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The Complexity of Negation-Limited Inverters

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Summary

We mainly consider the complexity of negation-limited inverters. We show an upper bound $d + 3\lceil \log(n+1) \rceil$ on depth for negation-limited inverter whose size is $O(ns + n^2)$, where $d$ and $s$ is, respectively, the depth and the size of a monotone sorting network with optimal depth. If the optimal depth of monotone circuit for the majority function is not less than that for any other threshold function, this upper bound match the lower bound [11] and a superlinear lower bound $\Omega(n \log n)$ on size of depth-optimal negation-limited inverter on basis \{AND, OR, NAND, NOR\} is shown.

1 Introduction

1.1 Background and Motivation

We do not know much about the complexities of combinational circuits (i.e., circuits with AND, OR, and NEGATION gates) for explicitly defined functions. Even a superlinear lower bound for the size of combinational circuits is not known. The complexities of monotone circuits (i.e., combinational circuits without NEGATION gates) for many explicitly defined functions are well understood. For example, exponential lower bounds for the size of monotone circuits are known [2, 8, 9]. Exponential gaps between monotone and combinational circuit complexity are also shown [8, 12]. So, we cannot generally derive strong lower bounds for the combinational circuit complexity using those bounds for the monotone circuit complexity.

In this situation, there is no doubt that it is necessary to understand the effect of NEGATION gates in order to obtain strong lower bounds for combinational circuit complexity. This is a motivation for the study of negation-limited circuit complexity, the complexity for circuits in which the number of NEGATION gates are restricted.

Markov [7] gave the number of NEGATION gates which is necessary and sufficient to compute a system of boolean functions. Especially, he showed that $\lceil \log(n + 1) \rceil$ NEGATION gates are sufficient to compute any system of boolean functions (all logarithms and exponents in this paper

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Inverters of Negation-Limited Preliminaries whose computing returns decrease depth-optimal Nishino, negation-lattice studied Nishino number the function function function of necessary when Wilson NEGA-contains smallest monotone threshold functions. Especially, maximum the is of outputs inverter boolean negation-limited any function gates). Beals, Tanaka (number parity of the inputs the define the size determine the circuits. Owing to this result, for any n-input boolean function f, if there exists a polynomial size combinational circuit for f, there also exists a polynomial size circuit with only \[\log(n+1)\] NEGATION gates. Tanaka and Nishino [11], and Beals, Nishino, and Tanaka [3] improved the result of Fischer, and also gave lower bounds for the negation-limited circuit complexity of several functions. On the other hand, Santha and Wilson [10] studied the negation-limited complexity of constant depth circuits.

1.2 Definitions and Preliminaries

Let \(\tilde{x} = \{x_1, \ldots, x_n\}\) be the set of inputs. Let \(f = (f^1, \ldots, f^m)\) be an n-input m-output boolean function \(\{0, 1\}^n \to \{0, 1\}^m\). A theorem of Markov [7] precisely determines the number \(r\) of NEGATION gates necessary and sufficient to compute \(f\). A chain \(C\) in the boolean lattice \(\{0, 1\}^n\) is an increasing sequence \(a^1 < \ldots < a^k \in \{0, 1\}^n\). The decrease of \(f\) on \(C\) is the number of \(i \leq k\) such that \(f^j(a^{j-1}) > f^j(a^j)\) for some \(1 \leq j \leq m\). We define \(d(f)\) to be the maximum decrease of \(f\) on any chain \(C\). Markov has shown that \(\lceil\log(d(f) + 1)\rceil\) NEGATION gates are necessary and sufficient to compute any \(f\).

We denote by \(C(f)\) the circuit complexity of \(f\), i.e., the size (number of gates) of the smallest circuit of AND, OR, and NEGATION gates with inputs \(x_1, \ldots, x_n\) and outputs \(f^1(\tilde{x}), \ldots, f^m(\tilde{x})\). We call a circuit including at most \(r\) NEGATION gates an \(r\)-circuit. Especially, when \(r = 0\), an \(r\)-circuit is said to be monotone. We denote by \(C^r(f)\) the size of the smallest \(r\)-circuit computing \(f\) for \(r \geq \lceil\log(d(f) + 1)\rceil\).

