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RELATIVE PROJECTIVITY OF CARLSON MODULES

佐々木 洋城 (HIROKI SASAKI)
Department of Mathematical Sciences
Faculty of Science, Ehime University
Matsuyama, 790-8577, JAPAN

1. INTRODUCTION

The simple groups of 2-rank two were classified about 1970 by Alperin, Brauer, Gorenstein, Walter, and Lyons. See for example Alperin-Brauer-Gorenstein [3]. The 2-groups of rank two which can be Sylow 2-subgroups of finite simple groups are

(1) dihedral 2-groups (including four-groups);
(2) semidihedral 2-groups;
(3) wreathed 2-groups;
(4) special 2-group which is a Sylow 2-subgroup of $SU(3,4)$.

All finite groups with these Sylow 2-subgroups above were determined in those works.

The cohomology algebras of finite simple groups of 2-rank two have been known, depending on the classification theorems and on the fact that the cohomology algebras of some classical groups were calculated. A nice overview of these results is in the work by Adem-Milgram [1].

TABLE 1. Finite simple groups of 2-rank 2 and cohomology algebras

Sylow 2-subgroup (non-abelian)	Simple Groups	Cohomology Algebras
dihedral	$PSL(2,q)$, q odd A_7	$k[\varepsilon_2,\zeta_3,\theta_3]/(\zeta\theta)$
semidihedral	$PSL(3,q)$, $q\equiv 3 \pmod 4$ $PSU(3,q)$, $q\equiv 1 \pmod 4$ M_{11}	$k[\beta_3,\gamma_4,\delta_5]/(\beta^2\gamma-\delta^2)$
wreathed	$PSL(3,q)$, $q\equiv 1 \pmod 4$ $PSU(3,q)$, $q\equiv 3 \pmod 4$	$k[\theta_3,\rho_4,\theta_5,\sigma_6]/(\theta_3^2,\theta_5^2)$
special of order 64	$SU(3,4)$	

Remark 1.1. In Table 1 the subscript of a cohomology class indicates the degree. For example ρ_4 is of degree 4, σ_6 is of degree 6, and so on.

From the late 1980's the mod 2 cohomology algebras of those finite groups with dihedral, semidihedral, and quaternion Sylow 2-subgroups have been calculated:

(1) dihedral and quaternion case by Martino-Priddy [21], 1991; by Asai-Sasaki [5], 1993;
(2) semidihedral case by Martino [20], 1988; by Sasaki [24], 1994.

The works by Martino and Priddy dealt with the classifying spaces, and, as a consequence, obtained the cohomology algebras. On the other hand, the works by Asai and Sasaki depend on the theory of cohomology varieties of modules and on the modular representation theory of finite groups. Especially the theory of relative projectivity of modules played a crucial role. The theory of projectivity of modules relative to subgroups is fundamental in the theory of modular representations of finite groups. In [17] R. Knörr introduced the notion of projective covers of modules relative to subgroups. In the works [4] and [5] an injective hull of the trivial module relative to subgroups gave almost all information of the cohomology algebras. In [23] T. Okuyama introduced the notion of projectivity of modules relative to "modules". In the work [24] an injective hull of the trivial module relative to modules was essentially important. (Carlson pointed out in his lecture note [11] that the definition of projectivity relative to modules is just a special case of the relative homological algebra that can be defined for a projective class of epimorphism.)

The purpose of this report is to show that our method can be applied to finite groups whose Sylow subgroups are
(1) extraspecial p-groups of order p^3 and of exponent p;
(2) wreathed 2-groups.

This work was done with Professor Tetsuro Okuyama.

For H a subgroup of a finite group G and an element ζ in $H^*(G,k)$ we shall often write ζ_H or $\zeta_{|H}$ for the restriction $\mathrm{res}_H\,\zeta$.

2. Relative projectivity of modules

2.1. Projectivity relative to subgroups. First we state some results concerning projectivity of Carlson modules relative to subgroups. The following lemma is easy to prove and well known. This can be used to show divisibility by a homogeneous element.

Lemma 2.1. *Let*

$$E_\rho : 0 \longrightarrow k \longrightarrow \Omega^{-1}(L_\rho) \xrightarrow{f} \Omega^{r-1}(k) \longrightarrow 0$$

be the extension corresponding to an element ρ in $H^r(G,k)$. Suppose that the Carlson module L_ρ is relatively $\mathcal{H}$-projective, where $\mathcal{H}$ is a set of subgroups of G. If an element ξ in $H^{n+r}(G,k)$ satisfies

$$\mathrm{res}_H\, f^*(\xi) = 0 \quad \textit{for every } H \textit{ in } \mathcal{H},$$

where $f^* : \mathrm{Ext}^{n+r}_{kG}(k,k) \longrightarrow \mathrm{Ext}^n_{kG}(L_\rho,k)$, then there exists an element η in $H^n(G,k)$ such that

$$\xi = \rho\eta.$$

The Green correspondence is one of the important tools for analyzing indecomposable modules. The theorem below is of fundamental importance in investigation of indecomposable direct summands of Carlson modules by Green correspondence.

Theorem 2.2. *Let ρ in $H^n(G,k)$ be a homogeneous element. Let U be an indecomposable direct summand of the Carlson module L_ρ of ρ with vertex D. Let H be a subgroup of G containing the normalizer $N_G(D)$ and let V be a Green correspondent of U with respect to (G,D,H). Then the Green correspondent V is a direct summand of the Carlson module $L_{(\rho_H)}$ of the restriction $\rho_H = \operatorname{res}_H \rho$ to the subgroup H; moreover the multiplicity of the direct summand U in L_ρ is the same as the multiplicity of V in $L_{(\rho_H)}$.*

2.2. **Projectivity relative to modules.** In the rest of this section we deal with the theory of projectivity relative to modules. See Okuyama [23] or Carlson [11] for details.

Definition 2.1. For V a kG-module let

$$\mathcal{P}(V) = \{\, X \mid X \text{ is a direct summand of } V \otimes A \text{ for some } kG\text{-module } A\,\}.$$

A module in $\mathcal{P}(V)$ is said to be $\mathcal{P}(V)$-*projective* or *projective relative to* $\mathcal{P}(V)$. It is also said to be V-*projective* or *projective relative to* V for short. A module is said to be $\mathcal{P}(V)$-*injective*, *injective relative to* $\mathcal{P}(V)$, V-*injective* or *injective relative to* V if it is $\mathcal{P}(V)$-projective.

