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Smoothing effect in Gevrey classes for Schrödinger equations  
(Structure of Solutions for Partial Differential Equations)

Author(s)  
Kajitani, Kunihiko

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Smoothing effect in Gevrey classes for Schrödinger equations

Kunihiko Kajitani
Institute of Mathematics
University of Tsukuba
305 Tsukuba Ibaraki Japan

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Introduction

We shall investigate Gevrey smoothing effects of the solutions to the Cauchy problem for Schrödinger type equations. Roughly speaking, we shall prove that if the initial data decay as \( e^{-c|x|^\kappa} (0 < \kappa \leq 1, c > 0) \), then the solutions belong to Gevrey class \( \gamma^{1/\kappa} \) with respect to the space variables. Let \( T > 0 \). We consider the following Cauchy problem,

(1) \[ \frac{\partial}{\partial t} u(t, x) - i \Delta u(t, x) - b(t, x, D)u(t, x) = 0, \quad t \in [-T, T], \quad x \in \mathbb{R}^n, \]

(2) \[ u(0, x) = u_0(x), \quad x \in \mathbb{R}^n, \]

where

(3) \[ b(t, x, D)u = \sum_{j=1}^{n} b_j(t, x)D_j u + b_0(t, x)u, \]

and \( D_j = -i \frac{\partial}{\partial x_j} \). We assume that the coefficients \( b_j(t, x) \) satisfy

(4) \[ |D_x^\alpha b_j(t, x)| \leq C_b (\rho_b < x >)^{-|\alpha|} |\alpha|!^s, \]

for \( (t, x) \in [-T, T] \times \mathbb{R}^n, \alpha \in \mathbb{N}^n \), where \( < x > = (1 + |x|^2)^{1/2} \). Moreover we assume that there is \( \kappa \in (0, 1] \) such that

(5) \[ \lim_{|x| \to \infty} Re b_j(t, x) < x >^{1-\kappa} = 0, \text{ uniformly in } t \in [-T, T]. \]

For \( \rho \geq 0 \) let define an exponential operator \( e^{\rho<D>^\kappa} \) as follows,

\[ e^{\rho<D>^\kappa} u(x) = \int_{\mathbb{R}^n} e^{i\phi(x, \xi) + \rho <\xi>^\kappa} \hat{u}(\xi)d\xi \]

where \( \hat{u}(\xi) \) stands for a Fourier transform of \( u \) and \( d\xi = (2\pi)^{-n}d\xi \). For \( \varepsilon \in \mathbb{R} \) denote \( \phi_\varepsilon = x\xi - i\varepsilon x\xi < x >^\sigma - < \xi >^\delta -1, \) where \( \sigma + \delta = \kappa \) and we define

\[ I_{\phi_\varepsilon}(x, D)u(x) = \int_{\mathbb{R}^n} e^{i\phi_\varepsilon(x, \xi)} \hat{u}(\xi)d\xi. \]

Then our main theorem follows.
Theorem. Assume (4)-(5) are valid and there is \( \varepsilon > 0 \) such that \( I_{\phi}, u_0 \in L^2(R^n) \). Then if \( \delta \leq 1 \), there exists a solution of (1)-(2) satisfying that there are \( C > 0, \rho > 0 \) and \( \delta > 0 \) such that

\[
|\partial_x^\alpha u(t, x)| \leq C(\rho|t|)^{-|\alpha||x|\rho^2}e^{\delta|x|^\kappa},
\]

for \( (t, x) \in [-T, T]\setminus[0 \times R^n, \alpha \in N^n] \).

Remark. (i) Kato T. and Yajima in [12] considered the smoothing effect phenomena. A. Jensen in [6] and Hayashi, Nakamitsu & Tsutsumi in [5] showed that if \( \varphi \in x \geq 0 \) \( u_0(x) \in L^2(R^n) \), the solution \( u \) of (1)-(2) belongs to \( H^k_{lo} \) for \( \varphi \neq 0 \), Hayashi & Saitoh in [4] proved that if \( \varphi^{<x>^\kappa}u_0 \) \( \delta > 0 \) is in \( L^2(R^n) \), the solution \( u \) is analytic in \( x \) for \( \varphi \neq 0 \) and De Bouard, Hayashi & Kato in [1], Kato & Taniguti in [11] show that if \( u_0 \) satisfies \( \|(x \cdot \nabla)^2u_0\| \leq \delta^{j+1} \) for \( j = 0, 1, 2, \ldots \), then the solution belongs to Gevrey \( \gamma^\kappa \) with respect to \( x \) for \( \varphi \neq 0 \). Theorem 1 is proved by Kajitani in [8] and [10], when \( \sigma = \kappa = 1 \).

1 Weighted Sobolev spaces

We introduce some Sobolev spaces with weights. Let \( \rho, \delta \) be real numbers and \( \kappa \in (0, 1] \). Define

\[
\tilde{H}^\kappa = \{u \in L^2_{lo}(R^n); \varphi^{<x>^\kappa}u \in L^2(R^n)\}.
\]

For \( \rho \geq 0 \) let define

\[
H^\kappa_{\rho} = \{u \in L^2(R^n); Fu(\xi) \in \tilde{H}^\kappa_{\rho}(R^n)\},
\]

where \( Fu \) stands for the Fourier transform of \( u \). For \( \rho < 0 \) we define \( H^\kappa_{-\rho} \) as the dual space of \( H^\kappa_{\rho} \). Then the Fourier transform \( F \) becomes bijective from \( H^\kappa_{\rho} \) to \( \tilde{H}^\kappa \). We define the operator \( \varphi^{<D>^\kappa} \) mapping continuously from \( H^\kappa_{\rho,\delta} \) to \( H^\kappa_{\rho,1} \) as follows;

\[
\varphi^{<D>^\kappa}u(x) = F^{-1}(\varphi^{<\xi>^\kappa}Fu(\xi))(x),
\]

for \( u \in H^\kappa_{\rho,1} \) and \( \varphi^{<x>^\kappa} \) maps continuously from \( \tilde{H}^\kappa_{1} \) to \( \tilde{H}^\kappa_{1,1} \). We define for \( \delta \geq 0 \) and \( \rho \in R \)

\[
H^\kappa_{\rho,\delta} = \{u \in H^\kappa_{\rho}; \varphi^{<D>^\kappa}u \in \tilde{H}^\kappa_{\delta}\}
\]

For \( \delta < 0 \) we define \( H^\kappa_{\rho,\delta} \) as the dual space of \( \tilde{H}^\kappa_{\rho,1} \). We note that \( H^\kappa_{\rho,0} = H^\kappa_{\rho}, H^\kappa_{0,\delta} = \tilde{H}^\kappa_{\delta} \) and \( H^\kappa_{0,0} = L^2(R^n) \). Furthermore we define for \( \rho \geq 0 \) and \( \delta \in R \)

\[
H^\kappa_{\rho,\delta} = \{u \in \tilde{H}^\kappa_{\delta}; \varphi^{<x>^\kappa}u \in H^\kappa_{\rho}\}
\]

and for \( \rho < 0 \) define \( \tilde{H}^\kappa_{\rho,\delta} \) as the dual space of \( \tilde{H}^\kappa_{\rho,1} \). Denote by \( H' \) the dual space of a topological space \( H \). Then \( H'_{\rho,\delta} = H'_{\rho,1,\delta} \) and \( H'_{\rho,\delta} = \tilde{H}^\kappa_{\rho,1,\delta} \) hold for any \( \rho, \delta \in R \). We shall prove \( H'_{\rho,\delta} = H'_{\rho,\delta} \) later on (see Proposition 3.8).