Let \(|\tilde{x}|\) denote the number of ones in \(\tilde{x}\). We define the \(n\)-input parity function as a function which returns one if \(|\tilde{x}|\) is odd, and the \(n\)-input \(k\)-th threshold function as a function which returns one if \(|\tilde{x}| \geq k\) for \(1 \leq k \leq n\), i.e., the majority function is the \(\lceil n/2 \rceil\)-th threshold function.

1.3 Main Results

We first show an upper bound \(d + 3\lceil\log(n+1)\rceil\) on depth for negation-limited inverter whose size is \(O(ns + n^2)\), where \(d\) and \(s\) is, respectively, the depth and the size of a monotone sorting network with optimal depth. This upper bound matches the lower bound shown by Tanaka and Nishino [11] under the following natural assumption.

Assumption A The optimal depth of monotone circuit for the majority function is not less than that for any other threshold function.

Under this assumption, a superlinear lower bound \(\Omega(n \log n)\) on size of depth-optimal negation-limited inverter on basis \{AND, OR, NAND, NOR\} is shown.

2 Properties of Negation-Limited Inverters

Let \(n+1\) be a power of two, and let \(b(n) = \lceil\log(n+1)\rceil\). An inverter is an \(n\)-input \(n\)-output boolean function \(I_n = (I_n^1, \ldots, I_n^n)\), where \(I_n^i(\tilde{x}) = \neg x_i\) for all \(1 \leq i \leq n\). A theorem of Markov implies
that $b(n)$ NEGATION gates are necessary and sufficient to compute $I_n$. Tanaka and Nishino [11] showed the following properties (P1, P2, and P3) of NEGATION gates in any $b(n)$-circuit for $I_n$.

P1) There exists a path that goes through all $b(n)$ NEGATION gates.

P2) On any path between any two NEGATION gates or any NEGATION gate and any output, there are at least one AND gate and at least one OR gate.

From the properties P1 and P2, the lower bound on depth for negation-limited inverter is obtained.

Lemma 1 (Tanaka and Nishino [11]) Any negation-limited inverter must have depth at least $d_{M\Lambda\mathcal{J}} + 3b(n)$, where $d_{M\Lambda\mathcal{J}}$ is the optimal depth of monotone circuit for the majority function.

For $1 \leq \alpha \leq b(n)$, we denote the $\alpha$-th NEGATION gate by $N_{\alpha}$ on the path in property P1. Let $z_{\alpha}$ and $y_{\alpha}$, respectively, be the functions computed at the input and output of $N_{\alpha}$, i.e., $z_{\alpha} = \neg y_{\alpha}$.

P3) $z_1z_2\ldots z_r$ is the binary representation of $|\tilde{x}|$.

Observe that from property P3,

$$|\tilde{x}| + \sum_{\alpha=1}^{b(n)} y_{\alpha}(\tilde{x})2^{b(n)-\alpha} = n. \quad (1)$$

3 Upper Bound on Depth

Let $s$ be the size of monotone sorting network with the optimal depth $d$. From the previous result [3, 5, 11], we have the following proposition.

Proposition 2 $y_1(\tilde{x}), \ldots, y_{b(n)}(\tilde{x})$ can be computed by a $b(n)$-circuit with size $s + O(n)$ and $d + 3b(n) - 2$. Especially, $y_{\alpha}(\tilde{x})$ can be computed in depth $d + 3\alpha - 2$ for $1 \leq \alpha \leq b(n)$.