Definition 2.2. An exact sequence $E : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of kG-modules is said to be $\mathcal{P}(V)$-*split* (or V-*split* for short) if $V \otimes E : 0 \longrightarrow V \otimes A \xrightarrow{1\otimes f} V \otimes B \xrightarrow{1\otimes g} V \otimes C \longrightarrow 0$ splits.

Definition 2.3. Let M be a kG-module. A short exact sequence $E : 0 \longrightarrow X \longrightarrow R \longrightarrow M \longrightarrow 0$ is called a $\mathcal{P}(V)$-*projective cover* of M if

(1) R is $\mathcal{P}(V)$-projective;
(2) E is $\mathcal{P}(V)$-split;
(3) the kernel X has no $\mathcal{P}(V)$-projective direct summand.

If the exact sequence E above is a $\mathcal{P}(V)$-projective cover of M, then the kernel X is denoted by $\Omega_{\mathcal{P}(V)}(M)$. A $\mathcal{P}(V)$-projective cover is also called a V-projective cover and the kernel is also denoted by $\Omega_V(M)$. Similarly the notion of a $\mathcal{P}(V)$-*injective hull* is defined. If $F : 0 \longrightarrow M \longrightarrow S \longrightarrow Y \longrightarrow 0$ is a $\mathcal{P}(V)$-injective hull of M, then the cokernel Y is denoted by $\Omega^{-1}_{\mathcal{P}(V)}(M)$.

Theorem 2.3. *Every kG-module has a $\mathcal{P}(V)$-projective cover, which is uniquely determined up to isomorphism of sequences.*

The following lemma is of fundamental importance in investigation of the projectivity relative to modules by Green correspondence.

Lemma 2.4. *Let X be a kG-module. Let*

$$0 \longrightarrow \Omega_{\mathcal{P}(X)}(M) \longrightarrow R \longrightarrow M \longrightarrow 0$$

be a $\mathcal{P}(X)$-projective cover of a kG-module M and let U be an indecomposable direct summand of R with vertex D. Let H be a subgroup of G containing the normalizer $N_G(D)$ and let V be a Green correspondent of U with respect to (G,D,H). We let moreover

$$0 \longrightarrow \Omega_{\mathcal{P}(X_H)}(M_H) \longrightarrow S \longrightarrow M_H \longrightarrow 0$$

be a $\mathcal{P}(X_H)$-projective cover of the restriction M_H. Then the Green correspondent V is a direct summand of S; moreover the multiplicity of V in S is the same as the multiplicity of U in R.

Definition 2.4 (Carlson [11]). An element ζ in $H^n(G,k) - \{0\}$ is said to be *productive* if the exact sequence

$$E_\zeta : 0 \longrightarrow k \longrightarrow \Omega^{-1}(L_\zeta) \longrightarrow \Omega^{n-1}(k) \longrightarrow 0$$

is a $\mathcal{P}(L_\zeta)$-injective hull of the trivial module k, or equivalently the extension E_ζ is a $\mathcal{P}(L_\zeta)$-projective cover of the syzygy $\Omega^{n-1}(k)$. This condition is equivalent to the condition that $\zeta\,\mathrm{Ext}^*_{kG}(L_\zeta, L_\zeta) = 0$.

It is known that any homogeneous element of odd degree is productive when the prime p is odd (See for example Benson [6] Proposition 5.9.6 (ii)). However, for $p = 2$ no such general facts are known.

For a homogeneous element ζ in $H^*(G,k)$ and a subgroup H of G we see that $L_{\rho|H} = L_{(\rho_H)} \oplus$ (projective). The lemma below is a kind of converse of this fact and is useful to show a productive element in the cohomology algebra of a subgroup of a finite group G containing a Sylow normalizer to be stable under G.

Lemma 2.5. *Let S be a Sylow p-subgroup of a finite group G and let H be a subgroup of G containing the normalizer $N_G(S)$. Suppose that an element ρ in $H^r(H,k)$ is productive, namely, the extension*

$$0 \longrightarrow k_H \longrightarrow \Omega^{-1}(L_\rho) \longrightarrow \Omega^{r-1}(k_H) \longrightarrow 0$$

is a $\mathcal{P}(L_\rho)$-projective cover of the syzygy $\Omega^{r-1}(k_H)$. Assume that there exists a kG-module X such that

$$X_H \simeq L_\rho \oplus (\text{projective}).$$

Then there exists a productive element $\widetilde{\rho}$ in $H^r(G,k)$ such that

$$\mathrm{res}_H\, \widetilde{\rho} = \rho.$$

3. System of Parameters of Cohomology Algebras

We first state a theorem of Carlson on system of parameters of cohomology algebras and a corollary, which is a key fact for investigation of the projectivity relative to subgroups of Carlson modules.

Let G be a finite group of p-rank r and let S be a Sylow p-subgroup of G, where p is a prime number. For $i = 1, \dots, r$ let

$$\mathcal{H}_i(G) = \{\, C_G(E) \mid S \geq E \text{ is elementary abelian of rank } i \,\}.$$

Let k be a field of characteristic p.

Theorem 3.1 (Carlson [10] Proposition 2.4). *The cohomology algebra $H^*(G,k)$ has a homogeneous system $\{\zeta_1, \dots, \zeta_r\}$ of parameters with the property that for every $i = 1, \dots, r$*

$$\zeta_i \in \sum_{H \in \mathcal{H}_i(G)} \mathrm{tr}_H^G\, H^*(H,k).$$

Corollary 3.2 (Okuyama). *If a homogeneous system $\{\zeta_1, \ldots, \zeta_r\}$ of parameters is taken as in the theorem above, then the tensor product $L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_{r-1}}$ is $\mathcal{H}_r(G)$-projective.*

In particular, if $r = 2$, then L_{ζ_1} is $\mathcal{H}_2(G)$-projective and the element ζ_1 is regular in $H^*(G, k)$.

The following theorem shows that a system of parameters can be obtained from a productive cohomology element when the p-rank is two.

