数理解析研究所講究録
1056 巻 1998 年 68-74

Magnetic Scattering at Low Energy in Two Dimensions

岡山大理　田村 英男 (Hideo Tamura)

We consider the low–energy scattering for Schrödinger operators with magnetic fields compactly supported in two dimensions. We study the asymptotic behavior at low energy of scattering amplitudes. As a direct application, we also discuss the behavior of scattering amplitudes for scattering by magnetic fields with small support. The results obtained strongly depend on the total flux of magnetic fields under consideration.

1. Magnetic scattering at low energy

We work in the two dimensional space $\boldsymbol{R}^2$ with generic point $x=(x_1,x_2)$. Let $b(x),\ V(x)\in C_0^\infty(\boldsymbol{R}^2\to\boldsymbol{R})$ be given magnetic and electric fields with compact support and let

$$A(x)=(a_1(x),a_2(x))\in C^\infty(\boldsymbol{R}^2\to\boldsymbol{R}^2),\quad \nabla\times A=\partial_1a_2-\partial_2a_1=b,$$

be a magnetic potential associated with b. We consider the Hamiltonian

$$H=H(A,V)=(-i\nabla-A)^2+V=\sum_{j=1}^{2}(-i\partial_j-a_j)^2+V.$$

This operator admits a unique self–adjoint realization in $L^2(\boldsymbol{R}^2)$. We denote by the same notation H this realization with domain $\mathcal{D}(H)=H^2(\boldsymbol{R}^2)$ (Sobolev space). The total flux α of field b is defined by

$$\alpha=(2\pi)^{-1}\int b(x)\,dx.$$

For brevity, we assume that

$$0<\alpha<1.$$

The argument extends to the case that $\alpha\notin\boldsymbol{Z}$ is not an integer.

The magnetic potential $A(x)$ associated with field $b(x)$ is not uniquely determined, but the Hamiltonians with the same magnetic field are unitarily equivalent under the gauge transformation $A\to A+\nabla g$. Thus we fix the magnetic potential as

$$A(x)=(-\partial_2\varphi(x),\partial_1\varphi(x)),$$

where

$$\varphi(x)=(2\pi)^{-1}\int\log|x-y|\,b(y)\,dy.$$

As is easily seen, $A(x)$ behaves like

$$A(x)=B_\alpha(x)+O(|x|^{-2}),\quad B_\alpha(x)=\alpha(-x_2/|x|^2,x_1/|x|^2),$$

as $|x| \to \infty$. It should be noted that $A(x)$ never decays faster than $O(|x|^{-1})$ at infinity, even if $b(x)$ is assumed to be of compact support. Thus the difference $H - H_0$ between H and the free Hamiltonian $H_0 = -\Delta$ belongs to the long–range perturbation class. Nevertheless we know ([5]) that the ordinary wave operators

$$W_\pm(H, H_0) = s - \lim_{t\to\pm\infty} \exp(itH)\exp(-itH_0) : L^2(\boldsymbol{R}^2) \to L^2(\boldsymbol{R}^2)$$

exit and are asymptotically complete

$$\mathrm{Ran}(W_-(H, H_0)) = \mathrm{Ran}(W_+(H, H_0)).$$

Hence the scattering matrix $S(\lambda; H, H_0) : L^2(S^1) \to L^2(S^1)$ at energy $\lambda > 0$, S^1 being the unit circle, can be defined as a unitary operator. Let $S(\omega', \omega; \lambda)$, $(\omega', \omega) \in S^1 \times S^1$, denote the integral kernel of $S(\lambda; H, H_0)$. Then the scattering amplitude $f(\omega \to \omega'; \lambda)$ for scattering from incident direction ω into final one ω' at energy $\lambda > 0$ is defined by

$$f(\omega \to \omega'; \lambda) = c(\lambda)\,(S(\omega', \omega; \lambda) - \delta(w' - \omega))$$

with $c(\lambda) = (2\pi/i\sqrt{\lambda})^{1/2}$.

