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Smoothing effect in Gevrey classes for Schrödinger equations

Kunihiko Kajitani
Institute of Mathematics
University of Tsukuba
305 Tsukuba Ibaraki Japan

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Introduction

We shall investigate Gevrey smoothing effects of the solutions to the Cauchy problem for Schrödinger type equations. Roughly speaking, we shall prove that if the initial data decay as $e^{-c<x>}$ ($0 < \kappa \leq 1, c > 0$), then the solutions belong to Gevrey class $\gamma^{1/\kappa}$ with respect to the space variables. Let $T > 0$. We consider the following Cauchy problem,

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, x) - i\Delta u(t, x) - b(t, x, D)u(t, x) &= 0, & t \in [-T, T], x \in \mathbb{R}^n, \\
u(0, x) &= u_0(x), & x \in \mathbb{R}^n,
\end{align*}
\]

where

\[
b(t, x, D)u = \sum_{j=1}^{n} b_j(t, x)D_j u + b_0(t, x)u,
\]

and $D_j = -i \frac{\partial}{\partial x_j}$. We assume that the coefficients $b_j(t, x)$ satisfy

\[
|D_x^\alpha b_j(t, x)| \leq C_b(x)^{-|\alpha|}|\alpha|^s,
\]

for $(t, x) \in [-T, T] \times \mathbb{R}^n, \alpha \in \mathbb{N}^n$, where $<x> = (1 + |x|^2)^{1/2}$. Moreover we assume that there is $\kappa \in (0, 1]$ such that

\[
\lim_{|x| \to \infty} \text{Re} b_j(t, x)<x>^{1-\kappa} = 0, \text{ uniformly in } t \in [-T, T].
\]

For $\rho \geq 0$ let define a exponential operator $e^{\rho<\frac{\phi}{D}>}$ as follows,

\[
e^{\rho<\frac{\phi}{D}>\gamma}(x) = \int_{\mathbb{R}^n} e^{i\phi(x, \xi)} u(x, \xi) d\xi,
\]

where $\hat{u}(\xi)$ stands for a Fourier transform of $u$ and $d\xi = (2\pi)^{-n} d\xi$. For $\varepsilon \in \mathbb{R}$ denote $\phi_\varepsilon = x\xi - i\varepsilon x\xi < x >^{\sigma-1} < \xi >^{\delta-1}$, where $\sigma + \delta = \kappa$ and we define

\[
I_{\phi_\varepsilon}(x, D)u(x) = \int_{\mathbb{R}^n} e^{i\phi_\varepsilon(x, \xi)} \hat{u}(\xi) d\xi.
\]

Then our main theorem follows.
Theorem. Assume (4)-(5) are valid and there is \( \varepsilon > 0 \) such that \( I_{\phi}u_{0} \in L^{2}(R^{n}) \). Then if \( d\kappa \leq 1 \), there exists a solution of (1)-(2) satisfying that there are \( C > 0, \rho > 0 \) and \( \delta > 0 \) such that

\[
|\partial_{t}^{\alpha}u(t, x)| \leq C(\rho|t|)^{-|\alpha|}||u||_{p}^{2}e^{\delta<\varepsilon>^{2}},
\]

for \( (t, x) \in [-T, T]\backslash 0 \times R^{n}, \alpha \in N^{n} \).

Remark. (i) Kato T. and Yajima in [12] considered the smoothing effect phenomena. A. Jensen in [6] and Hayashi, Nakamitsu & Tsutsumi in [5] showed that if \( \dot{u} \in L^{2}(R^{n}) \), the solution \( u \) of (1)-(2) belongs to \( H_{\kappa_{\rho}}^{\tau} \) for \( \tau \neq 0 \). Hayashi & Saitoh in [4] proved that if \( \hat{H}_{\kappa_{\rho}}^{\tau}u_{0} \) \( (\delta > 0) \) is in \( L^{2}(R^{n}) \), the solution \( u \) is analytic in \( x \) for \( t \neq 0 \) and De Bouard, Hayashi & Kato in [1], Kato & Taniguti in [11] show that if \( u_{0} \) satisfies \( \|u_{0}\|_{p} \leq C_{j}+1j^{1} \), then the solution belongs to Gevrey \( \gamma^{j/2} \) with respect to \( x \) for \( t \neq 0 \). Theorem 1 is proved by Kayitani in [8] and [10], when \( \sigma = \kappa = 1 \).

1 Weighted Sobolev spaces

We introduce some Sobolev spaces with weights. Let \( \rho, \delta \) be real numbers and \( \kappa \in (0, 1] \). Define

\[
\tilde{H}_{\kappa}^{\rho} = \{u \in L^{2}_{\kappa_{\rho}}(R^{n}); e^{\delta<\varepsilon>^{2}}u(x) \in L^{2}(R^{n})\}.
\]

For \( \rho \geq 0 \) let define

\[
H_{\kappa}^{\rho} = \{u \in L^{2}(R^{n}); Fu(\xi) \in \hat{H}_{\rho}(R^{n})\},
\]

where \( Fu \) stands for the Fourier transform of \( u \). For \( \rho < 0 \) we define \( H_{\kappa_{\rho}}^{\tau} \) as the dual space of \( H_{\kappa_{\rho}}^{\tau} \). Then the Fourier transform \( F \) becomes bijective from \( H_{\kappa_{\rho}}^{\tau} \) to \( \hat{H}_{\rho}^{\tau} \). We define the operator \( e^{\rho<\varepsilon>^{2}} \) mapping continuously from \( H_{\kappa_{\rho}}^{\tau} \) to \( H_{\kappa_{\rho}}^{\tau} \) as follows;

\[
e^{\rho<\varepsilon>^{2}}u(x) = F^{-1}(e^{\rho<\varepsilon>^{2}}Fu(\xi))(x),
\]

for \( u \in H_{\kappa_{\rho}}^{\tau} \) and \( \delta < 0 \) maps continuously from \( \tilde{H}_{\kappa_{\rho}}^{\tau} \) to \( \tilde{H}_{\kappa_{\rho}}^{\tau} \). We define for \( \delta \geq 0 \) and \( \rho \in R \)

\[
H_{\kappa_{\rho}}^{\delta} = \{u \in H_{\rho}; e^{\rho<\varepsilon>^{2}}u \in \tilde{H}_{\kappa_{\rho}}^{\delta}\}.
\]

For \( \delta < 0 \) we define \( H_{\kappa_{\rho}}^{\delta} \) as the dual space of \( H_{\kappa_{\rho}}^{\delta} \). We note that \( H_{\kappa,0}^{\delta} = H_{\rho}^{\kappa} \), \( H_{0,0}^{\delta} = \tilde{H}_{\kappa}^{\rho} \) and \( H_{0,0}^{\delta} = L^{2}(R^{n}) \). Furthermore we define for \( \rho \geq 0 \) and \( \delta \in R \)

\[
\tilde{H}_{\kappa_{\rho}}^{\delta} = \{u \in \tilde{H}_{\kappa_{\rho}}^{\delta}; e^{\rho<\varepsilon>^{2}}u \in H_{\rho}^{\kappa}\}
\]

and for \( \rho < 0 \) define \( \tilde{H}_{\kappa_{\rho}}^{\delta} \) as the dual space of \( \tilde{H}_{\kappa_{\rho}}^{\delta} \). Denote by \( H' \) the dual space of a topological space \( H \). Then \( H_{\kappa_{\rho}}^{\delta} = H_{\kappa_{\rho}}^{\delta} \) and \( \tilde{H}_{\kappa_{\rho}}^{\delta} = \tilde{H}_{\kappa_{\rho}}^{\delta} \) hold for any \( \rho \) and \( \delta \in R \). We shall prove \( H_{\kappa_{\rho}}^{\delta} = \tilde{H}_{\kappa_{\rho}}^{\delta} \) later on (see Proposition 3.8).