Theorem 3.3. *Let G be a finite group of p-rank two. Let ρ in $H^r(G, k)$ be a regular element in $H^*(G, k)$. Assume that the element ρ is productive, that is, the extension*

$$E_\rho : 0 \longrightarrow k_G \longrightarrow \Omega^{-1}(L_\rho) \xrightarrow{f} \Omega^{r-1}(k_G) \longrightarrow 0$$

is a $\mathcal{P}(L_\rho)$-injective hull of the trivial kG-module k_G and that for a number s with $s \geq r-1$

$$\Omega^s(L_\rho) \simeq L_\rho.$$

Then there exists an inverse image σ in $\mathrm{Ext}^s_{kG}(k, k)$ of $\Omega^{-r+1}f : \Omega^{s-r}(L_\rho) \longrightarrow k$ by the induced homomorphism $f^* : \mathrm{Ext}^s_{kG}(k, k) \longrightarrow \mathrm{Ext}^{s-r}_{kG}(L_\rho, k)$. The elements σ and ρ form a system of parameters for the cohomology algebra $H^*(G, k)$.

4. Extraspecial p-groups

Let p be an odd prime. In this section we consider the cohomology algebra of an extraspecial p-group

$$P = \langle\, a, b \mid a^p = b^p = [a, b]^p = 1, [[a, b], a] = [[a, b], b] = 1 \,\rangle$$

of order p^3 of exponent p; especially we shall choose a system of parameters whose members are universally stable. The mod p cohomology algebra of P was calculated by Leary [18]. Tezuka-Yagita [26] investigated the p-parts of integral cohomology algebras of finite groups G having P as Sylow p-subgroups. Tezuka-Yagita [25] studied the mod p cohomology algebra of the general linear group $\mathrm{GL}(3, \mathbf{F}_p)$, whose Sylow p-subgroup is our P above. We apply our results on relative projectivity of modules stated in Sections 2 and 3 to the p-group P and finite groups with P as Sylow p-subgroups.

4.1. System of parameters.

Definition 4.1. Let

$$c = [a, b].$$

Then $Z(P) = \langle\, c \,\rangle$. For i, $0 \leq i \leq p-1$, let

$$E_i = \langle\, ab^i, c \,\rangle; \; a_i = ab^i, \; b_i = b.$$

Let

$$E_\infty = \langle\, b, c \,\rangle; \; a_\infty = b, \; b_\infty = a.$$

We put

$$\Omega = \{\, 0, 1, \ldots, p-1, \infty \,\}; \; \mathcal{E} = \{\, E_i \mid i \in \Omega \,\}.$$

The set $\mathcal{E}$ is the collection of elementary abelian subgroups of P of rank two. We note that $C_P(E) = E$ for E in $\mathcal{E}$; hence $\mathcal{H}_2(P) = \mathcal{E}$.

Theorem 3.1 and Corollary 3.2 say that there exists a system $\{\xi_1, \xi_2\}$ of parameters such that

(1) $\xi_2 \in \sum_{E\in\mathcal{E}} \mathrm{tr}_E^P H^*(E,k)$;
(2) L_{ξ_1} is $\mathcal{E}$-projective;
(3) ξ_1 is regular in $H^*(P,k)$.

Of course there are many choices of system of parameters as above. The cohomology classes which we define below are good ones because of Lemma 4.1, Theorems 4.2 and 4.3. We have to mention that they and their properties would be seen or verified by similar arguments to those in the papers [18], [25], or [26].

Definition 4.2. For i with $0 \le i \le p-1$, regarding $H^1(E_i,k)$ as $\mathrm{Hom}(E_i,k)$, let

$$\lambda_i = a_i^*,\ \mu_i = b_i^*.$$

We also let

$$\lambda_\infty = -a_\infty^*,\ \mu_i = b_\infty^*.$$

For i in Ω we let

$$\alpha_i = \Delta(\lambda_i),\ \gamma_i = \Delta(\mu_i),$$

where $\Delta : H^1(E_i,k) \to H^2(E_i,k)$ is the Bockstein homomorphism. Then the element b_i acts on these elements as follows:

$$\alpha_i^{b_i} = \alpha_i,\ \gamma_i^{b_i} = -\alpha_i + \gamma_i.$$

Definition 4.3. Let

$$\nu = \mathrm{norm}_{E_\infty}^P(\gamma_\infty) \in H^{2p}(P,k).$$

For i in Ω let

$$\zeta_i = \mathrm{tr}_{E_i}^P(\gamma_i^{p-1}) \in H^{2(p-1)}(P,k)$$

and define

$$\zeta = \sum_{i\in\Omega} \zeta_i.$$

We define moreover

$$\rho = \nu^{p-1} - \zeta^p \in H^{2p(p-1)}(P,k),$$
$$\sigma = \nu^{p-1}\zeta \in H^{2(p^2-1)}(P,k).$$

Note that

$$\sigma \in \sum_{E\in\mathcal{E}} \mathrm{tr}_E^P H^{2(p^2-1)}(E,k).$$

For $E = E_i$ in $\mathcal{E}$ we shall often omit subscript i of the cohomologies γ_i and α_i.

Lemma 4.1. *For E in $\mathcal{E}$ one has*

$$\mathrm{res}_E\,\rho = \prod_{\eta\in\mathbf{F}_{p^2}\setminus\mathbf{F}_p} (\gamma - \eta\alpha), \quad \mathrm{res}_E\,\sigma = -\left(\alpha \prod_{i\in\mathbf{F}_p} (\gamma - i\alpha)\right)^{p-1}.$$

For E_i in $\mathcal{E}$ the factor group $P/E_i = \langle \overline{b_i} \rangle$, where $\overline{b_i} = E_i b_i$, acts on the set

$$\{ L_{\gamma-\eta\alpha} \mid \eta \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \}$$

by conjugation as follows

$$L_{\gamma-\eta\alpha}{}^{b_i} = L_{\gamma-(\eta+1)\alpha}.$$

This action induces the action of $P/E_i = \langle \overline{b_i} \rangle$ on the set $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ such that $\eta^{b_i} = 1+\eta$ for η in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$. Thus, if we write $(\mathbf{F}_{p^2} \setminus \mathbf{F}_p)/P$ for the quotient set of $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ under this action, then a complete set of representatives of the conjugation on $\{ L_{\gamma-\eta\alpha} \mid \eta \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \}$ can be written as

$$\{ L_{\gamma-\eta\alpha} \mid \eta \in (\mathbf{F}_{p^2} \setminus \mathbf{F}_p)/P \}.$$

Using Lemma 4.1 and Corollary 3.2, we can show the following.

