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Varieties of modules and p-blocks of finite groups

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1 Introduction

Let G be a finite group and k an algebraicaly closed field of characteristic $p>0$. For a finitely generated kG-module M, we denote the closed subvariety of $V_G(k)$ defined by the annihilator of $\mathrm{Ext}^*_{kG}(M,M)$ in $H^*(G,k)$ by $V_G(M)$, where $V_G(k)$ is the maximal ideal spectrum of the cohomology ring $H^*(G,k)$. In this note, we consider varieties of indecomposable modules in a p-block of kG. It is known that for any homogeneous closed subvariety V of $V_G(k)$, there is a kG-module M such that $V_G(M)=V$. On the other hand, if M is an indecomposable kG-module, then the variety $V_G(M)$ is connected (as a projective variety)([C]). Our main result is the following.

Theorem *Let B be a block of kG with defect group D. Let V be a connected homogeneous closed subvariety of $V_G(k)$. Then $V=V_G(L)$ for some indecomposable kG-module L in B if and only if $V=\mathrm{res}^*_{G,D}(W)$ for some connected homogeneous closed subvariety W of $V_D(k)$.*

Here, we say V is connected if it is connected as a projective variety.

In section 4, we study extensions of graded modules over a graded algebra A, which is a twisted algebra (in the sence of [Z]) of the group algebra of an elementary abelian 2-group. We state some results on the complexity $c(M)$ and the rate of growth of $\mathrm{Ext}^*_A(M,M)$ for a graded A-module M. It is known that these are equal if A is a group algebra.

2 Varieties of modules in B

Let B be a block of kG and D a defect group of B. We denote by V_B the union of varieties of all finitely generated kG-modules in B. It is easy to see that $V_B = \mathrm{res}_{G,D}^*(V_D(k))$, where $\mathrm{res}_{G,D}^*$ is the map induced by the restriction, $H^*(G,k) \longrightarrow H^*(D,k)$. Moreover, there is a finitely generated kG-module M in B such that $V_G(M) = V_B$. If V is a connected homogeneous closed subvariety of V_B, then V is not necessarily a variety of some indecomposable kG-module in B. The problem is that, in general, V does not come from a *connected* homogeneous closed subvariety of $V_D(k)$ (see Example 2.4).

Theorem 2.1 *Let H be a subgroup of G, M a kG-module and V a connected homogeneous closed subvariety of $V_G(k)$. Suppose that the trivial kH-module is a direct summand of M as a kH-module. If $V = \mathrm{res}_{G,H}^*(W)$ for some connected homogeneous closed subvariety W of $V_H(k)$, then there exists an indecomposable kG-module L such that $V_G(L) = V$ and* $\mathrm{Hom}_{kG}(M, L) \neq 0$.

Now, we consider the varieties of kG-modules in a block B. Since the varieties of kG-modules in B is contained in V_B, we consider only such a variety.

Corollary 2.2 *Let B be a block of kG with defect group D. Let V be a connected homogeneous closed subvariety of V_B. Then $V = V_G(L)$ for some indecomposable kG-module L in B if and only if $V = \mathrm{res}_{G,D}^*(W)$ for some connected homogeneous closed subvariety W of $V_D(k)$.*

Proof. Suppose that there exists an indecomposable kG-module L in B such that $V_G(L) = V$. Then there exists an indecomposable kD-module N such that $L|N\uparrow^G$ and $N|L\downarrow_D$. So we have that $\mathrm{res}_{G,D}^*(V_D(N)) = V$. Conversely, suppose that there exists a connected homogeneous closed subvariety W of $V_D(k)$ such that $\mathrm{res}_{G,D}^*(W) = V$. Note that there exists a kG-module M in B such that $k|M\downarrow_D$. By Theorem 2.1, there exists an indecomposable kG-module L such that $V_G(L) = V$ and $\mathrm{Hom}_{kG}(M, L) \neq 0$. In particular, L belongs to B.

Let V be a connected homogeneous closed subvariety of $V_G(k)$. If H is a Sylow p-subgroup of G, then it is easy to see that $V = \mathrm{res}_{G,H}^*(W)$ for some connected homogeneous closed subvariety W of $V_H(k)$. So we have,

Corollary 2.3([H1]) *Let V be a connected homogeneous closed subvariety of $V_G(k)$. Then there exists an indecomposable kG-module L such that $V_G(L) = V$ and* $\mathrm{Hom}_{kG}(k, L) \neq 0$.

Example 2.4 Let $p = 2$. Let G be a 2-nilpotent group generated by

$$x_i, y_i, z_i, u, v \quad (i = 1, 2)$$

with relations,

$$x_i^2 = y_i^3 = z_i^2 = u^2 = v^2 = 1, \; y_i^{x_i} = y_i^{-1},$$

$$x_i^v = x_j,\ y_i^v = y_j,\ z_i^v = z_j,$$
$$[a_i, b_j] = [a_i, z_i] = [a_i, u] = [v, u] = 1,$$
$$(a, b = x, y, z,\ 1 \leq i \neq j \leq 2).$$

So $G \cong ((S_3 \times C_2) \int C_2) \times C_2$, where we denote the symmetric group of degree 3 by S_3 and a cyclic group of order 2 by C_2. We set

$$D =< x_2, z_1, z_2, u >,\ E =< x_2 u, z_1 >,\ F =< x_2, z_2 >,$$

$$V = \mathrm{res}_{G,E}^*(V_E(k)) \cup \mathrm{res}_{G,F}^*(V_F(k)).$$

Then G has a 2-block B with defect group D. Moreover, V is a connected homogeneous closed subvariety of V_B. But there exists no connected homogeneous closed subvariety W of $V_D(k)$ such that $V = \mathrm{res}_{G,D}^*(W)$. So there exists no indecomposable kG-module L in B such that $V_G(L) = V$.