Lemma 1.1. Let \( \rho, \delta \in R \). Then

\[
(i) \quad H_{\kappa_{\rho}}^{\delta} = e^{-\rho<\varepsilon>^{2}}, e^{-\delta<\varepsilon>^{2}}L^{2} = e^{-\rho<\varepsilon>^{2}}\tilde{H}_{\kappa_{\rho}}^{\delta},
\]

\[
(ii) \quad \tilde{H}_{\kappa_{\rho}}^{\delta} = e^{-\rho<\varepsilon>^{2}}, e^{-\delta<\varepsilon>^{2}}L^{2} = e^{-\delta<\varepsilon>^{2}}H_{\kappa_{\rho}}^{\delta}.
\]

Lemma 1.2 Let \( 1 > \rho > 0, \delta \in R \) and \( u \in \tilde{H}_{\kappa_{\rho}}^{\delta} \). Then

\[
|D_{x}^{\alpha}u(x)| \leq C_{n}(1-\varepsilon)^{-n/2}||u||_{H_{\kappa_{\rho}}^{\delta}}(||u||_{H_{\kappa_{\rho}}^{\delta}}(\rho)-|\alpha||e^{<\varepsilon>^{2}}
\]

for \( x \in R^{n}, \alpha \in N^{n} \) and \( 0 < \varepsilon < 1 \).

We can prove these lemmas analogously to the case of \( \kappa = 1 \) which is proved in [10].
2 Almost analytic extension of symbols

Following Hörmander’s notation we define the symbol classes of pseudo-differential operators. Let \( m(x, \xi), \varphi(x, \xi), \psi(x, \xi) \) a weight and \( g = \varphi^{-2}dx^2 + \psi^{-2}d\xi^2 \) a Riemann metric. We denote by \( S(m, g) \) the set of symbols \( a(x, \xi) \) satisfying

\[
|a^{(\alpha)}_{(\beta)}(x, \xi)| \leq C_a m(x, \xi) \varphi^{-|\alpha|} \psi^{-|\beta|},
\]

for \( (x, \xi) \in \mathbb{R}^{2n}, \alpha, \beta \in \mathbb{N}^n \), where \( a^{(\alpha)}_{(\beta)} = \partial^\alpha_x \partial^\beta_\xi a \). Let \( d \geq 1 \). Moreover we call that a function \( a(x, \xi) \in S(m, g) \) belongs to \( \gamma^d S(m, g) \), if \( a(x, \xi) \) satisfies that there are \( C_a, \rho_a > 0 \) such that

\[
(2.1) \quad |a^{(\alpha)}_{(\beta)}(x, \xi)| \leq C_a \rho_a^{-|\alpha+\beta|} |\alpha + \beta||\alpha+\beta|^d \psi^{-|\beta|} \varphi^{-|\alpha|},
\]

for \( (x, \xi) \in \mathbb{R}^{2n}, \alpha, \beta \in \mathbb{N}^n \).

We denote \( g_0 = dx^2 + d\xi^2 \) and \( g_1 = <x>^{-2}dX^2 + <\xi>^{-2}d\xi^2 \).

We remark that the symbol class \( \gamma^1 S(m, g_i)(i=0,1) \) is introduced in [10] when \( d=1 \). Here we consider the case of \( d > 1 \).

Let \( d > 1 \), \( \chi(t) \in C_0^\infty((0, \infty)) \) satisfying \( \chi(t) = 0, t \leq 1/2, \chi(t) = 1, t \leq 1 \), and

\[
(2.2) \quad |D_t^k \chi(t)| \leq C_0 \rho_0^{-k} k^d,
\]

for \( t \in \mathbb{R}, k \in \mathbb{N} \). Then for a weight \( w(x, \xi) \in \gamma^d S(m, g_1) \) and a parameter \( b > 0 \) we can see easily that \( \chi(bw(x, \xi)) \in \gamma^d S(1, g_1) \) satisfying

\[
(2.3) \quad |D_x^\alpha D_\xi^\beta \chi(bw(x, \xi))| \leq C_1 \rho_1^{-|\alpha+\beta|} |\alpha + \beta||\alpha+\beta|^d <x>^{-|\beta|} <\xi>^{-|\alpha|},
\]

for \( (x, \xi) \in \mathbb{R}^{2n}, \alpha, \beta \in \mathbb{N}^n, b \geq 1 \).

Lemma 2.1. Let \( d \geq 1 \) and \( \{p_k(x, \xi)\}_{k=0}^\infty \) be a series of symbols satisfying

\[
(2.4) \quad |p_k^{(\alpha)}_{(\beta)}(x, \xi)| \leq m(x, \xi)(<x> \varphi^{-|\alpha+\beta|} |\alpha + \beta||\alpha+\beta|^d k!)^{d} <x>^{-|\beta|} <\xi>^{-|\alpha|},
\]

for \( (x, \xi) \in \mathbb{R}^{2n}, \alpha, \beta \in \mathbb{N}^n \) and \( k \geq 0 \). Then there is \( p(x, \xi) \in \gamma^d S(m, g_1) \) such that

\[
(2.5) \quad p(x, \xi) - \sum_{k=0}^{N-1} p_k(x, \xi) \in \gamma^d S(m((x) \varphi^{-N} N!)^d, g_1),
\]

for any integer \( N \geq 0 \).

Proof This lemma is essentially a result of [2]. The case of \( d = 1 \) is explained in [10]. Here we prove the lemma in the case of \( d > 1 \). Let \( b_k = \rho_p^{-1} k! M \) and \( M \geq 2 \).