Theorem 4.2. (1) *The set $\{ \rho, \sigma \}$ is a system of parameters of the cohomology algebra $H^*(P,k)$.*

(2) *The Carlson module L_ρ is $\mathcal{E}$-projective. In fact the module L_ρ decomposes as follows:*

$$L_\rho = \bigoplus_{E \in \mathcal{E}} \bigoplus_{\eta \in (\mathbf{F}_{p^2} \setminus \mathbf{F}_p)/P} L_{\gamma-\eta\alpha}{}^P.$$

(3) *The element ρ is regular in $H^*(P,k)$.*

4.2. **Finite group with P as a Sylow p-subgroup.** In the rest of this section we let G be a finite group with P as a Sylow p-subgroup. We can show that the elements ρ and σ are the restrictions from any such G. Namely

Theorem 4.3. *The cohomologies ρ and σ are universally stable.*

Definition 4.4. Since the cohomologies ρ and σ are universally stable, there exits an element $\widetilde{\rho}$ in $H^{2p(p-1)}(G,k)$ such that

$$\mathrm{res}_P(\widetilde{\rho}) = \rho$$

and an element $\widetilde{\sigma}$ in $H^{2(p^2-1)}(G,k)$ such that

$$\mathrm{res}_P(\widetilde{\sigma}) = \sigma.$$

Definition 4.5. The Carlson module $L_{\widetilde{\rho}}$ of the element $\widetilde{\rho}$ is projective relative to $\mathcal{H}_2(G) = \{ C_G(E) \mid E \in \mathcal{E} \}$ by Corollary 3.2. The centralizer $C_G(E)$ of E in $\mathcal{E}$ has a normal p-complement; hence $L_{\widetilde{\rho}}$ is $\mathcal{E}$-projective. Theorem 4.2 implies an indecomposable direct summand of $L_{\widetilde{\rho}}$ has vertex some E in $\mathcal{E}$ and source some $L_{\gamma-\eta\alpha}$, η in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$. For E in $\mathcal{E}/G$ we write

$$\{ X_i^{(E)} \mid i \in I^{(E)} \}$$

for the set of indecomposable direct summands of $L_{\widetilde{\rho}}$ whose vertices are E; we denote by $X^{(E)}$ their direct sum. Then we have

$$L_{\widetilde{\rho}} = \bigoplus_{E \in \mathcal{E}/G} X^{(E)}.$$

Theorem 4.4. *The Carlson module $L_{\widetilde{\rho}}$ decomposes as follows:*

$$L_{\widetilde{\rho}} = \bigoplus_{E \in \mathcal{E}/G} \bigoplus_{i \in I^{(E)}} X_i^{(E)},$$

where if $i \neq j$, then $X_i^{(E)}$ and $X_j^{(E)}$ have different sources.

Definition 4.6. Let

$Y_i^{(E)}$ be a Green correspondent of $X_i^{(E)}$ with respect to $(G, E, N_G(E))$.

By Theorem 2.2 the $kN_G(E)$-module $Y_i^{(E)}$ is an indecomposable direct summand of the Carlson module $L_{\rho'}$ of the restriction $\rho' = \mathrm{res}_{N_G(E)}\, \widetilde{\rho}$ with multiplicity one. Let us denote by $Y^{(E)}$ the direct sum of these modules:

$$Y^{(E)} = \bigoplus_{i \in I^{(E)}} Y_i^{(E)}.$$

Proposition 4.5. *One has*

$$(Y^{(E)})^G = X^{(E)} \oplus (\text{projective}) \ .$$

Corollary 4.6. *One has*

$$\mathrm{Ext}^*_{kG}(L_{\widetilde{\rho}}, k) \simeq \bigoplus_{E \in \mathcal{E}/G} \mathrm{Ext}^*_{kN_G(E)}(Y^{(E)}, k).$$

In particular

$$\dim H^{n+2p(p-1)}(G, k) = \dim H^n(G, k) + \sum_{E \in \mathcal{E}/G} \dim \mathrm{Ext}^*_{kN_G(E)}(Y^{(E)}, k).$$

Therefore we have to investigate the module $Y^{(E)}$, which is a direct summand of the Carlson module $L_{\rho'}$ of $\rho' = \mathrm{res}_{N_G(E)}\, \widetilde{\rho}$.

Definition 4.7. We write $\{ L_{\gamma-\eta_i\alpha} \mid i \in I^{(E)} \}$ for a set of complete set of representatives of the action of the factor group $N_G(E)/C_G(E)$ on the set $\{ L_{\gamma-\eta\alpha} \mid \eta \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \}$.

For i in $I^{(E)}$ the module $Y_i^{(E)}$ would be investigated in the following way. We omit the superscript $^{(E)}$ and subscript $_i$ in what follows, namely, we write Y for an indecomposable direct summand of $L_{\rho'}$ with vertex E and source $L_{\gamma-\eta\alpha}$.

(1) First investigate

$$H_\eta = \{ g \in N_G(E) \mid {L_{\gamma-\eta\alpha}}^g \simeq L_{\gamma-\eta\alpha} \}.$$

(2) Denote by L_C the extension of $L_{\gamma-\eta\alpha}$ to $C_G(E)$ in a natural way. Let

$${L_C}^{H_\eta} = \bigoplus_j M_j$$

be an indecomposable decomposition of the induced module ${L_C}^{H_\eta}$. The module Y is the induced module ${M_j}^{N_G(E)}$ of some indecomposable M_j.

(3) Determine M_j.

5. Wreathed 2-groups

Let

$$S = \langle a, b, t \mid a^{2^n} = b^{2^n} = t^2 = 1,\ ab = ba,\ tat = b \rangle, \quad n \geq 2$$

be a wreathed 2-group.

Let k be a field of characteristic 2 containing a cubic root of unity. We shall consider the cohomology algebras $H^*(G, k)$ of finite groups G having S above as Sylow 2-subgroups.

5.1. System of Parameters.

Definition 5.1. Let

$$c = ab,\ x = a^{2^{n-1}},\ y = b^{2^{n-1}},\ z = xy = c^{2^{n-1}}$$

and let

$$E = \langle x, y \rangle, \quad F = \langle z, t \rangle.$$

Then $\{ E, F \}$ is a complete set of representatives of the conjugacy classes of four-groups in S. Their centralizers are

$$C_S(E) = \langle a \rangle \times \langle b \rangle, \quad C_S(F) = \langle c \rangle \times \langle t \rangle.$$

We set

$$\langle a \rangle \times \langle b \rangle = U, \quad \langle c \rangle \times \langle t \rangle = V.$$

Then we have

$$\mathcal{H}_2(S) = \{ U,\ V \}.$$

By Theorem 3.1 and Corollary 3.2, the cohomology algebra $H^*(S, k)$ has a homogeneous system $\{ \xi_1, \xi_2 \}$ of parameters such that

(1) $\xi_2 \in \mathrm{tr}_U^S H^*(U, k) + \mathrm{tr}_V^S H^*(V, k)$;
(2) L_{ξ_1} is $\{ U,\ V \}$-projective;
(3) ξ_1 is regular in $H^*(S, k)$.