Let,

$$t_{0}^{\alpha} = |\tilde{x}| - x_{i}, \quad t_{\alpha}^{i} = t_{0}^{i} + \sum_{\gamma=1}^{\alpha} y_{\gamma}(\tilde{x})2^{b(n)-\gamma},$$

for $1 \leq \alpha \leq b(n)$, and let

$$T_{i,\alpha}^{k}(\tilde{x}) = \begin{cases} 1 & \text{if } t_{\alpha}^{i} \geq k, \\ 0 & \text{otherwise}, \end{cases}$$

for $1 \leq i, k \leq n$ and $0 \leq \alpha \leq b(n)$. Each $T_{i,\alpha}^{k}(\tilde{x})$ is a monotone function over $\tilde{x} \cup \{y_1(\tilde{x}), \ldots, y_{\alpha}(\tilde{x})\}$. Especially, $T_{i,0}^{k}(\tilde{x})$ is the $k$-th threshold function over $x - \{x_i\}$, and from (1)

$$T_{i,n}^{b(n)}(\tilde{x}) = \begin{cases} 1 & \text{if } t_{b(n)}^{i} = n - x_{i} \geq n, \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 3 Let $s$ be the size of monotone sorting network with the optimal depth $d$. There exists a negation-limited inverter with the depth $d + 3b(n)$ can be constructed in size $O(ns + n^2)$. 

Proof. In parallel, we construct $n$ monotone sorting networks, $S_1, \ldots, S_n$. $S_i$ sort the inputs in $\tilde{x} - \{x_i\}$ for $1 \leq i \leq n$. As an output of $S_i$, $T_{i,0}^k(\tilde{x})$ is computed in depth $d$ for $1 \leq k \leq n$. From Proposition 2, $y_1(\tilde{x}), \ldots, y_{b(n)}(\tilde{x})$ are computed by a $b(n)$-circuit as shown in the previous result [3, 5, 11].

Since

$$T_{i,\alpha}^k(\tilde{x}) = T_{i,\alpha-1}^k(\tilde{x}) \vee y_\alpha(\tilde{x}) \wedge T_{i,\alpha-1}^{k-2^{b(n)}-\alpha}(\tilde{x}),$$

$I^*_n(\tilde{x})$ can be computed from $y_\alpha(\tilde{x}), \ldots, y_{b(n)}(\tilde{x})$ and all $T_{i,\alpha-1}^k(\tilde{x})$'s for $n - 2^{b(n)} - \alpha + 1 < k \leq n$. By using $\tilde{y}(\tilde{z})$ for $n - 2^{b(n)} - \alpha < k \leq n$ can be obtained in depth $d + 3\alpha$, from $y_{\alpha}(\tilde{x})$ and $T_{i,\alpha-1}^k(\tilde{x})$ for $n - 2^{b(n)} - \alpha + 1 < k' \leq n$. Thus, $I^*_n(\tilde{x}) = T_{i,b(n)}^n(\tilde{x})$ for all $1 \leq i \leq n$ are computed in depth $d + 3b(n)$.

Finally, there are $n$ monotone sorting networks such that each has size $s$, a subcircuit computing $y_1(\tilde{x}), \ldots, y_{b(n)}(\tilde{x})$ with size $s + O(n)$, and $2n - 2$ gates for computing $T_{i}^n(\tilde{x})$ from all $T_{i,0}^k(\tilde{x})$ and $y_1(\tilde{x}), \ldots, y_{b(n)}(\tilde{x})$ for each $1 \leq i \leq n$. Hence, the circuit has size $O(ns + n^2)$ and depth $d + 3b(n)$.}

Using the $\mathcal{AKS}$ monotone sorting networks (see [1]) in our construction, we have the following corollary.

**Corollary 4** Let $d_{\mathcal{AKS}}$ be the depth of the $\mathcal{AKS}$ monotone sorting network. A negation-limited inverter with depth $d_{\mathcal{AKS}} + 3b(n)$ can be constructed with size $O(n^2 \log n)$.

## 4 Under Assumption A

Here, we consider the negation-limited inverter under the following natural assumption.

**Assumption A** The optimal depth of monotone circuit for the majority function is not less than that for any other threshold function.

Under this assumption, the following corollary is obtained from Lemma 1 and Theorem 3.

**Corollary 5** Under Assumption A, the optimal depth of any negation-limited inverter is $d_{\mathcal{MAJ}} + 3b(n)$.

Next, we consider the size of depth-optimal negation-limited inverter under Assumption A. We change, in this section, our circuit model from the previous works. Namely, in the new model we ignore the size and depth of the NEGATION gates. This is a natural formulation, since the new model can be considered as the model of circuits with basis \{AND, OR, NAND, NOR\}, where we can interpret circuits with a limited number of NAND and NOR gates as standard negation-limited circuits. An $\Omega(n \log n)$ lower bound is shown for any depth-optimal negation-limited inverter in this model.

