Degenerating Families of Meromorphic Functions on Compact Riemann Surfaces

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§0. Introduction. In this talk, I give a topological theory of degenerating families of meromorphic functions on compact Riemann surfaces. In [4], Matsumoto-Montesinos gave a topological theory of degenerating families of compact Riemann surfaces of genus $\geq 2$. My theory can be regarded as an analogy to their theory in some sense. But my theory starts from branched coverings and is very explicit. Moreover they assumed that the total spaces are non-singular, while I don't. So the results thus obtained are slightly different.

§1. Terminology. A (non-constant) meromorphic function on a compact Riemann surface $X$ of genus $g$ is nothing but a surjective holomorphic mapping

$$f: X \longrightarrow \mathbb{P}^1 = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$$
of $X$ onto the complex projective line $\mathbb{P}^1$. $f$ is a surjective proper finite mapping. So, $f$ can be regarded as a finite branched covering of $\mathbb{P}^1$. Put

$$R_f = \{ p \in X \mid f \text{ is not biholomorphic around } p \}, \quad B_f = f(R_f).$$

They are finite subsets of $X$ and $\mathbb{P}^1$, respectively and are called the ramification locus and branch locus of $f$, respectively.

$$f: X \rightarrow f^{-1}(B_f) \rightarrow \mathbb{P}^1 - B_f$$

is an unramified covering. Its mapping degree is denoted by $\deg f$ and is called the degree of $f$.

In general, for a given complex manifold $M$, a finite branched covering of $M$ is a finite proper holomorphic mapping

$$f: X \rightarrow M$$

of an irreducible normal complex space $X$ onto $M$. The ramification locus $R_f$, the branch locus $B_f$ and the degree of $f$, $\deg f$, are defined as above. $R_f$ and $B_f$ are hypersurfaces of $X$ and $M$, respectively. For a hypersurface $B$ of $M$, $f$ is said to branch at most at $B$ if $B_f$ is contained in $B$.

**Definition 1.** Branched coverings $f: X \rightarrow M$ and $f': X' \rightarrow M$ are isomorphic ($f \simeq f'$) if

$$\begin{array}{c}
X \xrightarrow{\exists \psi} X' \\
\downarrow f \quad \downarrow f'
\end{array}$$

where $\psi$ is a biholomorphic mapping.
**Definition 2.** Branched coverings \( f : X \to M \) and \( f' : X' \to M' \) are equivalent \((f \sim f')\) (resp. topologically equivalent \((f \sim f')^{\text{top}}\)) if \( X \xrightarrow{\psi} X' \), where \( \psi \) and \( \varphi \) are biholomorphic mappings (resp. orientation preserving homeomorphisms).


**Theorem 1 (Grauert-Remmert).** Let \( B \) be a hypersurface of a complex manifold \( M \) and \( f' : X' \to M - B \) be a finite unramified covering. Then there exists a unique (up to isomorphisms) finite covering \( f : X \to M \) which branches at most at \( B \) and is an extension of \( f' \).

This theorem asserts that the correspondence \( f' \leftrightarrow f \) gives a categorical equivalence between unbranched coverings of \( M - B \) and coverings of \( M \) branching at most at \( B \). So, we can apply some terminology of unbranched coverings to the case of branched coverings. For example, covering transformations, Galois coverings, abelian coverings, cyclic coverings, etc..

**Corollary.** There is a one to one correspondence between \( \{ f : X \to M \mid f \text{ is a finite covering branching at most at } B \} \sim \) and \( \{ \text{conjugacy classes of subgroups } H \text{ of finite index of the fundamental group } \pi_1(M - B, \delta_0) \} \).
§ 2. Monodromy representations and checked patterns.

Let \( f: X \rightarrow M \) be a finite covering of a complex manifold \( M \) of degree \( d \) branching at most at a hypersurface \( B \). The (permutation) monodromy representation

\[ \Phi_f: \pi_1(M-B, g_0) \rightarrow S_d \]  

(the \( d \)-th symmetric group) of \( f: X \setminus f^{-1}(B) \rightarrow M-B \) is called the monodromy representation of \( f \) (see Fig. 1).

\[ \Phi_f(\delta) = (1342) \]

Im \( \Phi_f \) is a transitive subgroup of \( S_d \), for \( X \setminus f^{-1}(B) \) is connected.

By Theorem 1 and its corollary

Theorem 2. For a given homomorphism \( \Phi: \pi_1(M-B, g_0) \rightarrow S_d \) such that Im \( \Phi \) is transitive, there exists a unique (up to isomorphisms) covering \( f: X \rightarrow M \) of degree \( d \) branching at most at \( B \) such that \( \Phi_f = \Phi \).

Problem of Realization. Construct concretely the
branched covering \( f: X \rightarrow M \) in the above theorem. This is a difficult problem, which is studied in number theory in the case

\[ M = \mathbb{P}^1 \] and \( B = \{0, 1, \infty\} \) (see Schneps [5]).

We consider the case \( M = \mathbb{P}^1 \) and construct \( f: X \rightarrow \mathbb{P}^1 \) topologically. The idea comes originally from Klein [8].

Let \( B = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \) be a set of \( n \) distinct points in \( \mathbb{P}^1 \). We draw a simple loop passing through every \( \gamma_j \) oriented clockwise as in Fig. 2.

![Fig. 2](image)

We regard the inside area as a continent, which contains the reference point \( \gamma_0 \), and the outside area as an ocean. We then pull back them over a covering \( f: X \rightarrow \mathbb{P}^1 \) of degree \( d \) branching at most at \( B \). Then we get a checked pattern of \( d \) continents and \( d \) oceans on \( X \) which is compatible with \( \Phi_f \).