In the rest of this report the subscript of a cohomology class indicates the degree. For example α_2 is of degree 2, ν_4 is of degree 4, and so on.

Definition 5.2. Let

$$\alpha_2 \in \inf^U H^2(U/\langle b \rangle, k),\ \beta_2 \in \inf^U H^2(U/\langle a \rangle, k)$$
$$\chi_2 \in \inf^V H^2(V/\langle t \rangle, k),\ \psi_2 \in \inf^V H^2(V/\langle c \rangle, k).$$

Let

$$\tau_1 \in \inf^S H^1(S/U, k)$$
$$\zeta_2 = \mathrm{tr}_U^S \alpha_2 \in H^2(S, k)$$
$$\nu_4 = \mathrm{norm}_U^S \alpha_2 \in H^4(S, k)$$

and let

$$\rho_4 = \tau_1^4 + \zeta_2^2 + \nu_4$$
$$\sigma_6 = (\tau_1^2 + \zeta_2)\nu_4.$$

Then we have

Theorem 5.1. (1) *The set* $\{\rho_4, \sigma_6\}$ *is a homogeneous system of parameters of* $H^*(S,k)$.

(2) $\sigma_6 \in \operatorname{tr}_U^S H^6(U,k) + \operatorname{tr}_V^S H^6(V,k)$;

(3) *The Carlson module* L_{ρ_4} *is* $\{U, V\}$*-projective. In fact*

$$L_{\rho_4} = L_{\alpha_2+\omega\beta_2}{}^S \oplus L_{\chi_2+\omega\psi_2}{}^S,$$

where $\omega = \sqrt[3]{1} \in k$.

(4) *The element* ρ_4 *is regular in* $H^*(S,k)$.

The element ρ_4 is universally stable. To show this, we use the theory of projectivity of modules relative to "modules". First we see

Theorem 5.2. *The element* ρ_4 *is productive. Namely, the extension*

$$0 \longrightarrow k \longrightarrow \Omega^{-1}L_\rho \longrightarrow \Omega^3 k \longrightarrow 0$$

induced by the element ρ_4 *in* $H^4(S,k)$ *is a* $\mathcal{P}(L_\rho)$*-injective hull of the trivial module* k.

5.2. **Finite group with S with a Sylow 2-subgroup.** Let G be a finite group which has S as a Sylow 2-subgroup. Structure of these groups had been deeply investigated in Brauer-Wong [9], Brauer [8], and Alperin-Brauer-Gorenstein [2].

The fusion of 2-elements can be described by behavior of several involutions and subgroups. Among them we use four-groups and their normalizers. The reason is of course Theorem 3.1 by Carlson and Corollary 3.2.

The fusion of 2-elements in G is indicated in Table 2.

TABLE 2. Fusion of 2-elements

	a: $N_G(E)/C_G(E) \simeq \mathbf{Z}/2\mathbf{Z}$ ($x \not\sim z$)	b: $N_G(E)/C_G(E) \simeq S_3$ ($x \sim z$)
1: $E \not\sim F$ ($x \not\sim t$)	$S \cap G' = S' = \langle ab^{-1} \rangle$	$S \cap G' = \langle a, b \rangle$
2: $E \sim F$ ($x \sim t$)	$S \cap G' = \langle ab^{-1}, xt \rangle$	$S \cap G' = S$

Following Alperin-Brauer-Gorenstein [2], we call a group of type 1b a "D-group"; a group of type 2a a "Q-group"; a group of type 2b a "QD-group".

The cohomology algebra of the wreathed 2-group is calculated by Nakaoka's theorem. The cohomology algebras of finite groups with wreathed Sylow 2-subgroups are obtained below. In the following the cohomology algebras of other types of groups are stated as subalgebras of that of the wreathed Sylow 2-subgroup S.

Theorem 5.3. *Let G be a finite group with S as a Sylow 2-subgroup.*

(1) *If G is of type 1a, then*

$$H^*(G,k) \simeq H^*(S,k)$$
$$= k[\zeta_1, \tau_1, \zeta_2, \nu_2, \zeta_3, \nu_4]/(\zeta_1^2, \nu_2^2, \zeta_3^2, \tau\zeta_1, \tau\zeta_2, \tau\zeta_3, \nu_2\zeta_1, \nu_2\zeta_3, \zeta_1\zeta_3 - \zeta_2\nu_2).$$

(2) *If G is a D-group, then*

$$H^*(G,k) = k[\tau_1, \nu_2, \theta_3, \rho_4, \theta_5, \sigma_6],$$

where

$$\theta_3 = \tau_1\nu_2 + \zeta_1\zeta_2 + \zeta_3,\ \ \rho_4 = \tau_1^4 + \zeta_2^2 + \nu_4,\ \ \theta_5 = \tau^3\nu_2 + \zeta_1\rho_4 + \zeta_2\zeta_3,\ \ \sigma_6 = (\tau_1^2 + \zeta_2)\nu_4.$$

(3) *If G is a Q-group, then*

$$H^*(G,k) = k[\zeta_1, \sigma_2, \theta_3, \rho_4],$$

where

$$\sigma_2 = \tau_1^2 + \zeta_2.$$

(4) *If G is a QD-group, then*

$$H^*(G,k) = k[\theta_3, \rho_4, \theta_5, \sigma_6].$$

Remark 5.1. The elements ζ_1, ν_2, and ζ_3 above will be stated in subsection 5.5.