Define

\[
(2.6) \quad p(x, \xi) = \sum_{k=0}^{\infty} p_k(x, \xi) \chi(b_k((x) \varphi^{-1}))^{-1},
\]

Then we have

\[
|p_k^{(\alpha)}_{(\beta)}(x, \xi)| = \sum_k \sum_{\alpha', \beta'} (\alpha) (\beta) p_k^{(\alpha')}_{(\beta')}(x)((b_k((x) \varphi^{-1})^{(\alpha+\alpha')})^{(\alpha+\alpha')} \leq \sum_k \sum_{\alpha', \beta'} (\alpha) (\beta) m(x, \xi) \rho_p^{-|\alpha'+\beta'|} |\alpha' + \beta'|^d <x>^{-|\beta|} <\xi>^{-|\alpha|} \times M^{-k} C_0 \rho_0^{-|\alpha-\alpha' + \beta - \beta'|} |\alpha - \alpha' + \beta = \beta'|^d
\]
for \((x, \xi) \in \mathbb{R}^{2n}, \alpha, \beta \in \mathbb{N}^{n}\). Here we used the following inequality
\[
\sum_{\alpha' \leq \alpha} \left( \frac{\alpha}{\alpha'} \right) \rho_{p}^{-|\alpha'|} |\alpha'|!^{d} \rho_{0} |\alpha - \alpha'|!^{d} \leq \frac{\rho_{0}}{\rho_{0} - \rho_{p}} |\alpha|!^{d},
\]
for \(\rho_{0} > \rho_{p}\). Moreover we can write
\[
p(x, \xi) = \sum_{k=0}^{N-1} p_{k}(x, \xi)
\]

\[
= : I + II.
\]
Noting that \(\rho_{p}^{-k}k!^{d}(M\langle x\rangle\langle \xi\rangle)^{-N} \leq 1\) on \(supp \chi(b_{k}(\langle_{X}\rangle\langle_{\xi}\rangle)^{-1})\) for \(k \geq N\) and \(\rho_{p}^{-k}k!^{d}(M\langle x\rangle\langle \xi\rangle)^{-N} \geq 1/2\) on \(supp(1 - \chi(b_{k}(\langle_{X}\rangle\langle_{\xi}\rangle)^{-1}))\) for \(k \leq N - 1\) respectively, we can see that I and II belong to \(\gamma^{d}S(m(\langle X\rangle\langle \xi\rangle\rho_{p})^{-N}N!^{d}, g)\). Q.E.D.

Let \(a(x, \xi) \in \gamma^{d}(m, g_{1})\), that is, \(a(x, \xi)\) satisfies (2.1). Denote \(b_{\alpha}(x) = B\rho_{a}^{-1}4^{n}\langle x\rangle^{-1}|\alpha|!^{\frac{d-1}{|\alpha|}}\) for \(x \in \mathbb{R}^{n}\). We define an almost analytic extension of \(a(x, \xi)\) as follows,
\[
a(x+iy, \xi+i\eta) = \sum_{\alpha, \beta} a^{(\alpha)(\beta)}(x, \xi)\chi(b_{\alpha}(\langle x\rangle\langle \xi\rangle)^{-1})x(b_{\alpha}(\xi)|\eta|)|\chi\alpha!\beta!|^{-1},
\]
for \(x, y, \xi, \eta \in \mathbb{R}^{n}\), where \(a^{(\alpha)(\beta)}(x, \xi) = \partial_{\xi}^{\alpha}(-\partial_{x})^{\beta}a(x, \xi)\). Then we can prove easily

**Proposition 2.2** Let \(a(x, \xi) \in \gamma^{d}S(m, g_{1})\). Then the function \(a(x+iy, \xi+i\eta)\) defined by (2.8) satisfies the following properties.

(i) \(|D^{\alpha}_{x} D^{\beta}_{y} a(x+iy, \xi+i\eta)| \leq Cm(x, \xi)(C\rho_{a}^{-|\alpha+\beta+\gamma+\delta|}(x)|\beta(\xi)|^{-|\gamma|}(\eta)|^{-|\delta|}|\alpha+\beta+\gamma+\delta|!^{d}d|.

(ii) \(|(\partial_{\xi_{j}}+i\partial_{\eta_{j}})D^{\alpha}_{x} D^{\beta}_{y} a(x+iy, \xi+i\eta)| \leq Cm(x, \xi)(C\rho_{a}^{-|\alpha+\beta+\gamma+\delta|}e^{-c_{\rho}(\frac{|y|}{\langle y\rangle^{-1}})^{2}}(x)|\beta(\xi)|^{-|\gamma|}(\eta)|^{-|\delta|}|\alpha+\beta+\gamma+\delta|!^{d}d|.

(iii) \(|(\partial_{\xi_{j}}+i\partial_{\eta_{j}})D^{\alpha}_{x} D^{\beta}_{y} a(x+iy, \xi+i\eta)| \leq Cm(x, \xi)(C\rho_{a}^{-|\alpha+\beta+\gamma+\delta|}e^{-c_{\rho}(\frac{|y|}{\langle y\rangle^{2}})^{2}}(x)|\beta(\xi)|^{-\alpha}(\eta)|^{-|\gamma|}(\eta)|^{-|\delta|}|\alpha+\beta+\gamma+\delta|!^{d}d|.

For simplicity denote \(\gamma^{1/\kappa}S(\varepsilon(\varepsilon)^{r}+\rho(\xi)^{r}, \delta_{0})\) by \(A_{\rho, \delta}^{\kappa}\), where \(\delta_{0} = d\alpha^{2} + d\xi^{2}\). For \(a_{i} \in A_{\rho, \delta}^{\kappa}(i = 1, 2)\) we define a product of \(a_{1}\) and \(a_{2}\) as follows,
\[
(a_{1} \circ a_{2})(x, \xi) = os - \int_{\mathbb{R}^{2n}} e^{-iy_{0}}a_{1}(x, \xi + \eta)a_{2}(x + y, \xi)dy d\eta,
\]

\[
= \lim_{\epsilon \to 0} \int_{\mathbb{R}^{2n}} e^{-iy_{0}-\epsilon(y^{2}+\eta^{2})}a_{1}(x, \xi + \eta)a_{2}(x + y, \xi)dy d\eta,
\]
where \(d\eta = (2\pi)^{-n}d\eta\). Then we can show the proposition below.
Proposition 2.3. (i) Let $\kappa \leq 1$ and $a_i \in A^*_{\rho_i, \delta_i}$, $i = 1, 2$. Then there is $\epsilon_0 > 0$ such that if $|\rho_1|, |\delta_2| \leq \epsilon_0$, the product $a_1 \circ a_2$ belongs to $A^*_{\rho_1+\rho_2, \delta_1+\delta_2}$.

(ii) Let $a_i \in A^*_{\rho_i, \delta_i}$, $i = 1, 2, 3$. Then if $|\rho_i|(i = 1, 2), |\delta_i|(i = 2, 3) \leq \epsilon_0/2$, we have $(a_1 \circ a_2) \circ a_3 = a_1 \circ (a_2 \circ a_3)$.

Proposition 2.4 Let $d \geq 1$ and $a_1 \in \gamma^d S((x)^{m_1}(\xi)^{\ell_1}, g_1)$, $i = 1, 2$. Then $a_1 \circ a_2$ belongs to $S((x)^{m_1+m_2}(\xi)^{\ell_1+\ell_2}, g_1)$ and moreover we can decompose

\[ a_1 \circ a_2 (x, \xi) = p(x, \xi) + r(x, \xi), \]

where $p(x, \xi) \in \gamma^d S((x)^{m_1+m_2}(\xi)^{\ell_1+\ell_2}, g_1)$ satisfies that there are $C > 0$ and $\epsilon_0 > 0$ such that

\[ p(x, \xi) - \sum_{|\gamma| < N} \gamma^{-1} a_1^{(\gamma)}(x, \xi) a_2^{(\gamma)}(x, \xi) \in \gamma^d S(C^{1+N} N! (x)^{m_1+m_2-N}(\xi)^{\ell_1+\ell_2-N}, g), \]

for any non-negative integer $N$, and $r(x, \xi)$ belongs to $A^{1/d}_{-\epsilon_0, -\epsilon_0}$.

