$ that $\pi' \geq \pi$, $|\pi'| = |\pi| + 2^{b(n)-\alpha}$ and $x_i|_{\pi'} = 1$. Then, we have $y_\alpha(\pi') = 0$ and $y_\gamma(\pi') = y_\gamma(\pi)$ for $\gamma \neq \alpha$. By the monotonicity of h_1, we have $h_1(\pi') = 1$. Thus, we have $1 = g(\pi') \neq I_n^i(\pi') = 0$. ∎

By the duality of AND and OR, the following lemma is obtained.

Lemma 8 *Suppose G is an AND gate and G' is an OR gate. That is, $g(\tilde{x}) = h_1(\tilde{x}) \wedge (h_2(\tilde{x}) \vee y_\alpha(\tilde{x}))$. Then, $h_1(\tilde{x}) \vee y_\alpha(\tilde{x}) = 1$ over $A_{\alpha,\beta}$.* ∎

From Lemma 7 and Lemma 8, we obtain the following lemma immediately.

Lemma 9 *$h_2(\tilde{x}) = I_n^i(\tilde{x})$ over $A_{\alpha-1,\beta'}$, where $\beta' \in \{\beta, \beta + 2^{b(n)-\alpha}\}$.*

Proof. Note that $A_{\alpha-1,\beta} = \{\pi \mid \pi \in A_{\alpha,\beta} \ \wedge \ y_\alpha(\pi) = 1\}$, and $A_{\alpha-1,\beta+2^{b(n)-\alpha}} = \{\pi \mid \pi \in A_{\alpha,\beta} \ \wedge \ y_\alpha(\pi) = 0\}$. Thus, if G is an OR gate, from Lemma 7 $h_2(\tilde{x}) = I_n^i(\tilde{x})$ on $A_{\alpha-1,\beta}$. Otherwise, if G is an AND gate, from Lemma 8, $h_2(\tilde{x}) = I_n^i(\tilde{x})$ on $A_{\alpha-1,\beta+2^{b(n)-\alpha}}$. ∎

Lemma 10 *There exists a path from a gate in depth $d_{\mathcal{MAJ}} + 1$ or $d_{\mathcal{MAJ}} + 2$ to the i-th output gate with $2b(n) - 1$ gates, for all $1 \leq i \leq n$.*

Proof. For each $1 \leq i \leq n$, start with a path with one gate in depth $d_{\mathcal{MAJ}} + 2b(n)$ that computes I_n^i over $A_{b(n),0} = \{0,1\}^n$, i.e., the i-th output gate. Such a path can be extended by using Lemma 6 and Lemma 9 as follow.

For $2 \leq \alpha \leq b(n)$ and $0 \leq \beta < 2^{b(n)-\alpha}$, let G be a gate in depth $d_{\mathcal{MAJ}} + 2\alpha$ that at the bottom of such a path and computes a function g with $g(\tilde{x}) = I_n^i(\tilde{x})$ on $A_{\alpha,\beta}$. From Lemma 6, there exists a gate G' that is a predecessor of G and a successor of N_α. From Lemma 9, there exists a gate H that is a predecessor of G' other than N_α, such that H computes a function h with $h(\tilde{x}) = I_n^i(\tilde{x})$ over $A_{\alpha-1,\beta'}$, where $\beta' \in \{\beta, \beta + 2^{b(n)-\alpha}\}$. Then, we extend the path by including 2 gates, G' and H, to it. Again, from Lemma 6, H is in depth $d_{\mathcal{MAJ}} + 2\alpha - 2$ for $\alpha \geq 3$. Otherwise, if $\alpha = 2$, i.e., in the last extension, depth of H is $d_{\mathcal{MAJ}} + 1$ or $d_{\mathcal{MAJ}} + 2$, That is, H has a successor in depth $d_{\mathcal{MAJ}} + 3$ and there exists a path from N_1 to H (since $h(\tilde{x})$ is not a monotone function over $x_1, \ldots, x_n$).