In a depth-optimal negation-limited inverter in this model, $N_1$ is in depth $d_{\mathcal{MAJ}}$, that is, $N_\alpha$ in depth $d_{\mathcal{MAJ}} + 2\alpha - 2$ for all $1 \leq \alpha \leq b(n)$ and the circuit has depth $d_{\mathcal{MAJ}} + 2b(n)$. We find $n$ disjoint paths with no NEGATION gate from $n$ gates in depth $d_{\mathcal{MAJ}} + 1$ or $d_{\mathcal{MAJ}} + 2$ to the $n$ output gates. Each of these paths has $2b(n) - 1$ gates on it.
We say two assignments $\pi$ and $\pi'$ that $\pi \geq \pi'$ iff $x_i|_\pi \geq x_i|_{\pi'}$ for all $1 \leq i \leq n$. For any functions $g(\hat{x})$ and $g'(\hat{x})$ and a set of assignments $A$, we say that $g(\hat{x}) = g'(\hat{x})$ over $A$ iff $g(\pi) = g'(\pi)$ for all $\pi \in A$. Let $A_{\gamma,\alpha} \subseteq \{0, 1\}^n$ be a set of assignment that for all $\pi \in A_{\gamma,\alpha}$, $|\pi| \mod 2^{b(n) - \alpha} \equiv \beta$, for all $1 \leq \alpha \leq b(n)$ and $0 \leq \beta < 2^{b(n) - \alpha}$.

**Lemma 6** Suppose there exists a gate $G$ which computes a function $g(\hat{x})$, that $g(\hat{x}) = I_{i}^{\alpha}(\hat{x})$ over $A_{\gamma,\alpha}$. For $2 \leq \alpha \leq b(n)$ and $0 \leq \beta < 2^{b(n) - \alpha}$,

1. There are at least one path from $N_{\gamma}$ to $G$ with no NEGATION gate except $N_{\alpha}$.
2. On each of such paths, there are at least one AND gate and at least one OR gate.
3. If $G$ is in depth $d_{MAJ} + 2\alpha$, then there are exactly one such path.

**Proof.**

1. Suppose there is no such path from $N_{\alpha}$ to $G$ with no NEGATION gate except $N_{\alpha}$. Then, $g$ is a monotone function on $x_1, \ldots, x_n$ and $y_{\gamma}(\hat{x})$ for all $\gamma \neq \alpha$. There exists an assignment $\pi \in A_{\alpha,\beta}$ that $|\pi| = \beta < n$ and $x_i|_\pi = 0$. Then, we have $y_{\alpha}(\pi) = 1$ and $g(\pi) = I_{i}^{\alpha}(\pi) = 1$. For such an assignment $\pi$, another assignment $\pi' \in A_{\alpha,\beta}$ that $\pi' \geq \pi$, $|\pi'| = |\pi| + 2^{b(n) - \alpha} > 0$ and $x_i|_{\pi'} = 1$. We have $y_{\alpha}(\pi') = 0$ and $y_{\gamma}(\pi') = y_{\gamma}(\pi)$ for all $\gamma \neq \alpha$. By the monotonicity of $g$, we have $g(\pi') \geq g(\pi) = 1$. However, $1 = g(\pi') \neq I_{i}^{\alpha}(\pi') = 0$.

2. Suppose on one of such paths, all gates (including $G$) except $N_{\alpha}$ are AND gates. There exists an assignment $\pi'' \in A_{\alpha,\beta}$, that $|\pi''| = \beta + 2^{b(n) - \alpha} < n$, since $\beta + 2^{b(n) - \alpha} < 2^{b(n)} - 1 = n$. Then, we have $y_{\alpha}(\pi'') = 0$. However, since all gates including except $N_{\alpha}$ are AND gates on such a path, $y_{\alpha}(\pi'') = 0$ implies that $g(\pi'') = 0$. Again, $0 = g(\pi'') \neq I_{i}^{\alpha}(\pi'') = 1$.