The aim here is to study the behavior as $\lambda \to 0$ of scattering amplitude $f(\omega \to \omega'; \lambda)$. The behavior strongly depends on the resonance space $\mathcal{E}_1$ at zero energy of H. The resonance space $\mathcal{E}_1$ is defined as

$$\mathcal{E}_1 = \{u \in L^2_{-1}(\boldsymbol{R}^2) : Hu = 0\}/\mathcal{E}_0$$

where $L^2_s(\boldsymbol{R}^2)$ denotes the weighted L^2 space $L^2(\boldsymbol{R}^2; (1+|x|^2)^s dx)$ and $\mathcal{E}_0 = \{u \in L^2(\boldsymbol{R}^2) : Hu = 0\}$ is the zero eigenspace of H. If the flux α of field b is not an integer, then it is shown that $\dim \mathcal{E}_1 \le 2$. When $\dim \mathcal{E}_1 = 2$, $\mathcal{E}_1$ is spanned by a pair (ρ_0, ρ_1) of functions taking the form

$$\rho_l(x) = r^{-\nu} e^{il\theta} + g_l, \quad \nu = |l - \alpha|,$$

with some $g_l \in L^2(\boldsymbol{R}^2)$, where (r, θ) denotes the polar coordinates over $\boldsymbol{R}^2$. If $\dim \mathcal{E}_1 = 1$, then $\mathcal{E}_1$ is spanned by a linear combination

$$\rho(x) = c_0 r^{-\alpha} + c_1 r^{-(1-\alpha)} e^{i\theta} + g, \quad g \in L^2(\boldsymbol{R}^2),$$

and the asymptotic formula as $\lambda \to 0$ of $f(\omega \to \omega'; \lambda)$ takes various forms according to the value α and the ratio $c = c_1/c_0$.

Theorem 1. 1 *Assume that b, $V \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ are real smooth functions with compact support and that the total flux α of magnetic field b satisfies $0 < \alpha < 1$. Let $\mathcal{E}_1$ denote the resonance space at zero energy of $H = H(A, V)$. Set $c(\lambda) = (2\pi/i\sqrt{\lambda})^{1/2}$ again and define*

$$f_\alpha(\omega' - \omega) = (\cos\alpha\pi - 1)\delta(\omega' - \omega) - (\sin\alpha\pi/2\pi)F_0(\omega' - \omega),$$

where $F(\theta) = \mathrm{v.p.}\, e^{i\theta}/(e^{i\theta} - 1)$, and the coordinates over S^1 are identified with the azimuth angles from the positive x_1 axis. Then the scattering amplitude $f(\omega \to \omega'; \lambda)$ for scattering from initial direction ω to final one ω' at energy λ obeys the following asymptotic formula as $\lambda \to 0$.

(1) *If* $\dim \mathcal{E}_1 = 2$, *then*

$$f(\omega \to \omega'; \lambda) = c(\lambda)\left(f_\alpha(\omega' - \omega) + (i/\pi)\sin\alpha\pi\left(1 - e^{i(\omega'-\omega)}\right) + o(1)\right).$$

(2) *Assume that* $\dim \mathcal{E}_1 = 1$. *Let*

$$\rho = c_0 r^{-\alpha} + c_1 r^{-(1-\alpha)} e^{i\theta} + g, \quad g \in L^2(\boldsymbol{R}^2),$$

be a resonance function spanning $\mathcal{E}_1$. *Then one has :*

(*i*) *Assume that* $0 < \alpha < 1/2$. *If* $c_0 \neq 0$, *then*

$$f(\omega \to \omega'; \lambda) = c(\lambda)\left(f_\alpha(\omega' - \omega) + (i/\pi)\sin\alpha\pi + o(1)\right)$$

and if $c_0 = 0$, *then*

$$f(\omega \to \omega'; \lambda) = c(\lambda)\left(f_\alpha(\omega' - \omega) - (i/\pi)\sin\alpha\pi e^{i(\omega'-\omega)} + o(1)\right).$$