Lemma 1.1. Let \( \rho, \delta \in R \). Then

(i) \( H^\kappa_{\rho,\delta} = e^{-\rho<\xi>^\kappa} \tilde{H}^\kappa_{\rho,\delta} \)

(ii) \( \tilde{H}^\kappa_{\rho,\delta} = e^{-\rho<\xi>^\kappa} \tilde{H}^\kappa_{\rho,\delta} \)

Lemma 1.2 Let \( 1 > \rho > 0, \delta \in R \) and \( u \in \tilde{H}^\kappa_{\rho,\delta} \). Then

\[
|D^\alpha_x u(x)| \leq C_n(1 - \varepsilon)^{-\rho/2}||u||_{H^\kappa_{\rho,\delta}}(\varepsilon\rho)^{-|\alpha||x|\rho^2}e^{\delta|x|^\kappa},
\]

for \( x \in R^n, \alpha \in N^n \) and \( 0 < \varepsilon < 1 \).

We can prove these lemmas analogously to the case of \( \kappa = 1 \) which is proved in [10].
2 Almost analytic extension of symbols

Following Hörmander's notation we define the symbol classes of pseudo-differential operators. Let $m(x, \xi), \varphi(x, \xi), \psi(x, \xi)$ a weight and $g = \varphi^{-2} dx^2 + \psi^{-2} d\xi^2$ a Riemann metric. We denote by $S(m, g)$ the set of symbols $a(x, \xi)$ satisfying

$$|a^{(\alpha)}_\beta(x, \xi)| \leq C_{\alpha \beta} m(x, \xi) \psi^{-|\alpha|} \varphi^{-|\beta|},$$

for $(x, \xi) \in \mathbb{R}^{2n}, \alpha, \beta \in \mathbb{N}^n$, where $a^{(\alpha)}_\beta = \partial^{\alpha} \psi^{-|\beta|} a$. Let $d \geq 1$. Moreover we call that a function $a(x, \xi) \in S(m, g)$ belongs to $\gamma^d S(m, g)$, if $a(x, \xi)$ satisfies that there are $C_a, \rho_0 > 0$ such that

$$|a^{(\alpha)}_\beta(x, \xi)| \leq C_a \rho_0^{-|\alpha + \beta|} |\alpha + \beta|^d \psi^{-|\beta|} \varphi^{-|\alpha|}$$

for $(x, \xi) \in \mathbb{R}^{2n}, \alpha, \beta \in \mathbb{N}^n$. Let $d > 1$ and $\chi(t) \in C_0^\infty((0, \infty))$ satisfying that $\chi(t) = 0, t \leq 1/2, \chi(t) = 1, t \leq 1$, and

$$|D^k_t \chi(t)| \leq C_0 \rho_0^{-k} k!^d,$$

for $t \in \mathbb{R}, k \in \mathbb{N}$. Then for a weight $w(x, \xi) \in \gamma^d S(m, g_1)$ and a parameter $b > 0$ we can see easily that $\chi(bw(x, \xi)) \in \gamma^d S(1, g_1)$ satisfying

$$|D^k_x D^\xi \chi(bw(x, \xi)))| \leq C_1 \rho_0^{-|\alpha + \beta|} |\alpha + \beta|^d < x >^{-|\beta|} < \xi >^{-|\alpha|},$$

for $(x, \xi) \in \mathbb{R}^{2n}, \alpha, \beta \in \mathbb{N}^n, b \geq 1$.

**Lemma 2.1.** Let $d \geq 1$ and $\{p_k(x, \xi)\}_{k=1}^\infty$ be a series of symbols satisfying

$$|p^{(\alpha)}_{k(\beta)}(x, \xi)| \leq m(x, \xi) (< x > < \xi >)^k \rho_0^{-|\alpha + \beta| - k} |\alpha + \beta|^d k!^d < x >^{-|\beta|} < \xi >^{-|\alpha|},$$

for $(x, \xi) \in \mathbb{R}^{2n}, \alpha, \beta \in \mathbb{N}^n$ and $k \geq 0$. Then there is $p(x, \xi) \in \gamma^d S(m, g_1)$ such that

$$p(x, \xi) - \sum_{k=0}^{N-1} p_k(x, \xi) \in \gamma^d S(m((x) \langle \xi \rangle \rho_p)^{-N} N!^d, g_1),$$

for any integer $N \geq 0$.

**Proof** This lemma is essentially a result of [2]. The case of $d = 1$ is explained in [10]. Here we prove the lemma in the case of $d > 1$. Let $b_k = \rho_0^{-1} k!^d M$ and $M \geq 2$. Define

$$p(x, \xi) = \sum_{k=0}^\infty p_k(x, \xi) \chi(b_k((x) \langle \xi \rangle)^{-1}),$$

Then we have

$$|p^{(\alpha)}_{k(\beta)}(x, \xi)| = \sum_{k} \sum_{\alpha', \beta'} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) \left( \begin{array}{c} \beta \\ \beta' \end{array} \right) p^{(\alpha')}_{k(\beta')} \chi(b_k((x) \langle \xi \rangle)^{-1}))^{(\alpha = \alpha')} \leq \sum_{k} \sum_{\alpha', \beta'} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) \left( \begin{array}{c} \beta \\ \beta' \end{array} \right) m(x, \xi) \rho_k^{-|\alpha + \beta'|} |\alpha + \beta'|^d < x >^{-|\beta|} < \xi >^{-|\alpha|} \times M^{-k} C_0 \rho_0^{-|\alpha - \alpha' + \beta - \beta'|} |\alpha - \alpha' + \beta = \beta'|^d.$$
\[
\leq 2\frac{\rho_0}{\rho_0 - \rho_p} m(x, \xi) \rho^{-|\alpha + \beta|} |\alpha + \beta|^d (x)^{-|\beta|} (\xi)^{|-|\alpha|},
\]
for \((x, \xi) \in \mathbb{R}^{2n}, \alpha, \beta \in \mathbb{N}^n\). Here we used the following inequality
\[
(2.7) \quad \sum_{\alpha' \leq \alpha} (\frac{\alpha}{\alpha'}) \rho_p^{-|\alpha'|} |\alpha'|!^d \rho_0^{-|\alpha - \alpha'|} |\alpha - \alpha'|!^d \leq \frac{\rho_0}{\rho_0 - \rho_p} |\alpha|!^d,
\]
for \(\rho_0 > \rho_p\). Moreover we can write
\[
p(x, \xi) = \sum_{k=0}^{N-1} p_k(x, \xi)
\]
\[
= \sum_{k=N}^{\infty} p_k(x, \xi) \chi(b_k((x, \xi)^{-1}) - 1) + \sum_{k=0}^{N} p_k(x, \xi) (1 - \chi(b_k((x, \xi)^{-1})) - 1)
\]
\[
=: I + II.
\]
Noting that \(\rho_p^{-k} k!^d (M(x, \xi)^{-1}) \leq 1\) on \(\text{supp} \chi(b_k((x, \xi)^{-1}))\) for \(k \geq N\) and \(\rho_p^{-k} k!^d (M(x, \xi)^{-1}) \geq 1/2\) on \(\text{supp} (1 - \chi(b_k((x, \xi)^{-1}))\) for \(k \leq N - 1\) respectively, we can see that \(I\) and \(II\) belong to \(\gamma^d S(m(x, \xi)\rho_p)^{-N} N!^d, g\). Q.E.D.