3 Pseudo-differential operators

Let $\kappa \leq \frac{1}{2}$. Now we want to define a pseudo differential operator $a(x, D)$ for a symbol $a(x, \xi) \in A^*_{\rho, \delta}$, which operates from $H^\kappa_{p', \rho'}$ to $H^\kappa_{p, \rho \cdot \rho'}$. When $\rho$ and $\delta$ are non-positive, since $A^*_{\rho, \delta}$ is contained in the usual symbol class $S^0_{0, 0}$ (denote by $S^m_{p, \rho}$ the Hörmander's class), we can define

\[ a(x, D)u(x) = \int e^{i\xi \cdot x} a(x, \xi) \hat{u}(\xi) d\xi, \]

for $u \in L^2(R^n)$ and for $a \in A^*_{\rho, \delta}$. Moreover for $a_i \in A^*_{\rho_i, \delta_i}$, $i = 1, 2, (\rho_i$ and $\delta_i$ non positive) the symbol $\sigma(a_1(x, D) a_2(x, D))(x, \xi)$ of the product of $a_1(x, D)$ and $a_2(x, D)$ can be written as follows,

\[ \sigma(a_1(x, D) a_2(x, D))(x, \xi) = (a_1 \circ a_2)(x, \xi) \]

and we have

\[ a_1(x, D)(a_2(x, D) u)(x) = (a_1 \circ a_2)(x, D) u(x) \]

for $u \in L^2(R^n)$, where $a_1 \circ a_2$ is defined by (2.9). Next we shall show that (3.2) and (3.3) are valid for any $\rho_1, \delta_1$. So to do, we need some preparations. Let $a \in A^*_{\rho, \delta}$ and $u \in H^\kappa_{p'}$. Then we can define $a(x, D) u(x)$ which belongs to $H^\kappa_{p}$. In fact, put $\tilde{a}(z, \xi) = e^{-\delta(z)\langle \xi \rangle^2} a(x, \xi)$. Then $\tilde{a}(x, \xi) \in A^*_{\rho, \delta}$. Noting that $e^{\rho(z)^*} \tilde{a}(\xi)$ we can define

\[ e^{i\xi \cdot x} a(x, D) u(x) = \int e^{i\xi \cdot x} \tilde{a}(z, \xi) e^{\rho(z)^*} \tilde{a}(\xi) d\xi, \]

which is in $L^2$, that is, $a(x, D) u \in H^\kappa_{p}$. For $\epsilon > 0$ we denote $\chi_\epsilon(x) = e^{-\epsilon(x)^2}$ and $\chi_\epsilon(D) = e^{-\epsilon(D)^2}$.

Lemma 3.1. (i) Let $a \in A^*_{\rho, \delta}$, $u \in L^2$ and $\epsilon_0 > 0$ chosen in Proposition 2.3. Then for any $\epsilon > 0$

\[ a(x, D)(\chi_\epsilon(D) \chi_\epsilon(x) u)(x) = (a(x, \xi) \chi_\epsilon(\xi)) \circ \chi_\epsilon(x)(x, D) u(x) \]

and

\[ (a \chi_\epsilon(\xi)) \circ \chi_\epsilon(x) \in A^*_{\rho - \epsilon_0, \delta - \epsilon_0}, \]
(ii) Let \( u \in L^2 \) and \( \epsilon_0 > 0 \) chosen in Proposition 2.3. Then there is \( \epsilon_1 > 0 \) such that for any \( \epsilon > 0 \)

\[
(3.7) \quad e^{-\rho < \xi >^*} (e^{-\delta < x >^*} \chi_\epsilon(x) \chi_\epsilon(D) u)(x) = a_\epsilon(x, D) u(x),
\]

where

\[
(3.8) \quad a_\epsilon(x, \xi) = e^{-\rho < \xi >^*} \circ (e^{-\delta < x >^*} \chi_\epsilon(x) \chi_\epsilon(\xi)) \in A^\kappa_{\rho - \epsilon_0, -\delta - \epsilon_0},
\]

for \( |\rho| \leq \epsilon_0 \) and \( \rho < \epsilon_1 \). We can prove the following lemma by use of Lemma 3.1.

**Lemma 3.2.** Let \( u \in H^\kappa_{\rho, \delta} \) and \( |\rho|, |\delta| \leq \epsilon_0/2 \) (\( \epsilon_0 \) is given in Proposition 2.3). Then for any \( \epsilon > 0 \) there is \( u_\epsilon \in H^\kappa_{\epsilon_0/2, \epsilon_0/2} \) such that

\[
(3.9) \quad \| u - u_\epsilon \|_{H^\kappa_{\rho, \delta}} < \epsilon.
\]

**Lemma 3.3.** Let \( a \in A^\kappa_{\rho_1, \delta_1}, 0 < \epsilon_0' < \epsilon_0 \) is chosen in Proposition 2.3) and \( u \in H^\kappa_{\epsilon_0', \epsilon_0} \). Then there is \( \epsilon_2 > 0 \) independent of \( a, \rho \) and \( \delta \) such that \( a(x, D) u(x) \) belongs to \( H^\kappa_{\epsilon_0', \epsilon_0, \rho_1, \delta_1} \) if \( 0 < \epsilon_0' - \rho \leq \min\{\epsilon_0, \epsilon_2 \} \) and \( 0 < \epsilon_0 - \rho_1 - \delta \leq \epsilon_0 \).

**Lemma 3.4.** Let \( a_i \in A^\kappa_{\rho_i, \delta_i} \) \((i = 1, 2)\) and \( u \in H^\kappa_{\epsilon_0', \epsilon_0} \). Then if \( |\rho_1| \leq \epsilon_0, 0 < \epsilon_0' - \rho_2 \leq \epsilon_0 \min\{1, \rho_2\}, 0 < \epsilon_0' - \delta_2 \leq \epsilon_0 - \rho_2 - \rho_1 \leq \epsilon_0 \min\{1, \rho_2\} \) and \( 0 < \epsilon_0 - \delta_2 - \delta_1 \leq \epsilon_0 \) are valid \( (\epsilon_0 \) is given in Proposition 2.3), we have

\[
(3.10) \quad a_1(x, D) (a_2(x, D) u)(x) = (a_1 \circ a_2)(x, D) u(x),
\]

which is in \( H^\kappa_{\epsilon_0', \rho_1, \rho_2, \delta_0, \delta_1} \).