(*ii*) *Assume that* $\alpha = 1/2$. *Set* $c = c_1/c_0$ ($c = \infty$ *if* $c_0 = 0$). *Then*

$$f(\omega \to \omega'; \lambda) = c(\lambda)\left(f_\alpha(\omega' - \omega) + \frac{i(1 - \bar{c}e^{-i\omega})(1 + ce^{i\omega'})}{\pi(1 + |c|^2)} + o(1)\right).$$

(*iii*) *Assume that* $1/2 < \alpha < 1$. *If* $c_1 \neq 0$, *then*

$$f(\omega \to \omega'; \lambda) = c(\lambda)\left(f_\alpha(\omega' - \omega) - (i/\pi)\sin\alpha\pi e^{i(\omega'-\omega)} + o(1)\right)$$

and if $c_1 = 0$, *then*

$$f(\omega \to \omega'; \lambda) = c(\lambda)\left(f_\alpha(\omega' - \omega) + (i/\pi)\sin\alpha\pi + o(1)\right).$$

(3) *Assume that* $\dim \mathcal{E}_1 = 0$. *Then*

$$f(\omega \to \omega'; \lambda) = c(\lambda)\left(f_\alpha(\omega' - \omega) + o(1)\right).$$

If $V(x) = 0$, then we can show that the solution to equation $Hu = 0$ with $u \in L^2_{-1}(\boldsymbol{R}^2)$ identically vanishes. Hence $\dim \mathcal{E}_1 = 0$ and it follows that

$$f(\omega \to \omega'; \lambda) = c(\lambda)\left(f_\alpha(\omega' - \omega; \lambda) + o(1)\right), \quad \lambda \to 0.$$

The leading term just coincides with the scattering amplitude calculated by [1] (see [6] also) for the Hamiltonian

$$H_\alpha = (-i\nabla - B_\alpha)^2, \quad B_\alpha(x) = \alpha(-x_2/|x|^2, x_1/|x|^2), \tag{1.1}$$

with domain

$$\mathcal{D}(H_\alpha) = \{u \in L^2(\boldsymbol{R}^2) : H_\alpha u \in L^2(\boldsymbol{R}^2), \ \lim_{|x|\to 0} u = 0\},$$

where $H_\alpha u = (-i\nabla - B_\alpha)^2 u$ is understood in $\mathcal{D}'$ (in the distributional sense).

The scattering amplitude $f(\omega \to \omega'; \lambda)$ is represented through the resolvent

$$R(\lambda + i0; H) = \lim_{\varepsilon \to 0} R(\lambda + i\varepsilon; H), \quad R(z) = (H - z)^{-1},$$

and the problem is reduced to analysing the behavior at low energy of the resolvent $R(\lambda + i0; H)$. A lot of works have been already done on the behavior of resolvents at low energy in the case of short–range potential scattering. An extensive list of related literatures can be found in the book [3]. The proof of the above theorem is, in principle, based on the idea developed in Jensen–Kato [4], althogh several technical improvements are required at many stages. The standard way to analyse the behavior of resolvents at low energy is based on the relation

$$R(\lambda + i0; H) = (Id + R(\lambda + i0; H_0)(H - H_0))^{-1} R(\lambda + i0; H_0) \tag{1.2}$$

obtained from the resolvent identity, where Id stands for the identity operator. For the case of scattering by magnetic fields, the difference $H - H_0$ is not necessarily of short–range class even for the field $b(x)$ compactly suppoted, as previously stated. The resolvent identity does not work for the pair (H, H_0). On the other hand, $H - H_\alpha$ becomes a perturbation of short–range class for the Hamiltonian H_α defined by (1.1), but the domain $\mathcal{D}(H_\alpha)$ does not coincide with that of H. It should be noted that even the form domains of these operators are different. This makes it difficult to use the resolvent identity for the pair (H, H_α) also. Thus we take a slightly different approach. We introduce a certain auxiliary operator