Let \(a(x, \xi) \in \gamma^d S(m, g_1)\), that is, \(a(x, \xi)\) satisfies (2.1). Denote \(b_{\alpha}(x) = B \rho_\alpha^{-1} 4^n \langle x \rangle^{-1} |\alpha|!^{\frac{d-1}{|\alpha|}}\) for \(x \in \mathbb{R}^n\). We define an almost analytic extension of \(a(x, \xi)\) as follows,
\[
(2.8) \quad a(x + iy, \xi + i\eta) = \sum_{\alpha, \beta} a^{(\alpha)}((\beta)\xi\alpha) x, (-y)\beta(i\eta)\alpha(b_{\beta}(x)|y|)(\chi\alpha!\beta!)^{-1},
\]
for \(x, y, \xi, \eta \in \mathbb{R}^n\), where \(a^{(\alpha)}(x, \xi) = \partial_{\xi}^\alpha (-i \partial_x)^\beta a(x, \xi)\). Then we can prove easily

**Proposition 2.2** Let \(a(x, \xi) \in \gamma^d S(m, g_1)\). Then the function \(a(x + iy, \xi + i\eta)\) defined by (2.8) satisfies the following properties.

(i) \(|D^\alpha_{\xi} D^\beta_{\eta} a(x + iy, \xi + i\eta)| \leq C m(x, \xi) (C \rho_a)^{-|\alpha + \beta + \gamma + \delta|} (x)^{-|\beta|} (\xi)^{-|\alpha|} (y)^{-|\gamma|} (\eta)^{-|\delta|} |\alpha + \beta + \gamma + \delta|^d|\).

(ii) \(|(\partial_{\xi_j} + i \partial_{\eta_j}) D^\alpha_{\xi} D^\beta_{\eta} a(x + iy, \xi + i\eta)| \leq C m(x, \xi) (C \rho_a)^{-|\alpha + \beta + \gamma + \delta|} e^{-c_0 (\frac{|y|}{c_0})^{\frac{1}{d}}} (x)^{-|\beta|} (\xi)^{-|\alpha|} (y)^{-|\gamma|} (\eta)^{-|\delta|} |\alpha + \beta + \gamma + \delta|^d|\).

(iii) \(|(\partial_{\xi_j} + i \partial_{\eta_j}) D^\alpha_{\xi} D^\beta_{\eta} a(x + iy, \xi + i\eta)| \leq C m(x, \xi) (C \rho_a)^{-|\alpha + \beta + \gamma + \delta|} e^{-c_0 (\frac{|y|}{c_0})^{\frac{1}{d}}} (x)^{-|\beta|} (\xi)^{-|\alpha|} (y)^{-|\gamma|} (\eta)^{-|\delta|} |\alpha + \beta + \gamma + \delta|^d|\).

For simplicity denote \(\gamma^{1/k} S/(e^{(x, \xi)}^\ast + \rho(\xi)^\ast, g_0)\) by \(A_{\rho, \delta}^\kappa\), where \(g_0 = dx^2 + d\xi^2\). For \(a_i \in A_{\rho_i, \delta_i}^\kappa(i = 1, 2)\) we define a product of \(a_1\) and \(a_2\) as follows,
\[
(2.9) \quad (a_1 \circ a_2)(x, \xi) = \text{os} - \int \int_{\mathbb{R}^{2n}} e^{-i\eta \cdot y} a_1(x, \xi + \eta) a_2(x + y, \xi) d\eta d\bar{\eta},
\]
\[
= \lim_{\epsilon \to 0} \int \int_{\mathbb{R}^{2n}} e^{-i\eta \cdot y - (|y|^2 + |\eta|^2) \epsilon}\ a_1(x, \xi + \eta) a_2(x + y, \xi) d\eta d\bar{\eta},
\]
where \(d\eta = (2\pi)^{-n} d\eta\). Then we can show the proposition below.
Proposition 2.3. (i) Let $\kappa \leq 1$ and $a_i \in A^\kappa_{\rho_i, \delta_i}, i = 1, 2$. Then there is $\epsilon_0 > 0$ such that if $|\rho_1|, |\delta_2| \leq \epsilon_0$, the product $a_1 \circ a_2$ belongs to $A^\kappa_{\rho_1+\rho_2, \delta_1+\delta_2}$.

(ii) Let $a_i \in A^\kappa_{\rho_i, \delta_i}, i = 1, 2, 3$. Then if $|\rho_i|(i = 1, 2), |\delta_i|(i = 2, 3) \leq \epsilon_0/2$, we have $(a_1 \circ a_2) \circ a_3 = a_1 \circ (a_2 \circ a_3)$.

Proposition 2.4 Let $d \geq 1$ and $a_i \in \gamma^d S((x)^{m_1}(\xi)^{\ell_1} g_1), i = 1, 2$. Then $a_1 \circ a_2$ belongs to $S((x)^{m_1+m_2}(\xi)^{\ell_1+\ell_2} g_1)$ and moreover we can decompose

\[
(2.10) \quad a_1 \circ a_2(x, \xi) = p(x, \xi) + r(x, \xi),
\]

where $p(x, \xi) \in \gamma^d S((x)^{m_1+m_2}(\xi)^{\ell_1+\ell_2} g_1)$ satisfies that there are $C > 0$ and $\epsilon_0 > 0$ such that

\[
(2.11) \quad p(x, \xi) - \sum_{|\gamma| < N} \gamma!^{-1} a_1^{(\gamma)}(x, \xi) a_2^{(\gamma)}(x, \xi) \in \gamma^d S((C^{1+N} N! (x)^{m_1+m_2-N}(\xi)^{\ell_1+\ell_2-N}, g),
\]

for any non negative integer $N$, and $r(x, \xi) \in A^{1/d}_{\epsilon_0, -\epsilon_0}$.

