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1. INTRODUCTION

Let $P(x, D)$ be a differential operator of order m defined in an open set Ω in $\mathbf{R}^n$ with principal symbol $p(x, \xi)$ which can be factorized as

$$p(x,\xi) = \prod_{j=1}^{m} q_j(x,\xi), \quad q_j(x,\xi) = \xi_1 - \lambda_j(x,\xi')$$

where $q_j(x,\xi)$ are real valued pseudodifferential symbol of order 1 and $x = (x_1, x') = (x_1, x_2, ..., x_n)$, $\xi = (\xi_1, \xi') = (\xi_1, \xi_2, ..., \xi_n)$. We assume that the characteristics of q_j intersects normally and non-involutively each other, that is

$$\{q_i, q_j\} \neq 0 \quad \text{on} \quad q_i = q_j = 0 \quad \text{for} \quad i \neq j$$

where $\{q_i, q_j\}$ denotes the Poisson bracket of q_i and q_j.

According to Iwasaki [2], we define the signature of a triplet (q_i, q_j, q_k) at z^0 where $q_i(z^0) = q_j(z^0) = q_k(z^0) = 0$. Let us say that three real numbers a, b, c have the same sign if they are simultaneously positive or simultaneously negative. We say $\mathrm{sgn}(q_i, q_j, q_k)(z^0) = +$ if $\{q_i, q_j\}(z^0)$, $\{q_j, q_k\}(z^0)$, $\{q_k, q_i\}(z^0)$ have the same sign and $\mathrm{sgn}(q_i, q_j, q_k)(z^0) = -$ otherwise.

When $m = 3$, in [2], Iwasaki proved that in order that the Cauchy problem of $P(x, D)$ is well posed the lower order terms must verify additional conditions at z^0 more than Ivrii-Petkov condition if $\mathrm{sgn}(q_1, q_2, q_3)(z^0) = +$. On the other hand, in [3] we proved that if the propagation cone at every triple characteristic is transversal to the doubly characteristic set and the lower order terms verify the Ivrii-Petkov condition then the Cauchy problem is well posed.

Here we recall that the localization p_{z^0} of p is the first non-trivial term in the Taylor expansion of p at z^0 which is a hyperbolic polynomial on $T_{z^0}(T^*\Omega)$ with respect to $\Theta = -H_{x^1} \in T_{z^0}(T^*\Omega)$ (see [1]) where H_f denotes the Hamilton vector field of $f \in C^\infty(T^*\Omega)$. The propagation cone at z^0 is the dual cone of the hyperbolic cone $\Gamma(p_{z^0}, \Theta)$ (for the definition, see [1]) with respect to the canonical symplectic structure on $T_{z^0}(T^*\Omega)$ induced by the 2-form $d\xi \wedge dx$.

In this note we show that, when $m = 3$ the condition $\mathrm{sgn}(q_1, q_2, q_3)(z^0) = -$ is equivalent to that the propagation cone at z^0 is transversal to the doubly characteristic set (Corollary 2.4). Let $m > 3$. At z^0 where $q_j(z^0) = 0$, $1 \leq j \leq m$ we can induce an order relation on the set $\{q_1, ..., q_m\}$ provided that $\mathrm{sgn}(q_i, q_j, q_k)(z^0) = -$ for every triplet (q_i, q_j, q_k). Using this order relation we give another formulation of the condition that the propagation cone at z^0 intersects transversally to the doubly characteristic set (Theorem 2.3).