To show that the element ρ_4 in $H^4(S,k)$ is universally stable, we show the following

Theorem 5.4. *There exists a kG-module $\widetilde{X}$ such that*

$$\widetilde{X}_S = L_\rho \oplus \text{(projective)}.$$

Using Theorem 5.4, Lemma 2.5, and lemma 2.4, we can show

Theorem 5.5. *There exists a productive element $\widetilde{\rho}$ in $H^4(G,k)$ such that*

$$\operatorname{res}_S \widetilde{\rho} = \rho,\ \ L_{\widetilde{\rho}} = \widetilde{X}.$$

5.3. ***QD*-groups.** Our proof of Theorem 5.4 is slightly complicated. So we sketch our argument only for QD-groups G; namely suppose, in this subsection, that the four-groups E and F are conjugate in G and the quotient group $N_G(E)/C_G(E)$ is isomorphic with S_3. Let H be the subgroup of $N_G(E)$ of index two containing the centralizer $C_G(E)$. We may assume that there exists an element h in H such that

$$a^h = b,\ \ b^h = ab.$$

Suppose for the moment that there exits an element $\widetilde{\rho}$ in $H^4(G,k)$ such that

$$\operatorname{res}_S \widetilde{\rho} = \rho.$$

Then an indecomposable direct summand $\widetilde{X}$ of the Carlson module $L_{\widetilde{\rho}}$ would have vertex U and a source $L_{\alpha_2+\omega\beta_2}$ so that $\widetilde{X}$ would be the Green correspondent of an indecomposable direct summand X' of the Carlson module $L_{\rho'}$ of the restriction $\rho' = \operatorname{res}_{N_G(E)} \widetilde{\rho}$ with vertex U and a source $L_{\alpha_2+\omega\beta_2}$. We write $N = N_G(E)$ and $C = C_G(E)$. We know that the centralizer $C_G(E)$ has a normal 2-complement:

$$C_G(E) = U \ltimes O(C_G(E)).$$

Let

$$\varepsilon = \alpha_2 + \omega\beta_2.$$

Since $C_G(E) = U \ltimes O(C_G(E))$, the Carlson module L_ε can be extended to $C = C_G(E)$. We denote by L_C the extension of L_ε to $C_G(E)$:

$$L_{C|U} \simeq L_\varepsilon, \ L_{C|O(C_G(E))} = \text{trivial module}.$$

Because the module X' belongs to the principal kN-block, the module X' is a direct summand of the induced module ${L_C}^N$. The induced module ${L_C}^N$ can be analyzed by Clifford theory. First we have

$$H = \{\, x \in N \mid L_\varepsilon^x \simeq L_\varepsilon \,\}.$$

Thus if ${L_C}^H = \sum M_j$ is an indecomposable decomposition, then the induced modules ${M_j}^N$s are indecomposable and ${L_C}^N = \sum {M_j}^N$. Indeed the induced module ${L_C}^H$ decomposes as follows.

Lemma 5.6. *For $i = 0, 1, 2$, let k_i be the one-dimensional kH-module on which U acts trivially and the element h acts as multiplication by ω^i. Then the induced module ${L_C}^H$ has a decomposition*

$${L_C}^H = M_0 \oplus M_1 \oplus M_2$$

of indecomposable kH-modules M_0, M_1, M_2 such that

(1) $M_{i|U} \simeq L_\varepsilon$, $i = 0, 1, 2$;
(2) *M_is are periodic of period six;*
(3) $M_i \simeq M_0 \otimes k_i$, $i = 0, 1, 2$;
(4)

$$M_0/\operatorname{rad} M_0 = k_2 \oplus k_0, \qquad \operatorname{soc} M_0 = k_1 \oplus k_2;$$
$$\Omega(M_0)/\operatorname{rad}\Omega(M_0) = k_0 \oplus k_1, \ \operatorname{soc}\Omega(M_0) = k_2 \oplus k_0;$$

(5) $\Omega^2(M_0) = M_2$, $\Omega^2(M_1) = M_0$, $\Omega^2(M_2) = M_1$.

The indecomposable kN-module X' would be one of ${M_i}^N$s; we put, say,

$$X' = {M_i}^N.$$

Since the module ${M_i}^N$ would be a direct summand of $L_{\rho'}$, the module M_i would be a direct summand of the Carlson module $L_{\rho''}$ of the restriction $\rho'' = \operatorname{res}_H \rho'$.

Now let us return to construction of the kG-module $\widetilde{X}$.

Lemma 5.7. *The cohomology ρ_4 is $N_G(E)$-stable.*

Definition 5.3. Let us take ρ' in $H^4(N_G(E), k)$ such that

$$\operatorname{res}_S \rho' = \rho.$$

Then we have

$$\rho' = \operatorname{tr}_S^N \rho.$$

We also let

$$\rho'' = \operatorname{res}_H \rho'.$$

Then we have

Proposition 5.8. *The Carlson module $L_{\rho''}$ has a decomposition*

$$L_{\rho''} = M \oplus M^t$$

of indecomposable kH-modules M and M^t such that

(1) $M_U \simeq L_\varepsilon$;
(2) ${M^N}_U \simeq {L_\varepsilon}^S$;
(3) *M is periodic of period six;*
(4)

$$\begin{aligned} M/\operatorname{rad} M &= k_1 \oplus k_2, & \operatorname{soc} M &= k_0 \oplus k_1; \\ \Omega(M)/\operatorname{rad}\Omega(M) &= k_2 \oplus k_0, & \operatorname{soc}\Omega(M) &= k_1 \oplus k_2; \\ \Omega^2(M)/\operatorname{rad}\Omega^2(M) &= k_0 \oplus k_1, & \operatorname{soc}\Omega^2(M) &= k_2 \oplus k_0; \end{aligned}$$

(5) *For $i \geq 0$*

$$\Omega^{i+3}(M)/\operatorname{rad}\Omega^{i+3}(M) \simeq \Omega^i(M)/\operatorname{rad}\Omega^i(M);$$
$$\operatorname{soc}\Omega^{i+3}(M) \simeq \operatorname{soc}\Omega^i(M).$$

Definition 5.4. Let

$$X' = M^N.$$

The indecomposable kN-module X' has vertex U and source L_ε.

By our definition of the indecomposable kN-module X' we have

Proposition 5.9. (1) *The indecomposable kN-module X' is periodic of period six.*
(2) $X'_S = {L_\varepsilon}^S$.
(3) *The indecomposable kN-module X' is a direct summand of $L_{\rho'}$.*

Using the proposition above and the assumption that the four-groups E and F are conjugate, we obtain

Proposition 5.10. *It follows that*

$${X'^G}_S \equiv L_\rho \quad (\text{mod projective}).$$

We can now define the kG-module $\widetilde{X}$.