$$K_\alpha = (-i\nabla - \chi_\infty B_\alpha)^2,$$

where $\chi_\infty(r)$, $r = |x|$, is a smooth real function vanishing near the origin and taking the value $\chi_\infty = 1$ for $r \gg 1$ large enough. By definition, K_α has the same domain as H, and the difference $W = H - K_\alpha$ belongs to the short–range class. In addition, K_α admits the partial wave expansion in angular momentum. This enables us to expand $R(\lambda + i0; K_\alpha)$ asymptotically in λ, $0 < \lambda \ll 1$, small enough and to analyse the behavior at low energy of resolvent $R(\lambda + i0; H)$ in question through relation (1.2) applied to the pair (H, K_α). The details are discussed in [7].

2. Scattering by magnetic fields with small support

As a simple application of Theorem 1.1, we discuss the scattering by magnetic fields with small support. Let $b(x)$, $V(x) \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ and $A(x)$, $\nabla \times A = b$, be as above. We set

$$b_\varepsilon(x) = \varepsilon^{-2} b(x/\varepsilon), \quad V_\varepsilon(x) = \varepsilon^{-2} V(x/\varepsilon), \quad A_\varepsilon(x) = \varepsilon^{-1} A(x/\varepsilon)$$

for $0 < \varepsilon \ll 1$, and we consider the Hamiltonian

$$H_\varepsilon = H(A_\varepsilon, V_\varepsilon) = (-i\nabla - A_\varepsilon)^2 + V_\varepsilon.$$

As is easily seen, $\nabla \times A_\varepsilon = b_\varepsilon$ and the field b_ε preserves the flux

$$(2\pi)^{-1} \int b_\varepsilon(x)\, dx = \alpha.$$

We denote by $f_\varepsilon(\omega \to \omega'; \lambda)$ the scattering amplitude for the pair (H_ε, H_0). By making a change $x/\varepsilon \to x$ of variables, $f_\varepsilon(\omega \to \omega'; \lambda)$ can be easily shown to satisfy the relation

$$f_\varepsilon(\omega \to \omega'; \lambda) = \sqrt{\varepsilon} f(\omega \to \omega'; \lambda\varepsilon^2)$$

and hence the asymptotic behavior as $\varepsilon \to 0$ of $f_\varepsilon(\omega \to \omega'; \lambda)$, $\lambda > 0$ being fixed, is obtained as an immediate consequence of Theorem 1.1. As $\varepsilon \to 0$, $A_\varepsilon(x) \to B_\alpha(x)$ and $b_\varepsilon(x) \to 2\pi\alpha\delta(x)$ in $\mathcal{D}'$. Thus the scaled Hamiltonian H_ε is formally convergent to the Hamiltonian

$$H_{0\alpha} = (-i\nabla - B_\alpha)^2, \quad \mathcal{D}(H_{0\alpha}) = C_0^\infty(\boldsymbol{R}^2 \setminus \{0\}),$$

with δ–like magnetic field at the origin. We are concerned with the relation between the limit $\lim_{\varepsilon\to 0} f_\varepsilon(\omega \to \omega'; \lambda)$ and the scattering amplitude for the Hamiltonian obtained as a self–adjoint extension of $H_{0\alpha}$.

We denote by $\overline{H}_{0\alpha}$ the closure of $H_{0\alpha}$. This is symmetric, but is not self–adjoint. Let $\Sigma_\pm = \mathrm{Ker}(\overline{H}_{0\alpha}^* \mp i)$. Then $\Sigma_\pm$ is spanned by $\psi_{\pm l}$, $l = 0, 1$, where

$$\psi_{+l}(x) = \tau_l K_\nu(e^{-i\pi/4} r) e^{il\theta}, \quad \psi_{-l}(x) = \tau_l e^{i\nu\pi/2} K_\nu(e^{i\pi/4} r) e^{il\theta}$$