2. Result

Let $q_i(x, \xi)$ be classical pseudodifferential symbols of order 1 defined near z^0 where $q_j(z^0) = 0$ and assume that the differentials dq_j are linearly independent at z^0. Let

$$p(x, \xi) = \prod_{j=1}^{m} q_j(x, \xi) \tag{2.1}$$

and assume that $p(x, \xi)$ is (microlocally) hyperbolic with respect to Θ. Thus we may suppose that $q_i(x, \xi)$ are real-valued and $dq_i(\Theta) > 0$ at z^0. Then it is clear that

$$\Gamma(p_{z^0}, \Theta) = \{X \in T_{z^0}(T^*\Omega) | dq_j(X) > 0, \forall j\}.$$

Denoting by $C(p_{z^0}, \Theta)$ the propagation cone at z^0 we easily see that

$$C(p_{z^0}, \Theta) = \{X \in T_{z^0}(T^*\Omega) | X = \sum \alpha_j H_{q_j}(z^0),\ \alpha_j \geq 0\}.$$

To simplify notations we write Γ_{z^0} and C_{z^0} for $\Gamma(p_{z^0}, \Theta)$ and $C(p_{z^0}, \Theta)$ respectively. Let $S_{ij} = \{(x, \xi) | q_i(x, \xi) = q_j(x, \xi) = 0\}$ for $i \neq j$ and define the map π_{ij};

$$\pi_{ij} : T_{z^0}(T^*\Omega) \ni X \mapsto (dq_i(X), dq_j(X)) \in \mathbf{R}^2.$$

Then we have

Lemma 2.1. *Assume that $\{q_\mu, q_\nu\}(z^0) \neq 0$ for every pair μ, ν, $\mu \neq \nu$. Then we have*

$$C_{z^0} \cap T_{z^0} S_{ij} = \{0\} \tag{2.2}$$

if and only if $\pi_{ij}(C_{z^0})$ *is a proper cone in* $\mathbf{R}^2$*. Moreover this is equivalent to*

$$-H_{c_iq_i+c_jq_j}(z^0)\in\Gamma_{z^0}$$

with some $c_i, c_j \in \mathbf{R}$.

Proof. We assume that $C_{z^0}\cap T_{z^0}S_{ij}=\{0\}$. We examine that $\pi_{ij}(C_{z^0})$ is a proper cone in $\mathbf{R}^2$. Otherwise $\pi_{ij}(C_{z^0})$ would contain a line. Thus there exist $0\neq X_k=\sum\alpha_{k\mu}\pi_{ij}(H_{q_\mu})$, $k=1,2$ such that $X_1+X_2=0$. This implies that

$$Y=\sum(\alpha_{1\mu}+\alpha_{2\mu})H_{q_\mu}\in T_{z^0}S_{ij}.$$

Since $\sum(\alpha_{1\mu}+\alpha_{2\mu})>0$ and hence $Y\neq 0$ this contradicts the assumption. Conversely assume that $\pi_{ij}(C_{z^0})$ is a proper cone in $\mathbf{R}^2$. Let $X=\sum\alpha_\mu H_{q_\mu}(z^0)\in C_{z^0}\cap T_{z^0}S_{ij}$. From $dq_i(X)=dq_j(X)=0$ it follows that

$$\sum\alpha_\mu\pi_{ij}(H_{q_\mu})=0.$$

Since $\pi_{ij}(H_{q_\mu})\neq 0$ we have $\alpha_\mu=0$ and hence $X=0$. That is $C_{z^0}\cap T_{z^0}S_{ij}=\{0\}$.

We next assume that $Z=-H_{c_iq_i+c_jq_j}(z^0)\in\Gamma_{z^0}$ with some $c_i,c_j\in\mathbf{R}$ so that

$$dq_\mu(Z)=c_idq_i(H_{q_\mu})+c_jdq_j(H_{q_\mu})>0,\ \forall\mu.$$

This shows that $\pi_{ij}(C_{z^0})$ is a proper cone in $\mathbf{R}^2$. Conversely we assume that $\pi_{ij}(C_{z^0})$ is a proper cone. Then we can choose c_i, c_j such that

$$<(c_i,c_j),\pi_{ij}(H_{q_\mu})>=(c_idq_i+c_jdq_j)(H_{q_\mu})>0,\ \forall\mu.$$

This proves $-H_{c_iq_i+c_jq_j}(z^0)\in\Gamma_{z^0}$. Hence the assertion.