Definition 5.5. Let

$\widetilde{X}$ be a Green correspondent of X' with respect to $(G, U, N_G(E))$.

By means of Propositions 5.9, 5.10 and the properties of the Green correspondence we see that the kG-module $\widetilde{X}$ is the desired one.

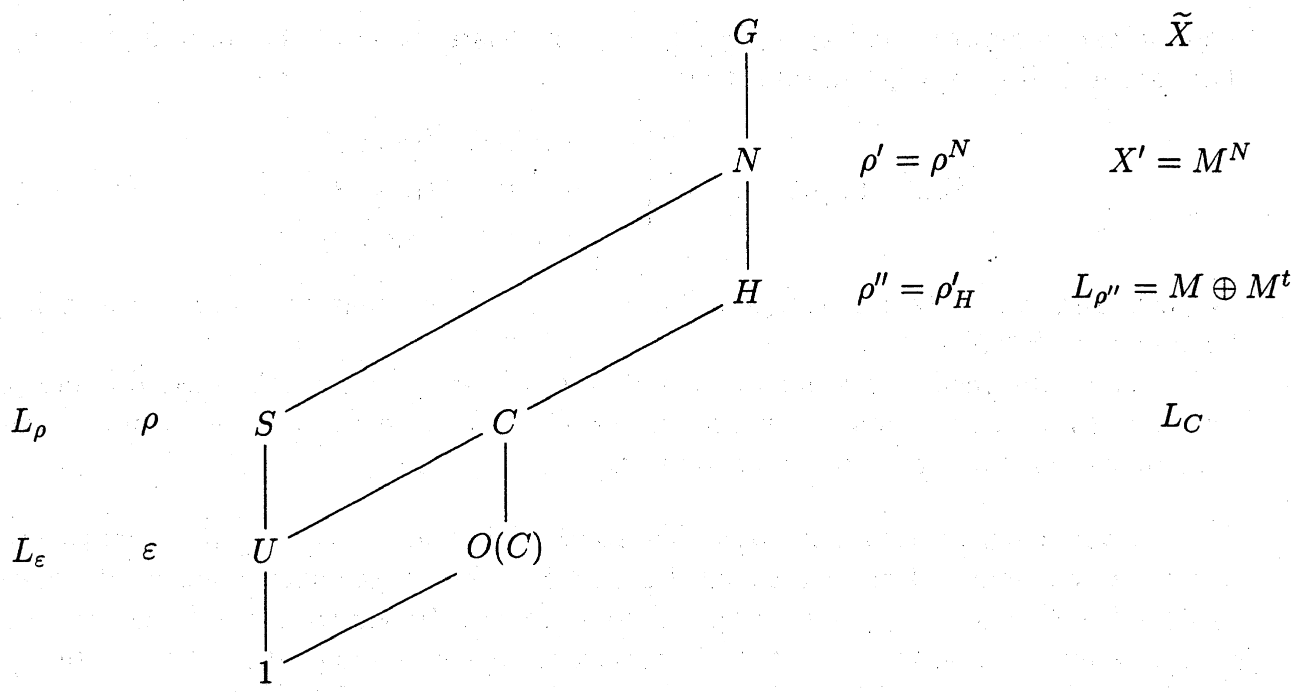
Theorem 5.11. *We have*

(1) *the kG-module $\widetilde{X}$ is periodic of period six;*
(2)

$$X'^G = \widetilde{X} \oplus (\text{projective});$$

(3)

$$\widetilde{X}_S = L_\rho \oplus (\text{projective}).$$



Remark 5.2. When the four-groups E and F are not conjugate in G, the kG-module $\widetilde{X}$ in Theorem 5.4 is not indecomposable.

5.4. **Relative injective hull of the trivial module.** We resume our situation that G is a finite group with wreathed Sylow 2-subgroup S. The element $\widetilde{\rho}_4$ is productive by Theorem 5.5. The $L_{\widetilde{\rho}}$-injective hull

$$0 \longrightarrow k_G \longrightarrow \Omega^{-1}(L_{\widetilde{\rho}}) \longrightarrow \Omega^3(k_G) \longrightarrow 0$$

gives us much information about the cohomology algebra. First we can deduce the following theorem from Theorem 3.3.

Theorem 5.12. *The element σ_6 is universally stable. Namely there exists an element $\widetilde{\sigma}_6 \in H^6(G,k)$ such that*

$$\operatorname{res}_S \widetilde{\sigma}_6 = \sigma_6.$$

Consequently the set

$$\{\rho_4, \ \sigma_6\}$$

is a homogeneous system of parameters for $H^*(G,k)$ for every G.

Second we can obtain dimension formulae for the cohomology groups $H^*(G,k)$. Applying the cohomology functor $\operatorname{Ext}_{kG}(-,k)$ to the extension

$$0 \longrightarrow k_G \longrightarrow \Omega^{-1}L_{\widetilde{\rho}} \longrightarrow \Omega^3 k_G \longrightarrow 0,$$

we obtain the short exact sequences

$$0 \longrightarrow \operatorname{Hom}_{kG}(\Omega^3 k, k) \longrightarrow \operatorname{Hom}_{kG}(\Omega^{-1}L_{\widetilde{\rho}_4}, k) \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Ext}^n_{kG}(k,k) \longrightarrow \operatorname{Ext}^{n+1}_{kG}(\Omega^3 k, k) \longrightarrow \operatorname{Ext}^{n+1}_{kG}(\Omega^{-1}L_{\widetilde{\rho}_4}, k) \longrightarrow 0, \quad n \geq 0.$$

In particular we have a formula

$$\dim \operatorname{Ext}^{n+4}_{kG}(k,k) = \dim \operatorname{Ext}^n_{kG}(k,k) + \dim \operatorname{Ext}^n_{kG}(L_{\widetilde{\rho}_4}, k)$$

and we can compute $\dim \mathrm{Ext}^n_{kG}(L_{\widetilde{\rho}_4}, k)$ by our construction of the module $\widetilde{X} = L_{\widetilde{\rho}}$. For example, if G is a QD-group, then

$$\dim \mathrm{Ext}^n_{kG}(L_{\widetilde{\rho}_4}, k) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \\ 1 & \text{if } n \equiv 1 \pmod 3 \\ 1 & \text{if } n \equiv 2 \pmod 3 \end{cases} .$$

We can also calculate $\dim \mathrm{Ext}^n_{kG}(k, k)$, $n = 1, 2, 3$, so that we obtain dimension formulae for $H^*(G, k)$.