Let \( a \in A^\kappa_{\rho_0, \delta_0} \) \((|\rho|, |\delta| \leq \epsilon_0/4)\), \( u \in H^\kappa_{\epsilon_0/2, \epsilon_0/2} \) and \( |\rho_1|, |\delta_1| < \epsilon_0/4 \). Put \( w = e^{\delta_1 < x >^*} e^{\rho_1 < D >^*} u \), which is in \( H^\kappa_{\epsilon_0/2, -\rho_1, -\delta_0, \delta_1} \). Since we can write \( u = e^{-\rho < D >^*} (e^{-\delta_1 < x >^*} w) \), we get by Lemma 3.4 with \( \epsilon_0' = \epsilon_0/2 - \rho_1, \epsilon_0 = \epsilon_0/2 - \delta_1, a_1 = a(x, \xi) e^{-\rho_1 < D >^*} \) and \( a_2 = e^{-\delta_1 < x >^*} \), we have

\[
(3.11) \quad \| a u \|_{H^\kappa_{\rho_1, \rho_2, \delta_0, \delta_1}} = \| a_1(x, D) w \|_{L^2} \leq C \| w \|_{L^2} = C \| u \|_{H^\kappa_{\rho_1} - \rho_2, \delta_1}
\]

for any \( u \in H^\kappa_{\epsilon_0/2, \epsilon_0/2} \). Since \( H^\kappa_{\rho_0/2, \epsilon_0/2} \) is dense in \( H^\kappa_{\rho_1, \delta_1} \) from Lemma 3.2, we get the following theorem.

**Theorem 3.5** Let \( a \in A^\kappa_{\rho_0, \delta_0} \) \((|\rho|, |\delta| \leq \epsilon_0/4)\), \( |\rho_1|, |\delta_1| < \epsilon_0/4 \), where \( \epsilon_0 \) is given in Proposition 2.3. Then \( a(x, D) \) maps from \( H^\kappa_{\rho_1, \delta_1} \) to \( H^\kappa_{\rho_1, \rho_1, \delta_1} \) and satisfies the following inequality

\[
(3.12) \quad \| a u \|_{H^\kappa_{\rho_1, \rho_1, \delta_1}} \leq C \| a \|_{H^\kappa_{\rho_1, \delta_1}}
\]

for any \( u \in H^\kappa_{\rho_1, \delta_1} \). For \( a \in A^\kappa_{\rho_0, \delta_0} \), we define

\[
(3.13) \quad a^t(x, \xi) = \text{os} - \int \int e^{i\eta y} a(x + y, \xi + \eta) dy d\eta,
\]
and \( a^*(x, \xi) = a^t(\bar{x}, \xi) \). Then we can prove the following lemma, by the same way as that of the proof (i) of Proposition 2.3.

**Lemma 3.6.** Let \( a \in A_{p,\delta}^\kappa \) and \(|\rho|, |\delta| \leq \epsilon_0\). Then \( a^t(x, \xi) \) defined in (2.29) belongs to \( A_{p,\delta}^\kappa \). Moreover it holds

(3.14), \[
(a^t(x, D)u, \varphi)_{L^2} = (u, a(x, D)\varphi)_{L^2},
\]
and

(3.15), \[
(a^*(x, D)u, \varphi)_{L^2} = (u, a(x, D)\varphi)_{L^2},
\]
for any \( u, \varphi \in H^\kappa_{\epsilon_0} \).

The relation (3.14) and the inequality (3.12) yield

\[
|a^t u| \leq \|u\|_{H_{\rho\rho\iota^\delta 1}^\kappa} ||\bar{a}\varphi||_{H_{\rho\rho\iota^\delta 1}^\kappa} \leq C \|u\|_{H_{\rho\rho/2}^\kappa},
\]
if \(|\rho|, |\delta| \leq \epsilon_0/4 \) and \(|\rho_1|, |\delta_1| < \epsilon_0/4\). Therefore taking account that \( H^\kappa_{\epsilon_0/4} \) is dense in \( H^\kappa_{\rho,\delta_1} \), we get from (3.14)

(3.15) \[
\|a^t u\|_{H_{-\rho_1}^\kappa} \leq C \|u\|_{H_{\rho_1,\delta_1}^\kappa},
\]
for any \( u \in H^\kappa_{\rho_1,\delta_1} \). Thus we get the following proposition.

**Proposition 3.7.** Let \( a \in A_{p,\delta}^\kappa \) and \(|\rho|, |\delta| \leq \epsilon_0/4 \) and \(|\rho_1|, |\delta_1| < \epsilon_0/4\). Then the pseudodifferential operators \( a^t(x, D) \) and \( a^*(x, D) \) satisfy (3.15).

Noting that \( (e^{\delta<x>^\kappa}e^{\rho<D>^\kappa})^t = e^{\rho<D>^\kappa}e^{\delta<x>^\kappa} \), we have for \( u \in H^\kappa_{\rho,\delta} \)

\[
e^{\rho<D>^\kappa}e^{\delta<x>^\kappa}u(x) = (e^{\delta<x>^\kappa}e^{\rho<D>^\kappa})^t (e^{-\rho<D>^\kappa}e^{-\delta<x>^\kappa}e^{\delta<x>^\kappa}e^{\rho<D>^\kappa}u)(x)
\]

Moreover we can see from Proposition 2.3 and Lemma 2.9 that \( (e^{\delta<x>^\kappa}e^{\rho<x>^\kappa})^t (e^{-\delta<x>^\kappa}e^{-\rho<x>^\kappa})^t \) is in \( A_{0,0}^\kappa \). Hence we obtain the fact below.

**Proposition 3.8.** Let \(|\rho|, |\delta| \leq \epsilon_0/4\). Then \( u \) belongs to \( H_{p,\delta}^\kappa \) if and only if \( u \in \tilde{H}_{\rho,\delta}^\kappa \).

The following result on the multiple symbols of pseudodifferential operators is a special case of Lemma 2.2 of Chapter 7 in Kumanogo's book [12].

**Lemma 3.9.** Let \( r_j(x, \zeta) \in A_{0,0}^\kappa (j = 1, 2, \ldots, v) \) and put

\[
q_v(x, D) = r_1(x, D)r_2(x, D) \cdots r_v(x, D).
\]
Then the symbol \( q_v(x, \zeta) \) belongs to \( A_{0,0}^\kappa \) and satisfies

(3.16) \[
|q_v^{(\alpha)}(x, \zeta)| \leq C^v \prod_{j=1}^v C_{r_j} e^{\varepsilon_v |\alpha| + |\beta|},
\]
for \((x, \zeta) \in R^{2n}, \alpha, \beta \in N^n, \) where \( C \) is independent of \( v \) and \( \varepsilon_v = \min \{ \varepsilon_{r_j}/4 \} \).