For case that on such a path all gates except $N_{\alpha}$ are OR gates can be prove similarly.

3. Since $N_{\alpha}$ is in depth $d_{MAJ} + 2\alpha - 2$ and $G$ in depth $d_{MAJ} + 2\alpha$, on such a path from $N_{\alpha}$ to $G$, there are one NEGATION gate, $N_{\alpha}$, and including $G$ exactly one AND gate and one OR gate. Thus, there are at most 2 such paths from $N_{\alpha}$ to $G$. Suppose there are 3 such paths from $N_{\alpha}$ to $G$. That is, both predecessors of $G$ have a common predecessor, $N_{\alpha}$. If $G$ is an AND gate (OR gate), then both predecessors of $G$ are OR gates (AND gates). There exist an assignment $\pi \in A_{\alpha,\beta}$ that $|\pi| = \beta + 2^{b(n) - \alpha} > 0$ and $x_i|_{\pi} = 1$. Since $y_{\alpha}(\pi) = 1$, we have $1 = g(\pi) \neq I_{i}^{\alpha}(\pi) = 0$.

Let $G$ be a gate in depth $d_{MAJ} + 2\alpha$ which computes function $g$ with $g(\hat{x}) = I_{i}^{\alpha}(\hat{x})$ over $A_{\gamma,\alpha}$ ($2 \leq \alpha \leq b(n)$ and $0 \leq \beta < 2^{b(n) - \alpha}$). From Lemma 6, there exists exactly one path from $N_{\alpha}$ to $G$ that consists of $N_{\alpha}$, and including $G$ exactly one AND gate and one OR gate. Let $G'$ be the gate between $N_{\alpha}$ and $G$ on such a path. Thus, $G'$ is in depth $d_{MAJ} + 2\alpha - 1$, since $N_{\alpha}$ is in depth $d_{MAJ} + 2\alpha - 2$ and $G$ is in depth $d_{MAJ} + 2\alpha$. Let $H_1$ be the predecessor of $G$ other than $G'$, and $H_2$ be the predecessor of $G'$ other than $N_{\alpha}$. We denote the function that $H_i$ computes by $h_i(\hat{x})$ for $1 \in \{1, 2\}$. Note that $h_1(\hat{x})$ and $h_2(\hat{x})$ are monotone functions over $\hat{x} \cup \{y_{\gamma}(\hat{x})\}$ for $\gamma < \alpha$, since there is no path from $N_{\alpha}$ to $H_i$ for $i \in \{1, 2\}$. 

Lemma 7 Suppose $G$ is an OR gate and $G'$ is an AND gate. That is, $g(\tilde{x}) = h_1(\tilde{x}) \lor (h_2(\tilde{x}) \land y_\alpha(\tilde{x}))$. Then, $h_1(\tilde{x}) \land y_\alpha(\tilde{x}) = 0$ over $A_{\alpha,\beta}$.

Proof. Suppose there exist an assignment $\pi \in A_{\alpha,\beta}$ such that $h_1(\pi) = y_\alpha(\pi) = 1$. It implies that $g(\pi) = I^{\pi}_n(\pi) = 1$ and $x_{i|\pi} = 0$. Note that $y_\alpha(\pi) = 1$ implies that $|\pi| \leq n - 2^{b(n)-\alpha}$. There exists an assignment $\pi' \in A_{\alpha,\beta}$ that $\pi' \geq \pi$, $|\pi'| = |\pi| + 2^{b(n)-\alpha}$ and $x_{i|\pi'} = 1$. Then, we have $y_\alpha(\pi') = 0$ and $y_\gamma(\pi') = y_\gamma(\pi)$ for $\gamma \neq \alpha$. By the monotonicity of $h_1$, we have $h_1(\pi') = 1$. Thus, we have $1 = g(\pi') \neq I^{\pi}_n(\pi') = 0$. By the duality of AND and OR, the following lemma is obtained.