3 Some associated primes in $H^*(G, k)$

Let G be a p-group. The complexity $c(M)$ of a finitely generated kG-module M is the smallest nonnegative integer c such that

$$\lim_{n \to \infty} \frac{\dim_k \Omega^n(M)}{n^c} = 0.$$

It is known that $c(M) = \dim V_G(M) = \dim H^*(G,k)/I(M)$ where $I(M)$ is the annihilator of $H^*(G, M)$ in $H^*(G,k)$ ([B,Chapter 5]). So there exists a minimal associated prime P of $H^*(G,M)$ such that $\dim H^*(G,k)/P = c(M)$ ([M, Theorem 6.5]). Since P is an associated prime ideal, there exists a homogeneous element $x \in H^*(G, M)$ such that $P = \mathrm{ann}\ x$. In particular, $\dim H^*(G,M)/\mathrm{ann}\ x = c(M)$.

Definition Let G be a p-group and M a finitely generated kG-module. Suppose that $1 \leq i \leq c(M)$. We define $m_i(M)$ to be the smallest integer $m \geq 0$ such that $\dim H^*(G,k)/\mathrm{ann}\ x \geq i$ for some $x \in H^m(G, M)$. Then we have

$$m_1(M) \leq m_2(M) \leq \cdots \leq m_{c(M)}(M) < \infty$$

by the above argument.

Example 3.1 (1)Let $p = 2$ and $G = C_2 \times C_2$. If M is a nonprojective indecomposable kG-module, then M is either periodic or isomorphic to $\Omega^n(k)$ for some $n \in \mathrm{Z}$. If M is periodic, then $m_1(M) = 0$. On the other hand, we

have $m_2(\Omega^n(k)) = \max\{n, 0\}$.
(2)Let $p = 2$ and $G = C_2 \times C_2 \times C_2$. Fix any positive integer n. Then,

$$\sup\{m_2(M)|M : \text{f.g.}kG\text{-module}, c(M) = 3, \dim_k M \leq n\} < \infty.$$

Question 3.2 *Let G be a p-group. Fix $n, i > 0$. Then,*

$$\sup\{m_i(M)|M : \textit{f.g.}kG\textit{-module}, i \leq c(M), \dim_k M \leq n\} < \infty \ \ ?$$

4 Extensions of modules over some graded algebras

In this section, we assume that $p = 2$. Let

$$A = k < x_1, \dots, x_r > /(x_i^2, a_i x_i x_j + a_j x_j x_i, 1 \leq i, j \leq r)$$

for $a_i \in k, a_i \neq 0$. Then A is a finite dimensional local selfinjective graded k-algebra with deg $x_i = 1$ (see [H2],[Z] for more details). For a finitely generated A-module M, we define the rate of growth $\gamma(\mathrm{Ext}_A^*(M, M))$ of $\mathrm{Ext}_A^*(M, M)$ to be the smallest nonnegative integer s such that

$$\lim_{n \to \infty} \frac{\dim_k \mathrm{Ext}_A^n(M, M)}{n^s} = 0.$$

Then, $0 \leq \gamma(\mathrm{Ext}_A^*(M, M)) \leq c(M) \leq r$.

Theorem 4.1 *Let M be a finitely generated graded A-module. If $\gamma(\mathrm{Ext}_A^*(M, M)) = 0$, then $c(M) \leq r/2$.*

Remark (1)If $a_i = 1$ for every i, then A is a group algebra of an elementary abelian 2-groups. In this case, $\gamma(\mathrm{Ext}_A^*(M, M)) = c(M)$.
(2)([H2])If we take $a_1, \dots, a_r$ suitably, then A satisfies the following.

(*)*For any $1 \leq s \leq r/2$, there exists a graded A-module M such that $c(M) = s$ and $\gamma(\mathrm{Ext}_A^*(M, M)) = 0$.*

Suppose that $r = 3$. If M is a graded A-module and $c(M) = 3$, then $\gamma(\mathrm{Ext}_A^*(M, M)) \geq 1$ by Theorem 4.1. Using Example 3.1(2), we can prove the following.

Proposition 4.2 *Suppose that $r = 3$. If M is a graded A-module with*

$c(M) = 3$, *then* $\gamma(\mathrm{Ext}^*_A(M, M)) \geq 2$.

Question 4.3 *Suppose that* $r = 3$. *If* M *is a graded* A*-module with* $c(M) = 3$, *then* $\gamma(\mathrm{Ext}^*_A(M, M)) = 3$?

Suppose that Example 3.1(2) is true if we replace $m_2(M)$ by $m_3(M)$. Then the equality in Question 4.3 holds.

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