We have obtained a system of parameters $\{\widetilde{\rho}_4, \widetilde{\sigma}_6\}$ and established dimension formulae for the cohomology groups $H^*(G, k)$. We have to get generators of the cohomology algebras over the subalgebra $k[\widetilde{\rho}_4, \widetilde{\sigma}_6]$.

5.5. **Generators of Cohomology Algebras.** First let us state generators of the cohomology algebra of the wreathed 2-group S. The cohomology algebra $H^*(S, k)$ has $\{\sigma_2 (= \tau_1^2 + \zeta_2), \rho_4\}$ as a system of parameters. Hence we can take generators of degree up to 4. In fact, $H^*(S, k)$ is generated over the subalgebra $k[\sigma_2, \rho_4]$ by τ_1, which was defined in Section 2, and the elements $\zeta_1, \nu_2, \zeta_3 \in H^*(S, k)$. To state these elements, let $\alpha_1 \in \inf^U H^1(U/\langle b \rangle, k)$; and let us define

$$\begin{aligned} \zeta_1 &= \mathrm{tr}_U^S \alpha_1 \in H^1(S, k), \\ \nu_2 &= \mathrm{norm}_U^S \alpha_1 \in H^2(S, k), \\ \zeta_3 &= \mathrm{tr}_U^S (\alpha_1 \alpha_2) \in H^3(S, k). \end{aligned}$$

When the four-groups E and F are not conjugate in G, the cohomology algebras $H^*(G, k)$ and $H^*(N_G(E), k)$ are isomorphic. This can be seen by comparing the dimensions of the cohomology groups. On the other hand, when E and F are conjugate, one can take an element $g_0 \in C_G(c)$ such that $E^{g_0} = F$ and $U^{g_0} \cap S = V$. Then we can determine the stable elements by considering the subspaces

$$\{\xi \in H^n(S, k) \mid \xi^{g_0}{}_V = \xi_V\}, \quad n \leq 4.$$

Of course the element g_0 above plays an important role throughout in our investigation for those groups in which E and F are conjugate.

References

[1] A. Adem and R. Milgram, *The mod 2 cohomology rings of rank 3 simple groups are Cohen-Macaulay*, Prospects in topology (Princeton) (F. Quinn, ed.), Ann. Math. Studies, vol. 138, Princeton Univ. Press, 1995, pp. 3–12.

[2] J. L. Alperin, R. Brauer, and D. Gorenstein, *Finite groups with quasi-dihedral and wreathed Sylow 2-subgroups*, Trans. Amer. Math. Soc. **151** (1970), 1–261.

[3] ———, *Finite simple groups of 2-rank two*, Scripta Math. **29** (1973), 191–214.

[4] T. Asai, *A dimension formula for the cohomology rings of finite groups with dihedral Sylow 2-subgroups*, Comm. Algebra **19** (1991), 3173–3190.

[5] T. Asai and H. Sasaki, *The mod 2 cohomology algebras of finite groups with dihedral Sylow 2-subgroups*, Comm. Algebra **21** (1993), 2771–2790.

[6] D. J. Benson, *Representations and cohomology II: Cohomology of groups and modules*, Cambridge studies in advanced mathematics, vol. 31, Cambridge University Press, Cambridge, 1991.
[7] D. J. Benson and J. F. Carlson, *Diagrammatic methods for the modular representations and cohomology*, Comm. Algebra **15** (1987), 53–121.
[8] R. Brauer, *Character theory of finite groups with wreathed Sylow 2-subgroups*, J. Algebra **19** (1971), 547–592.
[9] R. Brauer and W. J. Wong, *Some properties of finite groups with wreathed Sylow 2-subgroups*, J. Algebra **19** (1971), 263–273.
[10] J. F. Carlson, *Depth and transfer maps in the cohomology of groups*, Math. Z. **218** (1995), 461–468.
[11] ———, *Modules and group algebras*, Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel/Boston/Berlin, 1996.
[12] L. Evens, *The cohomology of groups*, Oxford Mathematics Monograph, Oxford University Press, New York, 1991.
[13] L. Evens and S. Priddy, *The ring of universally stable elements*, Quart. J. Math. Oxford (2) **40** (1989), 399–407.
[14] S. Fiedorowizc and S. Priddy, *Homology of classical groups over finite fields and their associated infinite loop spaces*, Springer Lecture Notes in Math., vol. 674, Springer-Verlag, Berlin/New York/Heidelberg, 1978.
[15] D. Gorenstein, *Finite groups*, Harper & Row, Publishers, New York, 1968.
[16] D. Gorenstein and J. H. Walter, *The characterization of finite groups with dihedral sylow 2-subgroups, I, II, III*, J. Algebra **2** (1965), 85–151, 218–270, 334–393.
[17] R. Knörr, *Relative projective covers*, Proc. Symp. Modular Representations of Finite Groups, Various Publication Series, vol. 29, Aarhus University, 1978, pp. 28–32.
[18] I. J. Leary, *The mod-p cohomology rings of some p-groups*, Math. Proc. Cambridge Philos. Soc. **112** (1992), no. 1, 63–75.
[19] R. Lyons, *A characterization of* $U_3(4)$, Trans. Amer. Math. Soc. **164** (1972), 371–387.
[20] J. Martino, *Stable splittings of the Sylow 2-subgroups of* $\mathrm{SL}_3(\mathbb{F}_q), q$ *odd*, Ph.D. thesis, Northwestern University, 1988.
[21] J. Martino and S. Priddy, *Classification of* BG *for groups with dihedral or quaternion Sylow 2-subgroups*, J. Pure Appl. Algebra **73** (1991), 13–21.
[22] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press, New York, London, 1989.
[23] T. Okuyama, *A generalization of projective covers of modules over group algebras*, unpublished manuscript.
[24] H. Sasaki, *The mod 2 cohomology algebras of finite groups with semidihedral Sylow 2-subgroups*, Comm. Algebra **55** (1994), 243–275.
[25] M. Tezuka and N. Yagita, *The mod p cohomology of* $\mathrm{GL}_3(\mathbf{F}_p)$, J. Algebra **81** (1983), 295–303.
[26] M. Tezuka and N. Yagita, *On odd prime components of cohomologies sporadic simple groups and the rings of universal stable elements*, J. Algebra **186** (1996), no. 2, 483–513.