3 Pseudo-differential operators

Let $\kappa > 1$. Now we want to define a pseudo differential operator $a(x, D)$ for a symbol $a(x, \xi) \in A^\kappa$, which operates from $H^\kappa_{\rho, \delta}$ to $H^\kappa_{\rho, \delta}$, $a \in A^\kappa$. When $\rho$ and $\delta$ are non positive, since $A^\kappa$ is contained in the usual symbol class $S^0_{0,0}$ (denote by $S^m_{\rho, \delta}$ the Hörmander’s class), we can define

\[
(3.1) \quad a(x, D)u(x) = \int e^{ix\xi} a(x, \xi) \hat{u}(\xi) d\xi,
\]

for $u \in L^2(R^n)$ and for $a \in A^\kappa$. Moreover for $a_i \in A^\kappa_{\rho_i, \delta_i}, i = 1, 2$ ($\rho_i$ and $\delta_i$ non positive) the symbol $\sigma(a_1(x, D)a_2(x, D))(x, \xi)$ of the product of $a_1(x, D)$ and $a_2(x, D)$ can be written as follows,

\[
(3.2) \quad \sigma(a_1(x, D)a_2(x, D))(x, \xi) = (a_1 \circ a_2)(x, \xi)
\]

and we have

\[
(3.3) \quad a_1(x, D)(a_2(x, D)u)(x) = (a_1 \circ a_2)(x, D)u(x)
\]

for $u \in L^2(R^n)$, where $a_1 \circ a_2$ is defined by (2.9). Next we shall show that (3.2) and (3.3) are valid for any $\rho_i, \delta_i$. To do so, we need some preparations. Let $a \in A^\kappa_{\rho, \delta}$ and $u \in H^\kappa_{\rho}$. Then we can define $a(x, D)u(x)$ which belongs to $\tilde{H}^\kappa_{\rho}$. In fact, put $\tilde{a}(z, \xi) = e^{-\delta(z)^2 + \rho(\xi)^2} a(z, \xi)$. Then $\tilde{a}(x, \xi) \in A^\kappa_{0,0}$. Noting that $e^{\epsilon(\xi)^s} \hat{u}(\xi)$ we can define

\[
(3.4) \quad e^{-\epsilon(\xi)^s} a(x, D)u(x) = \int e^{ix\xi} \hat{a}(x, \xi) e^{\epsilon(\xi)^s} \hat{u}(\xi) d\xi,
\]

which is in $L^2$, that is, $a(x, D)u(x) \in \tilde{H}^\kappa_{\rho}$. For $\epsilon > 0$ we denote $\chi_\epsilon(x) = e^{-\epsilon(x)^2}$ and $\chi_\epsilon(D) = e^{-\epsilon(D)^2}$.

Lemma 3.1. (i) Let $a \in A^\kappa_{\rho, \delta} (\rho, \delta \in R), u \in L^2$ and $\epsilon_0 > 0$ chosen in Proposition 2.3. Then for any $\epsilon > 0$

\[
(3.5) \quad a(x, D)(\chi_\epsilon(D)\chi_\epsilon(x))u(x) = (a(x, \xi)\chi_\epsilon(\xi)) \circ \chi_\epsilon(x))(x, D)u(x)
\]

and

\[
(3.6) \quad (a\chi_\epsilon(\xi)) \circ \chi_\epsilon(x) \in A^\kappa_{-\epsilon_0, -\epsilon_0}.
\]
(ii) Let $u \in L^2$ and $\epsilon_0 > 0$ chosen in Proposition 2.3. Then there is $\epsilon_1 > 0$ such that for any $\epsilon > 0$

\begin{equation}
\epsilon^{-\rho<\xi>^*} (\epsilon^{-\delta<\xi>^*} \chi_\epsilon(x) \chi_\epsilon(D) u)(x) = a_\epsilon(x, D) u(x),
\end{equation}

where

\begin{equation}
a_\epsilon(x, \xi) = \epsilon^{-\rho<\xi>^*} \circ (\epsilon^{-\delta<\xi>^*} \chi_\epsilon(x) \chi_\epsilon(\xi)) \in A^\kappa_{-\rho-\epsilon_0, -\delta-\epsilon_0},
\end{equation}

for $|\rho| \leq \epsilon_0$ and $\rho < \epsilon_1$. We can prove the following lemma by use of Lemma 3.1.

**Lemma 3.2.** Let $u \in H^\kappa_{\rho_\delta}$ and $|\rho_1|, |\delta_1| \leq \epsilon_0/2$ ($\epsilon_0$ is given in Proposition 2.3). Then for any $\epsilon > 0$ there is $u_\epsilon \in H^\kappa_{\epsilon_0/2, \epsilon_0/2}$ such that

\begin{equation}
\|u - u_\epsilon\|_{H^\kappa_{\rho, \delta}} < \epsilon.
\end{equation}

**Lemma 3.3.** Let $a \in A^\kappa_{\rho_\delta}$, $0 < \epsilon_0', \epsilon_0 \leq \epsilon_0$ ($\epsilon_0$ is given in Proposition 2.3) and $u \in H^\kappa_{\epsilon_0', \epsilon_0}$. Then there is $\epsilon_2 > 0$ independent of $a, \rho$ and $\delta$ such that $a(x, D) u(x)$ belongs to $H^\kappa_{\epsilon_0', \epsilon_0}$ and $0 < \epsilon_0 - \delta \leq \epsilon_0$. We have

\begin{equation}
\epsilon^{-\rho<\xi>^*} (\epsilon^{-\delta<\xi>^*} \chi_\epsilon(x) \chi_\epsilon(D) u)(x) = a_\epsilon(x, D) u(x),
\end{equation}

which is in $H^\kappa_{\epsilon_0/\epsilon_0, \epsilon_0/\epsilon_0}$. Let $a \in A^\kappa_{\rho_\delta}$, $|\rho| \leq \epsilon_0/4$, $u \in H^\kappa_{\epsilon_0/2, \epsilon_0/2}$ and $|\rho_1|, |\delta_1| < \epsilon_0/4$. Put $w = \epsilon_1<\xi>^* e^{<\rho_1<\xi>^*} u$, which is in $H^\kappa_{\epsilon_0/2, \epsilon_0/2-\delta_1}$. Since we can write $u = \epsilon^{-\rho_1<\xi>^*} e^{<\rho_1<\xi>^*} w$, we get by use of Lemma 3.4 with $\epsilon_1' = \epsilon_0/2 - \rho_1, \epsilon_0 = \epsilon_0/2 - \delta_1, a_1 = a(x, \xi)e^{-\rho_1<\xi>^*}$ and $a_2 = e^{-\delta_1<\xi>^*} \in A^\kappa_{0,0}$, we obtain

\begin{equation}
\|au\|_{H^\kappa_{\rho_1\delta_1}} = \|a_1(x, D)u\|_{L^2} \leq C\|u\|_{H^\kappa_{\rho_1\delta_1}}
\end{equation}

for any $u \in H^\kappa_{\epsilon_0/2, \epsilon_0/2}$. Since $H^\kappa_{\epsilon_0/2, \epsilon_0/2}$ is dense in $H^\kappa_{\rho_1\delta_1}$, from Lemma 3.2, we get the following theorem.