In each extension, 2 gates are included in the bottom of the path. After $b(n) - 1$ extension, a path from a gate in depth $d_{\mathcal{MAJ}} + 1$ or $d_{\mathcal{MAJ}} + 2$ to the i-th output gate with $2b(n) - 1$ gates are found. The gate at the bottom of such a path computes a function g that $g(\tilde{x}) = I_n^i(\tilde{x})$ over $A_{1,\beta}$, where $0 \leq \beta < 2^{b(n)-1}$. ∎

From this lemma, for each $1 \leq i \leq n$. we find a path from a gate in depth $d_{\mathcal{MAJ}}+1$ or $d_{\mathcal{MAJ}}+2$ to the i-output gate. Let P_i be the path to the i-th output gate, and $G_1^i, G_2^i, \ldots, G_{2b(n)-1}^i$ be gates on P_i, such that G_1^i is at the bottom of P_i whose depth is $d_{\mathcal{MAJ}} + 1$ or $d_{\mathcal{MAJ}} + 2$, and for $2 \leq k < 2b(n) - 1$, G_k^i whose depth is $d_{\mathcal{MAJ}} + k + 2$ is a successor of G_{k-1}^i. Thus, for $1 \leq \alpha \leq b(n)$, $G_{2\alpha-1}^i$ is a gate that computes $I_n^i(\tilde{x})$ over $A_{\alpha,\beta}$ for some $0 \leq \beta < 2^{b(n)-\alpha}$, and for $1 \leq \alpha \leq b(n) - 1$, $G_{2\alpha}^i$ is a gate that is a successor of $G_{2\alpha-1}^i$ and N_α.

Lemma 11 *$P_1, \ldots, P_n$ are disjoint.*

Proof. Suppose P_i and P_j sharing a gate G. Note that P_i and P_j must share G at the same position. That is, if $G = G_l^i = G_m^j$, then $l = m$. Otherwise there exists a path from N_1 to the i-th or j-th output gate with length longer then $2b(n) - 1$.

Suppose $G = G_{2\alpha-1}^i = G_{2\alpha-1}^j$ for some $1 \leq \alpha \leq b(n)$. Let $g(\tilde{x})$ be the function that computed by G. Then, there exist an $1 \leq \alpha \leq b(n)$ and $0 \leq \beta, \beta' < 2^{b(n)-\alpha}$, $g(\tilde{x}) = I_n^i(\tilde{x})$ over $A_{\alpha,\beta}$, and $g(\tilde{x}) = I_n^j(\tilde{x})$ over $A_{\alpha,\beta'}$. It is obvious that $\beta \neq \beta'$, since there exists an assignment $\pi \in A_{\alpha,\beta}$ that $x_i|_\pi \neq x_j|_\pi$. Thus, assume withloss loss of generality that $\beta < \beta'$. For an assignment $\pi \in A_{\alpha,\beta}$ that $x_i|_\pi = 0$ and $x_j|_\pi = 1$. Then, we have $g(\pi) = I_n^i(\pi) = 1$. There exist an assignment $\pi' \in A_{\alpha,\beta'}$ that $\pi' \geq \pi$, i.e., $x_j|_{\pi'} \geq x_j|_\pi = 1$. Note that g is a monotone function on $x_1, \ldots, x_n$ and $y_\gamma(\tilde{x})$ for all $\gamma < \alpha$. Thus, we have $g(\pi') \geq g(\pi) = 1$. However, $1 = g(\pi') \neq I_n^j(\pi') = 0$.

Suppose $G = G_{2\alpha}^i = G_{2\alpha}^j$ for some $1 \leq \alpha \leq b(n) - 1$. Since N_α is a predecessor of G, the predecessor of G other than N_α is also shared by P_i and P_j, that is $G_{2\alpha-1}^i = G_{2\alpha-1}^j$. ∎

Thus, we have the following theorem and corollary.

Theorem 12 *Under Assumption A, any depth-optimal negation-limited inverter on basis* {AND, OR, NAND, NOR} *has size at least* $2n\log(n+1) + 3n - 4$,

Proof. From Lemma 10 and Lemma 11, $n(2b(n) - 1) = 2n\log(n+1) - n$ gates are found above the bottom NEGATION gate, N_1. Below N_1, majority function is computed by monotone subcircuit as input of N_1. There are at least $4(n-1)$ gates in such a subcircuit (see [4]). ∎

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