Lemma 8 Suppose $G$ is an AND gate and $G'$ is an OR gate. That is, $g(\tilde{x}) = h_1(\tilde{x}) \land (h_2(\tilde{x}) \lor y_\alpha(\tilde{x}))$. Then, $h_1(\tilde{x}) \lor y_\alpha(\tilde{x}) = 1$ over $A_{\alpha,\beta}$.

From Lemma 7 and Lemma 8, we obtain the following lemma immediately.

Lemma 9 $h_2(\tilde{x}) = I^{\tilde{x}}_n$ over $A_{\alpha-1,\beta'}$, where $\beta' \in \{\beta, \beta + 2^{b(n)-\alpha}\}$.

Proof. Note that $A_{\alpha-1,\beta} = \{\pi \mid \pi \in A_{\alpha,\beta} \land y_\alpha(\pi) = 1\}$, and $A_{\alpha-1,\beta+2^{b(n)-\alpha}} = \{\pi \mid \pi \in A_{\alpha,\beta} \land y_\alpha(\pi) = 0\}$. Thus, if $G$ is an OR gate, from Lemma 7 $h_2(\tilde{x}) = I^{\tilde{x}}_n(\tilde{x})$ on $A_{\alpha-1,\beta}$. Otherwise, if $G$ is an AND gate, from Lemma 8, $h_2(\tilde{x}) = I^{\tilde{x}}_n(\tilde{x})$ on $A_{\alpha-1,\beta+2^{b(n)-\alpha}}$.

Lemma 10 There exists a path from a gate in depth $d_{\mathcal{M}AJ} + 1$ or $d_{\mathcal{M}AJ} + 2$ to the $i$-th output gate with $2b(n) - 1$ gates, for all $1 \leq i \leq n$.

Proof. For each $1 \leq i \leq n$, start with a path with one gate in depth $d_{\mathcal{M}AJ} + 2b(n)$ that computes $I^n_{\alpha}$ over $A_{b(n),\alpha} = \{0,1\}^n$, i.e., the $i$-th output gate. Such a path can be extended by using Lemma 6 and Lemma 9 as follow.

For $2 \leq \alpha \leq b(n)$ and $0 \leq \beta < 2^{b(n)-\alpha}$, let $G$ be a gate in depth $d_{\mathcal{M}AJ} + 2\alpha$ that at the bottom of such a path and computes a function $g$ with $g(\tilde{x}) = I^n_{\alpha}(\tilde{x})$ on $A_{\alpha,\beta}$. From Lemma 6, there exists a gate $G'$ that is a predecessor of $G$ and a successor of $N_{\alpha}$. From Lemma 9, there exists a gate $H$ that is a predecessor of $G'$ other than $N_{\alpha}$, such that $H$ computes a function $h$ with $h(\tilde{x}) = I^{\tilde{x}}_n(\tilde{x})$ over $A_{\alpha-1,\beta'}$, where $\beta' \in \{\beta, \beta + 2^{b(n)-\alpha}\}$. Then, we extend the path by including 2 gates, $G'$ and $H$, to it. Again, from Lemma 6, $H$ is in depth $d_{\mathcal{M}AJ} + 2\alpha - 2$ for $\alpha \geq 3$. Otherwise, if $\alpha = 2$, i.e., in the last extension, depth of $H$ is $d_{\mathcal{M}AJ} + 1$ or $d_{\mathcal{M}AJ} + 2$. That is, $H$ has a successor in depth $d_{\mathcal{M}AJ} + 3$ and there exists a path from $N_1$ to $H$ (since $h(\tilde{x})$ is not a monotone function over $x_1, \ldots, x_n$).

In each extension, 2 gates are included in the bottom of the path. After $b(n) - 1$ extension, a path from a gate in depth $d_{\mathcal{M}AJ} + 1$ or $d_{\mathcal{M}AJ} + 2$ to the $i$-th output gate with $2b(n) - 1$ gates are found. The gate at the bottom of such a path computes a function $g$ that $g(\tilde{x}) = I^n_{\alpha}(\tilde{x})$ over $A_{1,\beta}$, where $0 \leq \beta < 2^{b(n)} - 1$.