**Theorem 3.5** Let $a \in A^\kappa_{\rho_\delta}$, $|\rho| \leq \epsilon_0/4$, $|\rho_1|, |\delta_1| < \epsilon_0/4$, where $\epsilon_0$ are given in Proposition 2.3. Then $a(x, D)$ maps from $H^\kappa_{\rho_1\delta_1}$ to $H^\kappa_{\rho_1-\rho, \delta_1-\delta}$ and satisfies the following inequality

\begin{equation}
\|au\|_{H^\kappa_{\rho_1-\rho, \delta_1-\delta}} \leq C\|u\|_{H^\kappa_{\rho_1\delta_1}},
\end{equation}

for any $u \in H^\kappa_{\rho_1\delta_1}$. For $a \in A^\kappa_{\rho_\delta}$, we define

\begin{equation}
a^t(x, \xi) = \text{osc} \int \int e^{iy\eta} a(x + y, \xi + y) dy\eta,
\end{equation}
and \( a^*(x, \xi) = a^t(\widetilde{x}, \xi) \). Then we can prove the following lemma, by the same way as that of the proof (i) of Proposition 2.3.

**Lemma 3.6.** Let \( a \in A_{\rho, \delta}^\kappa \) and \(|\rho|, |\delta| \leq \epsilon_0\). Then \( a^t(x, \xi) \) defined in (2.29) belongs to \( A_{\rho, \delta}^\kappa \). Moreover it holds

\[
(a^t(x, D)u, \varphi)_{L^2} = (u, a(x, D)\varphi)_{L^2},
\]

\[
(a^*(x, D)u, \varphi)_{L^2} = (u, a(x, D)\varphi)_{L^2},
\]

for any \( u, \varphi \in H_{\kappa}^\infty \).

The relation (3.14) and the inequality (3.12) yield

\[
|\langle a^t u, \varphi \rangle| \leq \|u\|_{H_{\rho + \rho_1, \delta_1, -\delta}^\kappa} \|\overline{a}\varphi\|_{H_{\rho, \rho_1, \delta, 1-\delta}^\kappa} \leq C \|u\|_{H_{\rho + \rho_1, \delta, 1-\delta}^\kappa} \|\varphi\|_{H_{\rho, \rho_1, \delta, 1}^\kappa},
\]

if \(|\rho|, |\delta| \leq \epsilon_0/4\) and \(|\rho_1|, |\delta_1| < \epsilon_0/4\). Therefore taking account that \( H_{\rho + \rho_1, \delta_1}^\kappa /2 / \delta_1 /2 \) is dense in \( H_{\rho, \delta_1}^\kappa \), we get from (3.14)

\[
\|a^t u\|_{H_{-\rho_1, \delta_1}^\kappa} \leq C \|u\|_{H_{\rho_1, \delta_1}^\kappa},
\]

for any \( u \in H_{\rho_1, \delta_1}^\kappa \). Thus we get the following proposition.

**Proposition 3.7.** Let \( a \in A_{\rho, \delta}^\kappa \) and \(|\rho|, |\delta| \leq \epsilon_0/4\) and \(|\rho_1|, |\delta_1| < \epsilon_0/4\). Then the pseudodifferential operators \( a^t(x, D) \) and \( a^*(x, D) \) satisfy (3.15).

Noting that \( (\delta^<x>^\kappa \epsilon^<\rho<D>^\kappa)^t = \epsilon^<\rho<D>^\kappa \delta^<x>^\kappa \), we have for \( u \in H_{\rho, \delta}^\kappa \)

\[
ep^<\rho<D>^\kappa \delta^<x>^\kappa u(x) = (\delta^<x>^\kappa \epsilon^<\rho<D>^\kappa)^t (\epsilon^<\rho<D>^\kappa \delta^<x>^\kappa \epsilon^<\rho<D>^\kappa) u(x) = e^<\rho<D>^\kappa \delta^<x>^\kappa \epsilon^<\rho<D>^\kappa \delta^<x>^\kappa u(x).
\]

Moreover we can see from Proposition 2.3 and Lemma 2.9 that \((\delta^<x>^\kappa \epsilon^<\rho<D>^\kappa)^t \in A_{\rho_1, 0}^\kappa \) is in \( A_{\rho_1, 0}^\kappa \). Hence we obtain the fact below.

**Proposition 3.8.** Let \(|\rho|, |\delta| \leq \epsilon_0/4\). Then \( u \) belongs to \( H_{\rho, \delta}^\kappa \) if and only if \( u \in H_{\rho, \delta}^\kappa \).

The following result on the multiple symbols of pseudodifferential operators is a special case of Lemma 2.2 of Chapter 7 in Kumanogo's book [12].

**Lemma 3.9.** Let \( r_j(x, \zeta) \in A_{\rho_0, 0}^\kappa (j = 1, 2, \ldots, v) \) and put 

\[
q_v(x, D) = r_1(x, D) r_2(x, D) \cdots r_v(x, D).
\]

Then the symbol \( q_v(x, \zeta) \) belongs to \( A_{\rho_0, 0}^\kappa \) and satisfies

\[
|q_v(\alpha, \beta)| \leq C_v \prod_{j=1}^v C_{r_j} \varepsilon_v^{-||\alpha + \beta||},
\]

for \((x, \zeta) \in R^{2n}, \alpha, \beta \in N^n\), where \( C \) is independent of \( v \) and \( \varepsilon_v = \min \{\varepsilon_{r_j}/4\} \).

We can prove easily the following lemma as a corollary of Lemma 3.9, by using the Neumann series method.
Lemma 3.10. Let \( r(x, \xi) \) be in \( A^\epsilon_{0,0} \). If \( C_r > 0 \) is sufficiently small, then there is the inverse \((I + r(x, D))^{-1}\) which is a pseudodifferential operator with its symbol contained in \( A^\epsilon_{0,0} \).

Lemma 3.11. Let \( j(x, \xi) \in \gamma^d S(\epsilon_1, g_1) \). Then if \( \epsilon_1 > 0 \) is small enough, there are \( k_1(x, \xi) \in \gamma^d S(\epsilon_1 < x >^{-1} < \xi >^{-1}, g_1), \epsilon_0 > 0 \) independent of \( \epsilon_1 \) and \( r_{\epsilon_0}(x, \xi) \in A^{1/d}_{\epsilon_0, -\epsilon_0} \) such that \((I + j(x, D))^{-1} = k(x, D) + k_1(x, D) + r_{\epsilon_0}(x, D)\), where \( k(x, \xi) = (1 + j(x, \xi))^{-1} \).