We can prove easily the following lemma as a corollary of Lemma 3.9, by using the Neumann series method.
Lemma 3.10. Let \( r(x, \xi) \) be in \( A^n_{\rho_0}. \) If \( C_r > 0 \) is sufficiently small, then there is the inverse \((I + r(x, D))^{-1}\) which is a pseudodifferential operator with its symbol contained in \( A^n_{\rho_0}. \)

Lemma 3.11. Let \( j(x, \xi) \in \gamma^dS(\varepsilon_1, g_1). \) Then if \( \varepsilon_1 > 0 \) is small enough, there are \( k_1(x, \xi) \in \gamma^dS(\varepsilon_1 < x >^{-1} < \xi >^{-1}, g_1), \varepsilon_0 > 0 \) independent of \( \varepsilon_1 \) and \( r_\infty(x, \xi) \in A^{1/d}_{-\varepsilon_0,-\varepsilon_0} \) such that \((I + j(x, D))^{-1} = k(x, D) + k_1(x, D) + r_\infty(x, D), \) where \( k(x, \xi) = (1 + j(x, \xi))^{-1}. \)

4 Fourier Integral Operators

For \( \vartheta \in AS(\rho_0, \xi > + \delta_0 < x >, g)(\rho_0, \delta_0 \geq 0), \) where \( d \leq 1, \) we denote

\[
\phi(x, \xi) = x\xi - i\vartheta(x, \xi).
\]

For \( a \in A^n_{\rho_0}, \) we define a Fourier integral operator with a phase function \( \phi(x, \xi) \) as follows,

\[
a_{\phi}(x, D)u(x) = \int_{\mathbb{R}^n} e^{i\phi(x, \xi)} a(x, \xi) \hat{u}(\xi) d\xi,
\]

for \( u \in H_{\epsilon_0, \epsilon_0}. \) Putting \( p(x, \xi) = a(x, \xi)e^{\vartheta(x, \xi)}, \) we can see \( p(x, \xi) \in A^n_{\rho_0, \delta_0}. \) Therefore we can regard \( a_{\phi}(x, D) \) as a pseudo differential operator with its symbol \( p = ae^{\vartheta} \) defined in \( \S 2 \) and consequently it follows from Theorem 3.5 that \( a_{\phi}(x, D) \) acts continuously from \( H^s_{\rho_0, \delta_0} \) to \( H^s_{\rho_0, \delta_0} \). However in order to construct the inverse operator of \( p(x, D) \) it is better to regard \( p(x, D) \) as a Fourier integral operator. In particular for \( a = 1 \) we denote

\[
I_{\phi}(x, D)u(x) = \int_{\mathbb{R}^n} e^{i\phi(x, \xi)} \hat{u}(\xi) d\xi,
\]

\[
I_{\phi}^R(x, D)v(x) = \int e^{ix\xi} d\xi \int e^{i\phi(y, \xi)} v(y) dy.
\]

Theorem 4.1. Let \( a \in \gamma^dS(\xi^m(\xi^\epsilon, g_1), \vartheta \in \gamma^dS(\rho_0, \xi^\epsilon + \delta_0(x)^\epsilon, g_1) \) and \( \phi = x\xi - i\vartheta(x, \xi). \) Assume \( d \leq 1. \) Then if \( \rho_0, \delta_0 \) are sufficiently small, \( \tilde{a}(x, D) = I_{\phi}(x, D)a(x, D)I_{\phi}^{-1} \) and \( \tilde{a}'(x, D) = I_{\phi}(x, D)^{-1}a(x, D)I_{\phi}(x, D) \) are pseudodifferential operators of which symbols are given by

\[
\tilde{a}(x, \xi) = p(x, \xi) + r(x, \xi),
\]

\[
\tilde{a}'(x, \xi) = p'(x, \xi) + r'(x, \xi),
\]

where

\[
p(x, \xi) - a(x - i\nabla_x \vartheta(x, \Phi), \xi + i\nabla_x \vartheta(x, \Phi)) \in \gamma^dS(\xi^m < x >^{m-1} < \xi >^{\epsilon-1}, g_1),
\]

\[
p'(x, \xi) - a(x + i\nabla_x \vartheta(\Phi', \xi), \xi - i\nabla_x \vartheta(\Phi', \xi)) \in \gamma^dS(\xi^m < x >^{m-1} < \xi >^{\epsilon-1}, g_1),
\]

where \( \Phi = \Phi(x, x, \xi) \) and \( \Phi' = \Phi'(x, x, \xi) \) are given by (4.6) and (4.19) respectively and \( r, r' \) belong to \( A^{n_{-\varepsilon_0,-\varepsilon_0}} \) for an \( \varepsilon_0 > 0 \) independent of \( \rho_0. \)

This theorem is proved in [10] in the case of \( d = \kappa = 1. \) We can prove it similar way as that of [10].

Next we consider a phase function \( \vartheta \in \gamma^dS(\xi^\sigma(\xi)^\sigma, g_1). \) When \( \sigma + \delta = \kappa = 1/d < 1 \) or \( \sigma + \delta = 1 \) and \( d = \min(\delta^{-1} \sigma^{-1}) \), Theorem 4.1 holds also, that is, we can prove Theorem 4.6 below. So far we consider
only $d, \sigma, \delta, \kappa$ above. We note that $d > 1$.

**Lemma 4.2.** Let $a(x, \xi) \in \gamma^d S(<x>^m <\xi>^\ell, g_1)$ and $\theta \in \gamma^d S(\rho_0 <\xi>^\ell <x>^\sigma, g_1) (\rho_0 \geq 0)$. Put $\phi = x\xi - i\theta(x, \xi)$ and $\tilde{a}(x, D) = a_\phi(x, D) I_{-\phi}^R(x, D)$. If $\rho_0$ is sufficiently small, then $\tilde{a}(x, \xi)$ belongs to $S(<x>^m <\xi>^\ell, g)$ and moreover satisfies

\[
\tilde{a}(x, \xi) = \tilde{p}(x, \xi) + r(x, \xi),
\]

for $x, \xi \in \mathbb{R}^n$, and

\[
\tilde{p}(x, \xi) = -\sum_{|\gamma| < N} \gamma^{-1} D^\gamma \Phi^\gamma \{a(x, \Phi(y, \eta))J(y, \eta)\}_{y=x, \eta=\xi}
\]

\[
\in \gamma^d S(C^{1+N} N^d <x>^m <\xi>^\ell, g_1)
\]

for any $N$, where $\Phi(x, y, \xi)$ is a solution of the following equation,

\[
\Phi(x, y, \xi) = \int_0^1 \nabla \theta(y + t(x - y), \xi) dt,
\]

\[
J(x, y, \xi) = \frac{D\Phi(y, \xi)}{Dy}
\]

is the Jacobian of $\Phi, r(x, \xi) \in A_{-\epsilon_0}^{1/d}, -60$, and $C > 0, \epsilon_0 > 0$ are independent of $\rho_0$.

**Lemma 4.3.** Let $a(x, \xi)$ and $\theta$ be satisfied with the same condition as one of Lemma 4.2. For $\phi = x\xi - i\theta(x, \xi)$ put $a'(x, \xi) = I_{-\phi}^R(x, D)a_\phi(x, D)$. Then if $\rho_0$ and $\delta_0$ are sufficiently small, $a'(x, \xi)$ belongs to $S(<x>^m <\xi>^\ell, g)$ and moreover satisfies

\[
a'(x, \xi) = p^J(x, \xi) + r'(x, \xi),
\]

\[
p^J(x, \xi) - i\nabla_x \theta(x, y, \Phi(x, y, \xi)) = \eta
\]

\[
in \gamma^d S(C^{1+N} N^d <x>^m <\xi>^\ell, g_1),
\]

for any non negative integer $N$, where $\Phi'(y, \xi, \eta)$ is a solution of the equation

\[
\Phi'(y, \xi, \eta) = \int_0^1 \nabla \theta(t(x - y), \xi, \eta) dt,
\]

\[
J'(y, \xi, \eta) = \frac{D\Phi'(y, \xi, \eta)}{Dy}
\]

and $r'(x, \xi) \in A_{-\epsilon_0}^{1/d} (\epsilon_0 > 0)$ is independent of $\rho_0$.