with the modified Bessel function $K_\nu(z) = (i\pi/2)e^{i\nu\pi/2} H_\nu(iz)$. The constant $\tau_l > 0$ is determined by normalization $\|\psi_{\pm l}\|_{L^2(\boldsymbol{R}^2)} = 1$, and the phase factor $e^{i\nu\pi/2}$ is taken so that $\psi_{+l} - \psi_{-l} \to 0$ as $r \to 0$. Thus the closure $\overline{H}_{0\alpha}$ has its deficiency indices $(2, 2)$. By the general theory due to Krein, $\overline{H}_{0\alpha}$ has a family of self–adjoint extensions parameterized by 2×2 unitary mapping from one deficiency space to the othere one. Let

$$U = U(\eta, a, b) = e^{i\eta} \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1, \quad \eta \in \boldsymbol{R}, \quad a, b \in \boldsymbol{C},$$

be a 2×2 unitary matrix. We denote by the same notation U the mapping $U : \Sigma_+ \to \Sigma_-$ defined by

$$U\psi_+ = \tilde{e}_0\psi_{-0} + \tilde{e}_1\psi_{-1}, \quad \psi_+ = e_0\psi_{+0} + e_1\psi_{+1},$$

with ${}^t(\tilde{e}_0, \tilde{e}_1) = U^t(e_0, e_1)$. Then, for given $U = U(\eta, a, b)$, there exists a self–adjoint extension H_α^U such that

$$H_\alpha^U u = \overline{H}_{0\alpha} v + i\psi_+ - iU\psi_+$$

with domain

$$\mathcal{D}(H_\alpha^U) = \{u \in L^2(\boldsymbol{R}^2) : u = v + \psi_+ + U\psi_+, \; v \in \mathcal{D}(\overline{H}_{0\alpha}), \; \psi_+ \in \Sigma_+\}.$$

The unitary matrix $U(\eta, a, b)$ specifies the boundary condition at the origin. If, for example, $U = U(0, -1, 0)$, then the domain $\mathcal{D}(H_\alpha^U)$ is given by

$$\mathcal{D}(H_\alpha^U) = \{u \in L^2(\boldsymbol{R}^2) : H_{0\alpha} u \in L^2(\boldsymbol{R}^2), \; \lim_{r\to 0} u(x) = 0\},$$

and this extension coincides with H_α defined by (1.1). We denote by $f^U(\omega \to w'; \lambda)$ the scattering amplitude for the pair (H_α^U, H_0). It is defined through the asymptotic behavior

$$u(x) = e^{i\sqrt{\lambda}x\cdot\omega} + f^U(\omega \to w'; \lambda) e^{i\sqrt{\lambda}r} r^{-1/2} (1 + o(1)), \quad x = r\omega', \quad |x| \to \infty,$$

of the solution $u(x)$ to equation $(H_{0\alpha} - \lambda)u = 0$, where $u(x)$ satisfies the boundary condition specified by $U(\eta, a, b)$ at the origin. The scattering amplitude $f^U(\omega \to w'; \lambda)$ has been calculated in the recent work [2] and it takes a rather complicated form. We do not copy the explicit form obtained there. If, in particular, $U = U(0, -1, 0)$, then $f^U(\omega \to \omega'; \lambda) = c(\lambda) f_\alpha(\omega' - \omega)$. As previously stated, this is just the scattering amplitude calculated by [1, 6]. According to the results in [2], we obtain the following theorem.

Theorem 2. 1 *Let the notations and assumptions be as in Theorem 1.1.*

(1) *If* $\dim \mathcal{E}_1 = 2$, *then*

$$\lim_{\varepsilon \to 0} f_\varepsilon(\omega \to \omega'; \lambda) = f^U(\omega \to \omega'; \lambda)$$

with $U = U(\pi/2, \exp(i(1/2 - \alpha)\pi), 0)$.

