2-radical subgroups of the Conway simple group $Co_1$ (Cohomology of Finite Groups and Related Topics)

Sawabe, Masato

数理解析研究所講究録 1998年 1057号 14-16

1998-08

http://hdl.handle.net/2433/62321

Departmental Bulletin Paper

Kyoto University
2-radical subgroups of the Conway simple group $Co_1$

Masato Sawabe

Department of Mathematics, Kumamoto University, Kumamoto 860-8555, Japan

1 Introduction

Let $G$ be a finite group and $p$ be an element of $\pi(G) = \{p : \text{prime} \mid p \text{ divides } |G|\}$. Put $\tilde{B}_p(G) = \{U : p\text{-subgroup } \subseteq G \mid O_p(N_G(U)) = U\}$ and $B_p(G) = \tilde{B}_p(G) - \{1\}$. An element of $B_p(G)$ is called a $p$-radical subgroup of $G$. $B_p(G)$ plays an important role in the various fields. For example, $\Delta(B_p(G))$ gives us a valuable information when we verify the Dade’s conjecture for $G$. Here $\Delta(B_p(G))$ is a simplicial complex whose vertex set is $B_p(G)$, and its simplex is each chain of elements of $B_p(G)$ with respect to natural inclusion in $B_p(G)$. $\Delta(B_p(G))$ is called the $p$-radical complex of $G$. Furthermore it is known that the alternating-sum decomposition of mod $p$ cohomology of $G$ is

$$\tilde{H}^n(G, \mathbb{Z}_p) = \sum_{\sigma \in \Delta(B_p(G))/G} (-1)^{\dim(\sigma)} \tilde{H}^n(G_\sigma, \mathbb{Z}_p),$$

where $n$ is any non-negative integer, $G_\sigma$ is the stabilizer of a simplex $\sigma$, and $\Delta(B_p(G))/G$ is a set of the representatives of $G$-orbits of $\Delta(B_p(G))$ (See [5]). Hence the calculation of a group cohomology reduces to the calculation of smaller groups. On the other hand, $\Delta(B_p(G))$ can be regarded as a geometry for $G$. Recently, for a sporadic simple groups $G$, $\Delta(B_p(G))$ is investigated in this direction very much, and it is closely connected with the essential $p$-local geometry for $G$. $\Delta(B_p(G))$ is determined by S. D. Smith, S. Yoshiara and et al. for some sporadic simple groups $G$ and $p \in \pi(G)$. The purpose of this note is to announce [3], namely determination of $B_2(Co_1)$ up to conjugacy, where $Co_1$ is the Conway simple group.

2 Known and new results about $p$-radical subgroups

The following lemma is one of the most basic results on $p$-radical subgroups.

**Lemma 1** ([4; Lemma1.10]) Let $G$ be a finite group and $p \in \pi(G)$. If $U \in B_p(G)$ with $N_G(U) \subseteq M$, where $M$ is a subgroup of $G$, then $O_p(M) \subseteq U$. In particular, If $O_p(M) \neq U$ then $U/O_p(M) \in B_p(M/O_p(M))$.

Lemma 1 implies that we can find $p$-radical subgroups inductively.
Corollary 1 Let $G$ be a finite simple group, $M$ be a maximal subgroup of $G$ and $p \in \pi(M)$. If $O_p(M) \neq 1$ then $B_p(M) = \{O_p(M), U \mid U/O_p(M) \in B_p(M/O_p(M))\}$.

Theorem 1 ([1]) Let $G$ be a group of Lie type over a field of characteristic $p$. Then $B_p(G) = \{O_p(U) \mid G \supseteq U = \text{parabolic subgroup}\}$.

Proposition 1 For $H$ and $K$ are finite groups and $p \in \pi(H \times K)$, $\tilde{B}_p(H \times K) = \{V \times K \mid V \in \tilde{B}_p(H), W \in \tilde{B}_p(K)\}$ holds.

Proposition 2 Let $A$ be a finite group with a normal subgroup $G$ of a prime index $p$. Then for any $U \in B_p(A)$, $U \cap G = \{1\}$ or $U \cap G \in B_p(G)$.

In this case we have $\{U \in B_p(A) \mid U \subseteq G\} \subseteq B_p(G)$. On the other hand, for $U \in B_p(A)$ with $U \not\subseteq G$, there exists an element $x \in G$ such that $U = (U \cap G)(x)$. We can easily see that $U_1 = U \cap G \in \tilde{B}_p(G)$ and $|U : U_1| = p$. Hence it suffices to determine $B_p(G)$ essentially.

Proposition 3 Let $G$ be a finite group of Lie type over a field of characteristic $p$, and $\sigma$ be a field automorphism of $G$ of order $p$. Then $\{U \in B_p(G(\sigma)) \mid U \subseteq G\} = B_p(G)$.

3 Application

We consider the case $G = Co_1$ and $p = 2$. Let $(\Lambda, q)$ be the Leech lattice, that is, $(\Lambda, q)$ is the 24-dimensional even unimodular lattice which has no vector $v$ with $q(v) = 2$. Let $\text{Aut}(\Lambda, q) := \{\sigma \in O(\mathbb{R}^{24}, q) \mid \Lambda^\sigma = \Lambda\}$. $\text{Aut}(\Lambda, q)$ is called the Conway group, which will be denoted $\cdot 0$. Its center $Z = Z(\cdot 0)$ is of order 2, and the factor group $Co_1 := \cdot 0/Z$ is a simple group, which is also called the Conway group. The following remark is straightforward from our definitions.