Let us set $a_{\mu\nu}=\{q_\mu,q_\nu\}(z^0)$ and we express the condition (2.2) in terms of $a_{\mu\nu}$. Note that $Z=a_{\alpha\beta}H_{q_\gamma}+a_{\beta\gamma}H_{q_\alpha}+a_{\gamma\alpha}H_{q_\beta}\in T_{z^0}S_{\alpha\beta\gamma}$ where

$$S_{\alpha\beta\gamma}=\{(x,\xi)|q_\alpha(x,\xi)=q_\beta(x,\xi)=q_\gamma(x,\xi)=0\}.$$

If $a_{\alpha\beta}$, $a_{\beta\gamma}$, $a_{\gamma\alpha}$ have the same sign then it follows that $Z\in C_{z^0}$ which contradicts to $C_{z^0}\cap T_{z^0}S_{\alpha\beta}=\{0\}$. Thus if (2.2) holds for every pair i,j $(i\neq j)$ then one has $\mathrm{sgn}(q_\alpha,q_\beta,q_\gamma)(z^0)=-$ for every triplet (α,β,γ) (cf. Lemma 4.1 in [4]).

We say that $q_\alpha\gg q_\beta$ if $a_{\alpha\beta}>0$ or $\alpha=\beta$. Assuming that $\mathrm{sgn}(q_\alpha,q_\beta,q_\gamma)(z^0)=-$ for every triplet (α,β,γ), the relation $\gg$ becomes an order relation which is easily verified.

Lemma 2.2. *Assume that* $\operatorname{sgn}(q_\alpha, q_\beta, q_\gamma)(z^0) = -$ *for every triplet* (α, β, γ). *Let* $q_i \gg q_j$. *Then* $\pi_{ij}(C_{z^0})$ *is a proper cone if and only if*

$$\frac{a_{\nu i}}{a_{\nu j}} > \frac{a_{\mu i}}{a_{\mu j}}$$

for every q_ν, q_μ *with* $q_\nu \ll q_j \ll q_i \ll q_\mu$.

Proof. Recall that $\pi_{ij}(H_{q_\kappa}) = (a_{\kappa i}, a_{\kappa j})$. Note that $(a_{\kappa i}, a_{\kappa j})$ lies in the first, the third and the second quadrant if $q_i \ll q_\kappa$, $q_\kappa \ll q_j$ and $q_j \ll q_\kappa \ll q_i$ respectively. Let

$$\min_{q_\mu, q_i \ll q_\mu} \frac{a_{\mu j}}{a_{\mu i}} = \frac{a_{\mu_0 j}}{a_{\mu_0 i}}.$$

Then it is easy to see that the condition

$$< (a_{\mu_0 j}, -a_{\mu_0 i}), (a_{\nu i}, a_{\nu j}) >= a_{\mu_0 j} a_{\nu i} - a_{\mu_0 i} a_{\nu j} > 0$$

for every ν with $q_\nu \ll q_j$ is necessary and sufficient for $\pi_{ij}(C_{z^0})$ to be a proper cone. This proves the assertion.

Theorem 2.3. *Let* $m > 3$. *Then the following two conditions are equivalent.*

(i) $C_{z^0} \cap T_{z^0} S_{ij} = \{0\}$ *for every* i, j $(i \neq j)$.

(ii) $\operatorname{sgn}(q_i, q_j, q_k)(z^0) = -$ *for every triplet* (i, j, k) *and for every quadruplet* $(\alpha, \beta, \gamma, \delta)$ *with* $q_\alpha \ll q_\beta \ll q_\gamma \ll q_\delta$ *we have*

$$\frac{\{q_\alpha, q_\gamma\}(z^0)}{\{q_\alpha, q_\beta\}(z^0)} > \frac{\{q_\delta, q_\gamma\}(z^0)}{\{q_\delta, q_\beta\}(z^0)}.$$

Proof. It is clear from Lemma 2.2.

Corollary 2.4. *Let* $m = 3$. *Then the following two conditions are equivalent.*

(i) $C_{z^0} \cap T_{z^0} S_{ij} = \{0\}$ *for every* i, j $(i \neq j)$.

(ii) $\operatorname{sgn}(q_1, q_2, q_3)(z^0) = -$.

References

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