4 Fourier Integral Operators

For \( \phi \in \gamma^d S(\rho_0 < x >, g)(\rho_0, \delta_0 > 0) \), we denote \( \phi(x, \xi) = x\xi - i\theta(x, \xi) \).

For \( a \in A^\epsilon_{0,0} \) we define a Fourier integral operator with a phase function \( \phi(x, \xi) \) as follows,

\[
\begin{align*}
&\phi(x, \xi) = x\xi - i\theta(x, \xi), \\
&a_{\phi}(x, D)u(x) = \int_{\mathbb{R}^n} e^{i\phi(x, \xi)}a(x, \xi)\hat{u}(\xi)d\xi,
\end{align*}
\]

for \( u \in H_{\epsilon_0, \epsilon_0} \). Putting \( p(x, \xi) = a(x, \xi)e^{\theta(x, \xi)} \), we can see \( p(x, \xi) \in A^\epsilon_{p_0, \delta_0} \). Therefore we can regard \( a_{\phi}(x, D) \) as a pseudodifferential operator with its symbol \( p = ae^\theta \) defined in \( \S 2 \) and consequently it follows from Theorem 3.5 that \( a_{\phi}(x, D) \) acts continuously from \( H_{\epsilon_0, \delta_0}^d \) to \( H_{\rho_0, \delta_0}^d \). However in order to construct the inverse operator of \( p(x, D) \) it is better to regard \( p(x, D) \) as a Fourier integral operator. In particular for \( a = 1 \) we denote

\[
\begin{align*}
&I_{\phi}(x, D)u(x) = \int_{\mathbb{R}^n} e^{i\phi(x, \xi)}\hat{u}(\xi)d\xi, \\
&I_{\phi}^S(x, D)v(x) = \int e^{ix\xi}d\xi \int e^{i\phi(y, \xi)}v(y)dy.
\end{align*}
\]

Theorem 4.1. Let \( a \in \gamma^d S((x)^{\alpha}(\xi)^{\beta}, g_1), \theta \in \gamma^d S(\rho_{\theta}(\xi)^{\beta} + \delta_{\theta}(x)^{\beta}, g_1) \) and \( \phi = x\xi - i\theta(x, \xi) \). Assume \( d\kappa \leq 1 \). Then if \( \rho_{\theta}, \delta_{\theta} \) are sufficiently small, \( \tilde{a}(x, D) = I_{\phi}(x, D)a(x, D)I_{\phi}^{-1} \) and \( \tilde{a}'(x, D) = I_{\phi}(x, D)^{-1}a(x, D)I_{\phi}(x, D) \) are pseudodifferential operators of which symbols are given by

\[
\begin{align*}
&\tilde{a}(x, \xi) = p(x, \xi) + r(x, \xi), \\
&a'(x, \xi) = p'(x, \xi) + r'(x, \xi),
\end{align*}
\]

where

\[
\begin{align*}
&p(x, \xi) - a(x - i\nabla x\theta(x, \Phi), \xi + i\nabla x\theta(x, \Phi)) \in \gamma^d S(< x >^{\alpha-1} < \xi >^{\beta-1}, g_1), \\
&p'(x, \xi) - a(x + i\nabla x\theta(\Phi', \xi), \xi - i\nabla x\theta(\Phi', \xi)) \in \gamma^d S(< x >^{\alpha-1} < \xi >^{\beta-1}, g_1),
\end{align*}
\]

where \( \Phi = \Phi(x, x, \xi) \) and \( \Phi' = \Phi'(x, \xi, \xi) \) are given by (4.6) and (4.19) respectively and \( r, r' \) belong to \( A^\epsilon_{\rho_{\theta}, \delta_{\theta}} \) for any \( \epsilon_0 > 0 \) independent of \( \rho_{\theta} \).

This theorem is proved in [10] in the case of \( d = \kappa = 1 \). We can prove it similarly as that of [10].

Next we consider a phase function \( \theta \in \gamma^d S(\sigma^\beta, \xi, g_1) \). When \( \sigma + \delta = \kappa = 1/d < 1 \) or \( \sigma + \delta = 1 \) and \( d = \min(\delta^{-1}, \sigma^{-1}) \), Theorem 4.1 holds also, that is, we can prove Theorem 4.6 below. So far we consider
only \(d, \sigma, \delta, \kappa\) above. We note that \(d > 1\).

**Lemma 4.2.** Let \(a(x, \xi) \in \gamma^d S(<x >^m <\xi >^\ell, g_1)\) and \(\theta \in \gamma^d S(\rho_\theta <\xi >^\sigma <x >^\sigma, g_1)(\rho_\theta \geq 0)\). Put \(\phi = x\xi - i\theta(x, \xi)\) and \(\tilde{a}(x, D) = a_\phi(x, D)I_{-\phi}^R(x, D)\). If \(\rho_\theta\) is sufficiently small, then \(\tilde{a}(x, \xi)\) belongs to \(S(<x >^m <\xi >^\ell, g)\) and moreover satisfies

\[
\tilde{a}(x, \xi) = \tilde{p}(x, \xi) + r(x, \xi),
\]

for \(x, \xi \in \mathbb{R}^n\), and

\[
\tilde{p}(x, \xi) = \sum_{|\gamma| < N} \gamma^{-1} D\gamma^\phi \partial_\eta^\gamma \{a(x, \Phi(x, y, \eta))\}J(x, y, \eta)\big|_{y=x, \eta=\xi}
\]

\[
\in \gamma^d S(C^{1+4N}N^d <x >^m-N <\xi >^{\ell-N}, g_1)
\]

for any \(N\), where \(\Phi(x, y, \xi)\) is a solution of the following equation,

\[
\Phi(x, y, \xi) - i\tilde{\nabla}_x \theta(x, y, \Phi(x, y, \xi)) = \xi,
\]

\[
\tilde{\nabla}_x \theta(x, y, \xi) = \int_0^1 \nabla \theta(y + t(x - y), \xi) dt,
\]

\[
J(x, y, \xi) = \frac{D\Phi(x, y, \xi)}{D\xi} \text{ is the Jacobian of } \Phi, r(x, \xi) \in A_{-\epsilon_0}^{1/d}, \epsilon_0 > 0\] is independent of \(\rho_\theta\).