From this lemma, for each $1 \leq i \leq n$, we find a path from a gate in depth $d_{\mathcal{M}AJ} + 1$ or $d_{\mathcal{M}AJ} + 2$ to the $i$-output gate. Let $P_i$ be the path to the $i$-th output gate, and $G^i_1, G^i_2, \ldots, G^i_{2b(n)-1}$ be gates.
on $P_i$, such that $G^i_1$ is at the bottom of $P_i$ whose depth is $d_{MAJ} + 1$ or $d_{MAJ} + 2$, and for $2 \leq k < 2b(n) - 1$, $G^i_k$ whose depth is $d_{MAJ} + k + 2$ is a successor of $G^i_{k-1}$. Thus, for $1 \leq \alpha \leq b(n)$, $G^i_{2\alpha-1}$ is a gate that computes $I^i_n(\tilde{x})$ over $A_{\alpha,\beta}$ for some $0 \leq \beta < 2^{b(n) - \alpha}$, and for $1 \leq \alpha \leq b(n) - 1$, $G^i_{2\alpha}$ is a gate that is a successor of $G^i_{2\alpha-1}$ and $N^i_\alpha$.

**Lemma 11** $P_1, \ldots, P_n$ are disjoint.

**Proof.** Suppose $P_i$ and $P_j$ sharing a gate $G$. Note that $P_i$ and $P_j$ must share $G$ at the same position. That is, if $G = G^i_1 = G^j_1$, then $l = m$. Otherwise there exists a path from $N_1$ to the $i$-th or $j$-th output gate with length longer then $2b(n) - 1$.

Suppose $G = G^i_{2\alpha-1} = G^j_{2\alpha-1}$ for some $1 \leq \alpha \leq b(n)$. Let $g(\tilde{x})$ be the function that computed by $G$. Then, there exist an $1 \leq \alpha \leq b(n)$ and $0 \leq \beta, \beta' < 2^{b(n) - \alpha}$, $g(\tilde{x}) = I^i_n(\tilde{x})$ over $A_{\alpha,\beta}$, and $g(\tilde{x}) = I^j_n(\tilde{x})$ over $A_{\alpha,\beta'}$. It is obvious that $\beta \neq \beta'$, since there exists an assignment $\pi \in A_{\alpha,\beta}$ that $x_i|_{\pi} \neq x_j|_{\pi}$. Thus, assume with loss of generality that $\beta < \beta'$. For an assignment $\pi \in A_{\alpha,\beta}$ that $x_i|_{\pi} = 0$ and $x_j|_{\pi} = 1$. Then, we have $g(\pi) = I^i_1(\pi) = 1$. There exist an $\pi' \in A_{\alpha,\beta'}$ that $\pi' \geq \pi$, i.e., $x_i|_{\pi'} \geq x_j|_{\pi'} = 1$. Note that $g$ is a monotone function on $x_1, \ldots, x_n$ and $y_\gamma(\tilde{x})$ for all $\gamma < \alpha$. Thus, we have $g(\pi') \geq g(\pi) = 1$. However, $1 = g(\pi') \not\equiv I^j_1(\pi') = 0$.

Suppose $G = G^i_{2\alpha} = G^j_{2\alpha}$ for some $1 \leq \alpha \leq b(n) - 1$. Since $N^i_\alpha$ is a predecessor of $G$, the predecessor of $G$ other than $N^i_\alpha$ is also shared by $P_i$ and $P_j$, that is $G^i_{2\alpha-1} = G^j_{2\alpha-1}$.

Thus, we have the following theorem and corollary.

**Theorem 12** Under Assumption A, any depth-optimal negation-limited inverter on basis $\{\text{AND}, \text{OR}, \text{NAND}, \text{NOR}\}$ has size at least $2n \log(n + 1) + 3n - 4$.

**Proof.** From Lemma 10 and Lemma 11, $n(2b(n) - 1) = 2n \log(n + 1) - n$ gates are found above the bottom NEGATION gate, $N_1$. Below $N_1$, majority function is computed by monotone subcircuit as input of $N_1$. There are at least $4(n - 1)$ gates in such a subcircuit (see [4]).

**References**