**Lemma 4.4.** Let $\phi(x, \xi) \in \gamma^d S(\rho_0 \langle X\rangle^\sigma \langle\xi\rangle^\delta, g_1)$. If $\rho_0$ and $\delta_0$ are sufficiently small, there is the inverse of $I_\phi(x, D)$, which maps continuously from $H_{\rho, \delta}$ to $H_{\rho_1 - \rho_0, \delta_1 - \delta_0}$ for $|\rho_1|, |\delta_1|$ small enough and satisfies

\[
I_\phi(x, D)^{-1} = I_{-\phi}^R(x, D)(I + j(x, D))^{-1} = (I + j'(x, D))^{-1} I_{-\phi}^R(x, D)
\]

\[
= I_{-\phi}^R(x, D)(k(x, D) + k_1(x, D) + r(x, D)) = (k'(x, \xi) + k'_1(x, D) + r'(x, D))) I_{-\phi}^R(x, D),
\]
where \( j(x, \xi) = J(x, 0, \xi) - 1 + r(x, \xi), j'(x, \xi) = J'(x, \xi, 0) - 1 + r_2(x, \xi), k(x, \xi) = J(x, 0, \xi)^{-1}, k'(x, \xi) = J'(x, \xi, 0)^{-1} \) and \( k_1, k'_1 \in \gamma^d S(<x>^{-1}<\xi>^{-1}, g_1) \) and \( r, r' \in A_{-\epsilon_0}^{1/d}, -\epsilon_0 \).

Lemma 4.5. Let \( a(x, \xi) \) and \( \vartheta \) be satisfied with the same condition as one of Lemma 3.3. Let \( \phi = x\xi - i\vartheta \). Then we have

\[
\sigma(I_\phi(x, D)a(x, D))(x, \xi) = I_\phi \circ a(x, \xi) = e^{\vartheta(x, \xi)}(q(x, \xi) + r(x, \xi)),
\]

\[
\sigma(a(x, D)I_\phi(x, D)(x, \xi) = a \circ I_\phi(x, \xi) = e^{\vartheta(x, \xi)}(q'(x, \xi) + r'(x, \xi)),
\]

where \( r, r' \) is in \( A_{-\epsilon_0}^{1/d}, -\epsilon_0 \) if \( \rho_\vartheta \) is sufficiently small, and \( q, q' \) satisfies

\[
q(x, \xi) - \sum_{|\gamma|<N} \gamma^{-1} \partial^{\gamma}_{x} \partial^{\gamma}_{\xi} \{ a(x + y - i\tilde{\nabla}_{\xi} \vartheta(x, \xi, \eta), \xi) \} \Big|_{y=\eta=0} \in \gamma^d S(C^1 + NN!d <x>^m <\xi>^l, g_1),
\]

\[
q'(x, \xi) - \sum_{|\gamma|<N} \gamma^{-1} \partial^{\gamma}_{x} \partial^{\gamma}_{\xi} \{ a(x + y - i\tilde{\nabla}_{\xi} \vartheta(x, y, \xi), \xi) \} \Big|_{y=\eta=0} \in \gamma^d S(C^1 + NN!d <x>^m <\xi>^l, g_1),
\]

for any positive integer \( N \), and \( C > 0 \) and \( \epsilon_0 > 0 \) are independent of \( \rho_\vartheta \), where \( \tilde{\nabla}_{\xi} \vartheta(x, \xi, \eta) = \int_0^1 \nabla_{\xi} \vartheta(x, \xi + t\eta) dt \) and \( \tilde{\nabla}_{x} \vartheta(x, y, \xi) = \int_0^1 \nabla_{x} \vartheta(x + ty, \xi) dt \).

Summing up Lemma 4.2-Lemma 4.5, we obtain the following theorem.

Theorem 4.6. Let \( a \in \gamma^d S(<x>^m <\xi>^l, g_1), \vartheta \in \gamma^d S(\rho_\vartheta <\xi>^\delta <x>^\sigma, g_1) \) and \( \phi = x\xi - i\vartheta(x, \xi) \). Assume that \( \sigma + \delta = \kappa = 1/d <1 \) or \( \sigma + \delta = \kappa = 1, d = \min(\delta^{-1}, \sigma^{-1}) \). Then if \( \rho_\vartheta, \delta_\vartheta \) are sufficiently small, \( \tilde{a}(x, D) = I_\phi(x, D)a(x, D)I_\phi^{-1} \) and \( \tilde{a}'(x, D) = I_\phi(x, D)^{-1}a(x, D)I_\phi(x, D) \) are pseudodifferential operators of which symbols are given by

\[
\tilde{a}(x, \xi) = p(x, \xi) + r(x, \xi),
\]

\[
\tilde{a}'(x, \xi) = p'(x, \xi) + r'(x, \xi),
\]

where

\[
p(x, \xi) - a(x - i\nabla_{\xi} \vartheta(x, \Phi), \xi + i\nabla_{x} \vartheta(x, \Phi)) \in \gamma^d S(<x>^m <\xi>^l, g_1),
\]

\[
p'(x, \xi) - a(x + i\nabla_{\xi} \vartheta(\Phi', \xi), \xi - i\nabla_{x} \vartheta(\Phi', \xi)) \in \gamma^d S(<x>^m <\xi>^l, g_1),
\]

where \( \Phi = \Phi(x, x, \xi) \) and \( \Phi' = \Phi'(x, \xi, \xi) \) are given by (4.6) and (4.10) respectively and \( r, r' \) belong to \( A_{-\epsilon_0}^{1/d} \) for an \( \epsilon_0 > 0 \) independent of \( \rho_\vartheta \).

5 Criterion to \( L^2 \)-well posed Cauchy problem

For \( T > 0 \) let consider the following Cauchy problem,

\[
\partial_t u(t, x) - i\Delta u(t, x) - b(t, x, D)u(t, x) = 0,
\]
$u(0, x) = u_0(x),$

for $(t, x) \in (0, T) \times \mathbb{R}^n$. We assume that $b(t, x, \xi)$ is in $C^0([0, T]; S^1_{1, 0})$. Moreover we suppose that there are $C, K > 0$ such that

$$
Reb(t, x, \xi) \leq C,
$$

for $x, \xi \in \mathbb{R}^n$ with $|x|, |\xi| \geq K$ and $t \in [0, T]$. Then we can prove the following theorem by use of the same method as that of [3] and [7].