(2) *Assume that* $\dim \mathcal{E}_1 = 1$. *Then one has :*

(*i*) *Assume that* $0 < \alpha < 1/2$. *If* $c_0 \neq 0$, *then*

$$\lim_{\varepsilon \to 0} f_\varepsilon(\omega \to \omega'; \lambda) = f^U(\omega \to \omega'; \lambda)$$

with $U = U((1 - \alpha/2)\pi, \exp(-i\alpha\pi/2), 0)$, *and if* $c_0 = 0$, *then*

$$\lim_{\varepsilon \to 0} f_\varepsilon(\omega \to \omega'; \lambda) = f^U(\omega \to \omega'; \lambda)$$

with $U = U((1/2 + \alpha/2)\pi, \exp(i(1 - \alpha)\pi/2), 0)$.

(*ii*) *Assume that* $\alpha = 1/2$ *and set* $c = c_1/c_0$ *again. Then*

$$\lim_{\varepsilon \to 0} f_\varepsilon(\omega \to \omega'; \lambda) = f^U(\omega \to \omega'; \lambda)$$

with $U = U(3\pi/4, a, b)$, *where*

$$a = \frac{1}{\sqrt{2}}\left(1 - i\frac{1 - |c|^2}{1 + |c|^2}\right), \quad b = \frac{2ic}{\sqrt{2}(1 + |c|^2)}.$$

(*iii*) *Assume that* $1/2 < \alpha < 1$. *If* $c_1 \neq 0$, *then*

$$\lim_{\varepsilon \to 0} f_\varepsilon(\omega \to \omega'; \lambda) = f^U(\omega \to \omega'; \lambda)$$

with $U = U((1/2 + \alpha/2)\pi, \exp(i(1 - \alpha)\pi/2), 0)$, *and if* $c_1 = 0$, *then*

$$\lim_{\varepsilon \to 0} f_\varepsilon(\omega \to \omega'; \lambda) = f^U(\omega \to \omega'; \lambda)$$

with $U = U((1 - \alpha/2)\pi, \exp(-i\alpha\pi/2), 0)$.

(3) *Assume that* $\dim \mathcal{E}_1 = 0$. *Then*

$$\lim_{\varepsilon \to 0} f_\varepsilon(\omega \to \omega'; \lambda) = f^U(\omega \to \omega'; \lambda)$$

with $U = U(0, -1, 0)$.

If $\alpha = 1/2$, then the theorem above shows that the limit Hamiltonian H_α^U is realized through the unitary matrix $U(\eta, a, b)$ with $b \neq 0$. This means the lack of conservation of angular momentum in the limit $\varepsilon \to 0$. For example, incoming particles with only $l = 0$ as angular momentum may have angular momentum $l = 1$ after scattering by the field b_ε with support small enough.

The low–energy analysis for magnetic Schrödinger operators with long–range perturbations is important in showing the resolvent convergence in norm of the scaled Hamiltonian H_ε to some self–adjoint extension of $H_{0\alpha}$ and in studying the spectral structure of H_ε. The matter will be discussed in detail elsewhere.

References

[1] Y. Aharonov and D. Bohm, Significance of electromagnetic potential in the quantum theory, *Phys. Rev., 115* : 485–491 (1959).

[2] R. Adami and A. Teta, On the Aharonov–Bohm Hamiltonian, Preprint, 1997.

[3] S. Albeverio, F. Gesztesy, R. Hoegh–Krohn and H. Holden, ***Solvable Models in Quantum Mechanics***, Springer–Verlag, 1988.

[4] A. Jensen and T. Kato, Spectral properties of Schrödinger operators and time–decay of the wave functions, *Duke Math. J., 46*: 583–611 (1979).

[5] M. Loss and B. Thaller, Scattering of particles by long–range magnetic fields, *Ann. of Phys., 176*: 159–180 (1987).

[6] S. N. M. Ruijsenaars, The Ahanorov–Bohm effect and scattering theory, *Ann. of Phys., 146*: 1–34 (1983).

[7] H. Tamura, Magnetic scattering at low energy in two dimendions, Preprint Okayama University, 1998. *Rev. Math. Phys., 7*: 443–480 (1995).