**Lemma 4.3.** Let \(a(x, \xi)\) and \(\theta\) be satisfied with the same condition as one of Lemma 4.2. For \(\phi = x\xi - i\theta(x, \xi)\) put \(a'(x, \xi) = I_{-\phi}^R(x, D)a_\phi(x, D)\). Then if \(\rho_\theta\) and \(\delta_\theta\) are sufficiently small, \(a'(x, \xi)\) belongs to \(S(<x >^m <\xi >^\ell, g)\) and moreover satisfies

\[
a'(x, \xi) = p'(x, \xi) + r'(x, \xi),
\]

\[
p'(x, \xi) = \sum_{|\gamma| < N} \gamma^{-1} D\gamma^\phi \partial_\eta^\gamma \{a(\Phi'(y, \xi, \eta))\}J'(y, \xi, \eta)\big|_{y=x, \eta=\xi}
\]

\[
\in \gamma^d S(C^{1+4N}N^d <x >^m-N <\xi >^{\ell-N}, g_1),
\]

for any non negative integer \(N\), where \(\Phi'(y, \xi, \eta)\) is a solution of the equation

\[
\Phi'(y, \xi, \eta) - i\tilde{\nabla}_\xi \theta(y, \Phi'(y, \xi, \eta), \eta) = y,
\]

\[
\tilde{\nabla}_\xi \theta(y, \xi, \eta) = \int_0^1 \nabla \theta(y + t(\xi - \eta), \xi) dt,
\]

and \(J'(y, \xi, \eta) = \frac{D\Phi'(y, \xi, \eta)}{D\xi}\), and \(r'(x, \xi) \in A_{-\epsilon_0}^{1/d}(\epsilon_0 > 0)\) is independent of \(\rho_\theta\).

**Lemma 4.4.** Let \(\vartheta(x, \xi) \in \gamma^d S(\rho_\vartheta (x) <\xi >^\delta, g_1)\). If \(\rho_\vartheta\) and \(\delta_\vartheta\) are sufficiently small, there is the inverse of \(I_\varphi(x, D)\), which maps continuously from \(H_{\rho_\vartheta, \delta_\vartheta}\) to \(H_{\rho_1-\rho_\vartheta, \delta_1-\delta_\vartheta}\) for \(|\rho_1|, |\delta_1|\) small enough and satisfies

\[
I_\varphi(x, D)^{-1} = I_{R_\varphi(x, D)}(I + j(x, D))^{-1} = (I + j'(x, D))^{-1}I_{R_\varphi(x, D)}(x, D)
\]

\[
= I_{R_\varphi(x, D)}(k(x, D) + k_1(x, D) + r(x, D)) = (k'(x, \xi) + k_1'(x, D) + r'(x, D)))I_{R_\varphi(x, D)}(x, D),
\]
where \( j(x, \xi) = J(x, 0, \xi) - 1 + r_1(x, \xi), j'(x, \xi) = J'(x, \xi, 0) - 1 + r_2(x, \xi), k(x, \xi) = J(x, 0, \xi)^{-1}, k'(x, \xi) = J'(x, \xi, 0)^{-1} \) and \( k_1, k'_1 \in \gamma^d S(<x>-1<\xi>-1, g_1) \) and \( r, r' \in A_{1/d}^{1/d} \).

**Lemma 4.5.** Let \( a(x, \xi) \) and \( \vartheta \) be satisfied with the same condition as one of Lemma 3.3. Let \( \phi = x\xi - i\theta \). Then we have

\[
\sigma(I_{\phi}(x, D)a(x, D))(x, \xi) = I_{\phi} \circ a(x, \xi) = e^{\theta(x, \xi)}(q(x, \xi) + r(x, \xi)),
\]

where \( r, r' \) is in \( A_{-\epsilon_0}^{1/d} \), if \( \rho_{\theta} \) is sufficiently small, and \( q, q' \) satisfies

\[
q(x, \xi) = \sum_{|\gamma| < N} \gamma^{-1} D^{\gamma}_{\xi} \partial^{\gamma}_{\eta} \{a(x + y - i\nabla_{\xi}\vartheta(x, \xi, \eta, \xi))\} \eta = \eta = 0 \\
\in \gamma^{d} S(C^{1} + NN!d(<x>^{-1}<\xi>^{-1}, g_1)),
\]

\[
q'(x, \xi) = \sum_{|\gamma| < N} \gamma^{-1} D^{\gamma}_{\xi} \partial^{\gamma}_{\eta} \{a(x, \xi + \eta - i\nabla_{x}\vartheta(x, y, \xi))\} \eta = \eta = 0 \\
\in \gamma^{d} S(C^{1} + NN!d(<x>^{-1}<\xi>^{-1}, g_1)),
\]

for any positive integer \( N \), and \( C > 0 \) and \( \epsilon_0 > 0 \) are independent of \( \rho_{\theta} \), where \( \nabla_{\xi}\vartheta(x, \xi, \eta) = \int_{0}^{1} \nabla_{\xi}\vartheta(x, \xi + t\eta) dt \) and \( \nabla_{x}\vartheta(x, y, \xi) = \int_{0}^{1} \nabla_{x}\vartheta(x + ty, \xi) dt \).

Summing up Lemma 4.2-Lemma 4.5, we obtain the following theorem.

**Theorem 4.6.** Let \( a \in \gamma^d S(<x>^{m}<\xi>^{l}, g_1) \), \( \vartheta \in \gamma^d S(\rho_{\theta}<\xi>^{\delta}<x>^{\sigma}, g_1) \) and \( \phi = x\xi - i\theta(x, \xi) \). Assume that \( \sigma + \delta = \kappa = 1/d < 1 \) or \( \sigma + \delta = \kappa = 1 \), \( d = \min(\delta^{-1}, \sigma^{-1}) \). Then if \( \rho_{\theta}, \delta_{\theta} \) are sufficiently small, \( \tilde{a}(x, D) = I_{\phi}(x, D)a(x, D)I_{\phi}^{-1} \) and \( \tilde{a}'(x, D) = I_{\phi}(x, D)-1(ax, D)I_{\phi}(x, D) \) are pseudodifferential operators of which symbols are given by

\[
\tilde{a}(x, \xi) = p(x, \xi) + r(x, \xi),
\]

\[
da'(x, \xi) = p'(x, \xi) + r'(x, \xi),
\]

where

\[
p(x, \xi) - a(x - i\nabla_{\xi}\vartheta(x, \Phi), \xi + i\nabla_{x}\vartheta(x, \Phi)) \in \gamma^{d} S(<x>^{m-1}<\xi>^{l-1}, g_1),
\]

\[
\tilde{p}'(x, \xi) - a(x + i\nabla_{\xi}\vartheta(\Phi', \xi), \xi - i\nabla_{x}\vartheta(\Phi', \xi)) \in \gamma^{d} S(<x>^{m-1}<\xi>^{l-1}, g_1),
\]

where \( \Phi = \Phi(x, x, \xi) \) and \( \Phi' = \Phi'(x, \xi, \xi) \) are given by (4.6) and (4.10) respectively and \( r, r' \) belong to \( A_{1/d}^{1/d} \) for an \( \epsilon_0 > 0 \) independent of \( \rho_{\theta} \).

5 **Criterion to \( L^2 \)-well posed Cauchy problem**

For \( T > 0 \) let consider the following Cauchy problem,

\[
\partial_t u(t, x) - i\Delta u(t, x) - b(t, x, D)u(t, x) = 0,
\]
for \((t, x) \in (0, T) \times \mathbb{R}^n\). We assume that \(b(t, x, \xi)\) is in \(C^0([0, T]; S^1_{1,0})\). Moreover we suppose that there are \(C \in \mathbb{R}, K > 0\) such that

\[
(5.3) \quad Re b(t, x, \xi) \leq C,
\]

for \(x, \xi \in \mathbb{R}^n\) with \(|x|, |\xi| \geq K\) and \(t \in [0, T]\). Then we can prove the following theorem by use of the same method as that of [3] and [7].