**Theorem 5.1.** Assume that the above conditions $(4.3)-(4.5)$ are valid. For any $u_0 \in L^2$ and $f \in C^0([0, T]; L^2)$ there exists a unique solution $u \in C^0([0, T]; L^2) \cap C^1([0, T]; H^{-2})$ of the Cauchy problem $(5.1)-(5.2)$.

### 6 Proof of Theorem

Assume that $u(t, x)$ satisfies $(1)-(2)$ in the introduction. Put $v(t, x) = e^{\rho t \langle D \rangle_\sigma} u(t, X)$. Then $v$ satisfies the following Cauchy problem,

$$
\frac{\partial}{\partial t} v(t, x) = (i\Delta + c(t, x, D))v(t, x),
$$

$$
v(0, x) = u_0(x),
$$

where

$$
c(t, x, D) = \rho\langle D \rangle^\kappa + e^{\rho\langle D \rangle^\kappa} b(t, XD) e^{-\rho\langle D \rangle^\kappa}
$$

and

$$
b_1(x, \xi) \in \gamma^d S(<\xi><x^{-1}, g_1), r_1(t, z, \xi) \in A_{-\epsilon_{\infty} + \epsilon_{\infty}}^\kappa - \mathcal{E} \text{ from Theorem 4.1.}
$$

Once more we change the unknown function $v$ to $w$ as follows,

$$
w(t, x) = I_\phi(x, D)v(t, X),
$$

where $\phi = x\xi - i\epsilon \theta(t, x, \xi)$ and $\theta$ is given by

$$
\theta(t, x, \xi) = \theta_0(x, \xi) \phi_0(\frac{x}{M\langle \xi \rangle}) + t(\xi)^{\sigma+\delta}(1 - \phi_0(\frac{x}{M\langle \xi \rangle})),
$$

$$
\phi_0(x, \xi) = \frac{x \cdot \xi}{(x)^{1-\sigma}(\xi)^{1-\delta} e_1} \phi_0(\frac{x \cdot \xi}{(x)(\xi)e_1}) + (\xi)^{\delta-\sigma} f(|x \cdot \xi|) [\phi_1(\frac{x \cdot \xi}{(x)(\xi)e_1}) - \phi_1(\frac{x \cdot \xi}{(x)(\xi)e_1})],
$$

$$
f(t) = \int_0^t (1 + s^2)^{\frac{\sigma-1}{2}} ds,
$$

and $\phi_\pm(t) = \chi(\pm t), \phi_0(t) = 1 - \phi_+(t) - \phi_-(t)$ and $\chi(t) \in \gamma^d(R)$ such that $\chi(t) = 1$ for $t \geq 1, \chi(t) = 0$ for $t \leq 1/2, \chi'(t) \geq 0$ and $0 \leq \chi(t) \leq 1$. Then we can see that $\theta(t, x, \xi)$ belongs to $\gamma^d S(|x|^\sigma(\xi)^\delta, g_1)$ and that there are $\epsilon_1 > 0, M > 0, K > 0, \epsilon_0 > 0$ such that $\theta$ satisfies

$$
(\partial_t + x \cdot \nabla_x)\theta(t, x, \xi) \geq \alpha_0(\langle \xi \rangle^{2\delta} |x|^{2\alpha-2} + \langle \xi \rangle^{\sigma+\delta} + \langle \xi \rangle |x|^{\sigma+\delta-1} - c_1,
$$

for $x, \xi \in \mathbb{R}^n$ with $|x|, |\xi| \geq K, |t| \leq T$.

It follows from Lemma 4.4 that if $|\epsilon|$ is sufficiently small, we have the inverse $I_\phi(x, D)^{-1}$. Therefore we get the following Cauchy problem of $w$ from $(6.1)-(6.2)$,
\begin{align}
\frac{\partial}{\partial t} w(t,x) &= (\partial_t I_\phi)I_\phi(x,D)^{-1} w(t,x) + I_\phi(i\Delta + c(t,x,D))I_\phi(x,D)^{-1} w(t,x), \\
 w(0,x) &= I_\phi(x,D) u_0(x).
\end{align}

Since \( \theta(t,x,\xi) \in \gamma^d S((x)^\sigma(\xi)^\delta, g_1) \), it follows from (4.10) that \( \nabla_x \theta(x, \Phi(x, \xi)) \in \gamma^d S(<x>^\sigma <\xi>^\delta, g_1) \), \( \nabla_\xi \theta(x, \Phi(x, \xi)) \in \gamma^d S(<x>^\sigma <\xi>^\delta, g_1) \), and \( \Phi(x, \xi) - \xi \in \gamma^d S(<x>^\sigma <\xi>^\delta, g_1) \). Hence we have from (4.16) in Theorem 4.6 and Proposition 2.3

\begin{align}
\sigma(I_\phi \Delta I_\phi^{-1})(x,\xi) &= -|\xi| + i\epsilon |\nabla_x \theta(x, \Phi)|^2 + a_1(x, D) + r_2(x, \xi), \\
&= -(|\xi|^2 + |\nabla_x \theta(t,x,\xi)|^2 + 2i \epsilon \cdot \nabla_x \theta(t,x,\xi)) + a_1'(x, \xi) + r_2(x, D)
\end{align}

where \( a_1 \in S(<\xi><x>^{-1}, g), a_1' \in S(<x>^{2\sigma-2} <\xi>^{2\delta} + (\xi)^{-1} g_1) \) and \( r_2 \in A^{1/2} \) for some \( c > 0 \) (independent of \( \epsilon \)). Here we choose \( \epsilon \) such that \( r_2 \) belongs to \( S(1,g) \). Thus we obtain the equation of \( w \) from (6.6)-(6.7),

\begin{align}
\frac{\partial w}{\partial t} &= (i\Delta + \rho(D)^{\sigma+\delta} + b(t,x,D) + c(\partial_t + \xi \cdot \nabla_x) \theta)(t,x,D)) + r_3(t,x,D))w(t,x), \\
\end{align}

\begin{align}
\frac{\partial w}{\partial t} &= (i\Delta + \rho(D)^{\sigma+\delta} + b(t,x,D) + c(\partial_t + \xi \cdot \nabla_x) \theta)(t,x,D)) + r_3(t,x,D))w(t,x), \\
&= \rho p(x,\xi) + \text{Re} \epsilon H_0 \theta(x,\xi) + \text{Re} \epsilon R_4(t,x,\xi) \leq 0,
\end{align}

for \( x,\xi \in \mathbb{R}^n \) with \( |x|, |\xi| \geq K \), where \( K > 0 \) is sufficiently large. Therefore we can solve the Cauchy problem (6.6)-(6.7) by use of Theorem 5.1, since \( w(0) = I_\phi(x,D) u_0 \) belongs to \( L^2 \), and consequently we get the solution \( u = e^{-\rho t(D)^{\sigma+\delta} I_\phi(x,D)^{-1} w(t,x)} = e(t,x,D)^{-1} I_\phi(x,D)^{-1} I_\phi(x,D)^{-1} u_0 \), which satisfies (6) from Lemma 1.2. This completes the proof of Theorem.

References