Remark 1 Let $G$ be a finite group and $p \in \pi(G)$. If $U \in B_p(G)$ with $N_G(U) \subseteq M$, where $M$ is a subgroup of $G$, then $U \in B_p(M)$.

The local subgroups of $Co_1$ have been classified by Curtis [2].

Theorem 2 ([2; Theorem 2.1]) For any elementary abelian 2-subgroup $E$ of $\cdot 0$, $N_{\cdot 0}(E)/Z$ is contained in a conjugate of one of the following seven groups.

\[L_1 = 2^{1+8} \cdot \Omega_8^+(2), \quad L_2 = 2^{2+12} \cdot (S_3 \times 3Sp_4(2)),\]
\[L_3 = 2^{2+12} :(S_3 \times L_4(2)), \quad L_4 = 2^{11} \cdot M_{24}, \quad L_5 = Co_2, \quad L_7 = (A_6 \times PSU_3(3)) : 2\]
\[L_6 = (A_4 \times G_2(4)) : 2,\]

Remark 1 and Theorem 2 imply $B_2(Co_1) \subseteq \{U^2 \mid g \in Co_1, U \in B_2(L_i) (1 \leq i \leq 7)\}$. We can determine $B_2(L_i)$ systematically by using the results in the previous section as follows.

$B_2(L_i) (1 \leq i \leq 5)$: It suffices to determine 2-radical subgroups of $\Omega_8^+(2)$, $S_3$, $3Sp_4(2)$, $L_4(2)$, $M_{24}$ and $Co_2$ by Corollary 1 and Proposition 1. We can find them from [4], [6] and Theorem 1.
$B_2(L_i)$ ($i = 6, 7$): Essentially it suffices to determine 2-radical subgroups of $A_4$, $A_6$, $G_2(4)$ and $PSU_3(3)$ by Propositions 1, 2 and 3. The cases $A_4$ and $A_6$ are straightforward. We can easily determine $B_2(G_2(4))$ and $B_2(PSU_3(3))$ by Theorem 1.

Now we find the candidates for $B_2(G)$, that is, we find $B_2(L_i)$ ($1 \leq i \leq 7$). Next we have to examine which element of $B_2(L_i)$ actually belongs to $B_2(G)$ for each $i$ ($1 \leq i \leq 7$). However when we examine we need detailed arguments. Then we have the following result.

$B_2(C_{0_1})$ consists of exactly 30 classes, and the representatives and the normalizers of them in $Co_1$ are as shown in Table 1, where $\{P_i\}_{1 \leq i \leq 15}$ and $\{N_i\}_{1 \leq i \leq 7}$ are the sets of representatives of $B_2(O^+_{8}(2))$ and $B_2(L_4(2))$ respectively.

<table>
<thead>
<tr>
<th>representative $T$</th>
<th>$N_{Co_1}(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R = ^{2+1}_{+}$</td>
<td>$R \cdot O^+_{8}(2)$</td>
</tr>
<tr>
<td>$R.P_i$ ($1 \leq i \leq 15$)</td>
<td>$R.N_{O^+_{8}(2)}(P_i)$</td>
</tr>
<tr>
<td>$E = ^{2+1}_{+}$</td>
<td>$E : M_{24}$</td>
</tr>
<tr>
<td>$Q = ^{2+1+12}_{+}$</td>
<td>$Q \cdot (S_3 \times 3S_6)$</td>
</tr>
<tr>
<td>$Q : S = 2^{4+12} : 2$</td>
<td>$Q \cdot (S \times 3S_6)$</td>
</tr>
<tr>
<td>$Q_1 = 2^{2+12}$</td>
<td>$Q_1 : (S_3 \times L_4(2))$</td>
</tr>
<tr>
<td>$Q_1 : N_i$ ($1 \leq i \leq 7$)</td>
<td>$Q_1 : (S_3 \times N_{L_4(2)}(N_i))$</td>
</tr>
<tr>
<td>$V = 2^2$</td>
<td>$(A_4 \times G_2(4)) : 2$</td>
</tr>
<tr>
<td>$V : \langle \sigma \rangle = 2^2 : 2$</td>
<td>$(V \times G_2(2)) : \langle \sigma \rangle$</td>
</tr>
<tr>
<td>$F = 2^2$</td>
<td>$(S_4 \times PSU_3(3)) : 2$</td>
</tr>
</tbody>
</table>

Remark. Let $G$ be a finite group and $p \in \pi(G)$. A $p$-subgroup chain $C : P_0 < P_1 < \cdots < P_n$ is called a radical $p$-chain of $G$ if it satisfies $P_0 = O_p(G)$ and $P_i = O_p(\cap_{j=0}^{i} N_G(P_j))$ for all $i$. We can easily determine all the radical 2-chains of $Co_1$ up to conjugacy by using Theorem 1, Proposition 1, [6] and the main result of this note.

References