**Theorem 5.1.** Assume that the above conditions (4.3)-(4.5) are valid. For any \(u_0 \in L^2\) and \(f \in C^0([0, T]; L^2)\) there exists a unique solution \(u \in C^0([0, T]; L^2) \cap C^1([0, T]; H^{-2})\) of the Cauchy problem (5.1)-(5.2).

## 6 Proof of Theorem

Assume that \(u(t, x)\) satisfies (1)-(2) in the introduction. Put \(v(t, x) = e^{{\rho}(D)x}u(t, X)\). Then \(v\) satisfies the following Cauchy problem,

\[
(6.1) \quad \frac{\partial}{\partial t}v(t, x) = (i\Delta + c(t, x, D))v(t, x),
\]

\[
(6.2) \quad v(0, x) = u_0(x),
\]

where

\[
c(t, x, D) = \rho(D)^{\kappa} + e^{\rho(D)^{\kappa}}b(t, XD)e^{-\rho(D)^{\kappa}}
\]

\[
= \rho(D)^{\kappa} + b(t, x, D) + b_1(t, x, D) + r_2(t, x, D),
\]

where \(b_1(x, \xi) \in \gamma^dS(<\xi>^{-1}, g_1), r_1(t, z, \xi) \in A_{-\epsilon_0+\epsilon_0,T,-\epsilon_0}^{\kappa}\) from Theorem 4.1. Once more we change the unknown function \(v\) to \(w\) as follows,

\[
(6.4) \quad w(t, x) = I_{\phi}(x, D)v(t, x),
\]

where \(\phi = x\xi - ie\theta(t, x, \xi)\) and \(\theta\) is given by

\[
\theta(t, x, \xi) = \theta_0(x, \xi)\phi_0\left(\frac{x}{M(\xi)}\right) + t\xi \sigma + f(t)\phi_1\left(\frac{x}{M(\xi)}\right) - \phi_2\left(\frac{x}{M(\xi)}\right),
\]

\[
f(t) = \int_0^t (1 + s^2)^{\frac{\sigma + 1}{2}} ds,
\]

and \(\phi_{\pm}(t) = \chi(\pm t), \phi_0(t) = 1 - \phi_+(t) - \phi_-(t)\) and \(\chi(t) \in \gamma^d(R)\) such that \(\chi(t) = 1\) for \(t \geq 1, \chi(t) = 0\) for \(t \leq 1/2, \chi'(t) \geq 0\) and \(0 \leq \chi(t) \leq 1\). Then we can see that \(\theta(t, x, \xi)\) belongs to \(\gamma^dS(x^{\sigma}(\xi)^{\delta}, g_1)\) and that there are \(\epsilon_1 > 0, M > 0, K > 0, c_0 > 0\) such that \(\theta\) satisfies

\[
(6.5) \quad (\partial_t + \xi \cdot \nabla_x)\theta(t, x, \xi) \geq c_0(\langle \xi \rangle^{2\delta} + \langle \xi \rangle^{\sigma + \delta}) - c_1,
\]

for \(x, \xi \in \mathbb{R}^n\) with \(|x|, |\xi| \geq K, |t| \leq T\).

It follows from Lemma 4.4 that if \(|\epsilon|\) is sufficiently small, we have the inverse \(I_{\phi}(x, D)^{-1}\). Therefore we get the following Cauchy problem of \(w\) from (6.1)-(6.2),
\( \frac{\partial}{\partial t} w(t, x) = (\partial_t I_{\phi}) I_{\phi}(x, D)^{-1} w(t, x) + I_{\phi}(i\Delta + c(t, x, D)) I_{\phi}(x, D)^{-1} w(t, x), \) 

(6.6) 

\[ w(0, x) = I_{\phi}(x, D) u_0(x). \]  

(6.7) 

Since \( \phi(t, x, \xi) \in \gamma^d S((x)^{\sigma}(\xi)^{\delta}, g_1) \), it follows from (4.10) that \( \nabla_x \phi(x, \Phi(x, \xi)) \in \gamma^d S(<x>^{\sigma}<\xi>^{\delta-1} + g_1), \nabla_x \phi(x, \Phi(x, \xi)) \in \gamma^d S(<x>^{\sigma-1}<\xi>^{\delta}, g_1) \), and \( \phi(x, \xi) - \xi \in \gamma^d S(<x>^{\sigma-1}<\xi>^{\delta}, g_1) \). Hence we have from (4.16) in Theorem 4.6 and Proposition 2.3

(6.8) 

\[ \sigma(I_{\phi} A_{-\epsilon}^{-1}) = -|\xi|^2 \frac{\partial}{\partial x} \phi(t, x, \xi) + a_1(x, D) + r_2(x, \xi), \]  

where \( a_1 \in S(<x>^{\sigma}<\xi>^{\delta}, g_1) \), \( a_1' \in S(<x>^{\sigma-1}<\xi>^{\delta}, g_1) \) and \( r_2 \in A_{-\epsilon_0+|\epsilon|}^{1/d} \) for some \( c > 0 \) (independent of \( \epsilon \)). Here we choose \( \epsilon \) such that \( r_2 \) belongs to \( S(1, g) \). Thus we obtain the equation of \( w \) from (6.6)-(6.7),

(5.10) 

\[ \frac{\partial w}{\partial t} = (\partial_t + c(t, x, D)) \phi(t, x, D)^{-1} w(t, x), \]  

(5.11) 

\[ w(0) = I_{\phi(0)} (x, D) u_0(x), \]  

where \( r_4 \in S(<x>^{2\sigma-2} + |\xi| < x >^{-1}, g_1) \). Moreover taking account of the assumptions (5) in the introduction and (6.5) we can choose conveniently \( K > 0, \epsilon \) and \( \rho \) such that we have

\[ \rho p(x, \xi) + \Re b(t, x, \xi) - \epsilon H \theta(ax, \xi) + R_{\epsilon} \leq 0, \]

for \( x, \xi \in \mathbb{R}^n \) with \( |x|, |\xi| \geq K, \) where \( K > 0 \) is sufficiently large. Therefore we can solve the Cauchy problem (6.6)-(6.7) by use of Theorem 5.1, since \( w(0) = I_{\phi(0)} u_0 \) belongs to \( L^2 \), and consequently we get the solution \( u = e^{-\rho t D} I_{\phi}(x, D)^{-1} w(t, x) = e(t, x, D)^{-1} I_{\phi(t)}(x, D)^{-1} I_{\phi}(x, D)^{-1} w(t, x) \), which satisfies (6) from Lemma 1.2. This completes the proof of Theorem.

References