Starting from \( \Phi: \pi_1(\mathbb{P}^1 - B, \gamma_0) \rightarrow \mathbb{S}_d \) such that \( \text{Im}\Phi \) is transitive, we do as follows: Put
\[ M_j = \Phi(r_j) \in S_d \quad \text{for } 1 \leq j \leq n, \]
where \( r_j \) are lassos as in Fig. 3.

Note that
\[ \pi_1(\mathbb{P}^1 - B, z_0) = \langle r_1, \ldots, r_n \mid r_n \cdots r_1 = 1 \rangle, \]
\[ M_n \cdots M_2 M_1 = 1 \in S_d. \]

Decompose each \( M_j \) into mutually prime cyclic permutations \( M_{j,k} \) whose length is \( e_{j,k} \). Put (by Riemann-Hurwitz formula)
\[ g = \frac{1}{2}\left\{ \sum_{j,k} (e_{j,k} - 1) - 2d \right\} + 1. \]
We prepare an oriented compact surface \( X \) of genus \( g \). We then draw a checked pattern of \( d \) continents and \( d \) oceans on \( X \) which is compatible with \( \Phi \). This pattern describes topologically \( f: X \to \mathbb{P}^1 \) such that \( \Phi_f = \Phi \).

**Example 1.** \( f: X \to \mathbb{P}^1, (z,w) \mapsto z \), where \( X \) is the Riemann surface of the algebraic function \( w = w(z) \) given by the equation \( w^3 - 3w - z = 0 \). (The genus of \( X \) is 0.)
Then \( n = 3, d = 3 \) and
\[ M_1 = \Phi_f(y_1) = (12), \quad M_2 = \Phi_f(y_2) = (13), \quad M_3 = \Phi_f(y_3) = (123). \]
See Fig. 4. The checked pattern in this case is as in Fig. 5.

(The points $j$ in Fig. 5 are in $f^{-1}(\mathcal{g}_j)$, while the number \(i\) in Fig. 5 denotes the $i$-th continent.)

Conversely we can read $\Phi_f$ from the checked pattern.

**Example 2.** $f: X \rightarrow \mathbb{P}^1$, $(z, w) \mapsto z$, where $X$ is given by the equation $27z^2w^2(w-1)^2-4(w^2-w+1)^3=0$. This is a Galois covering with $\text{Gal}(f) \cong S_3$. (The genus of $X$ is 0.) Then $n=3$, $d=6$ and

- $M_1 = \Phi_f(\gamma_1) = (154)(236)$,
- $M_2 = \Phi_f(\gamma_2) = (16)(24)(35)$,
- $M_3 = \Phi_f(\gamma_3) = (12)(34)(56)$.

See Fig. 6. The checked pattern in this case is as in Fig. 7.
Example 3. \( f: X \to \mathbb{P}^1, (z, w) \mapsto z \), where \( X \) is given by the equation \( w^3 - z^3 + 1 = 0 \). (The genus of \( X \) is 1.) Then \( n = 3 \), \( d = 3 \) and 
\[
M_j = \Phi_f(\delta_j) = (123) \quad \text{for } 1 \leq j \leq 3.
\]
See Fig. 8. The checked pattern in this case is in Fig. 9.

Example 4. \( f: X \to \mathbb{P}^1, (z, w) \mapsto z \), where \( X \) is given by the equation \( w^3 - z^2(z - 1)^2(z - 2) = 0 \). (The genus of \( X \) is 2.) Then \( n = 4 \), \( d = 3 \) and 
\[
M_1 = \Phi_f(\delta_1) = (132), \quad M_2 = \Phi_f(\delta_2) = (132), \\
M_3 = \Phi_f(\delta_3) = (123), \quad M_4 = \Phi_f(\delta_4) = (123).
\]
See Fig. 10. The checked pattern in this case is as in Fig. 11. (\( a_j \) and \( b_j \) for \( 1 \leq j \leq 2 \) form a symplectic homology basis.)
§3. Degenerating families of meromorphic functions.

Let $\Delta = \Delta(0, \varepsilon) = \{ t \in \mathbb{C} \mid |t| < \varepsilon \}$ be a disc and $\Delta^* = \Delta - \{ 0 \}$ be the punctured disc. A finite branched covering

$$f : X \longrightarrow \Delta \times \mathbb{P}^1$$

So far, we gave examples in which the equations of coverings are given. But as I noted above, even if only the monodromies $\Phi$ are given and the equation of the coverings $f$ are not given, then the topological picture (as Fig. 5 ~ Fig. 11) can be drawn.
is called a **degenerating family of meromorphic functions** (of compact Riemann surfaces) and is denoted by \( f = \{ f_t \} \) if the following 3 conditions are satisfied:

1. \( t \times \mathbb{P}^1 \not\subset B_f \) for every \( t \in \Delta \).

2. For every \( t \in \Delta^* \), \( t \times \mathbb{P}^1 \) meets at \( n \) points transversally with \( B_f \). (\( n \) is constant for \( t \in \Delta^* \)).

3. For every \( t \in \Delta^* \), \( f_t = f : X_t = f^{-1}(t \times \mathbb{P}^1) \rightarrow t \times \mathbb{P}^1 = \mathbb{P}^1 \) is a finite covering of degree \( d = \deg f \) branching at \( B_f \cap (t \times \mathbb{P}^1) = \{ \delta_1(t), \ldots, \delta_n(t) \} \).

See Fig. 12.

The center fiber \( X_0 = f^{-1}(0 \times \mathbb{P}^1) \) is a degeneration of \( X_t \).

**Example 5.** \( f = \{ f_t \}, \ f_t : X_t \rightarrow \mathbb{P}^1, (z, w) \mapsto z \),

where \( X_t : w^3 - 3tw - z = 0 \). \( M_1 = \Phi_t(\delta_1) = (12), \ M_2 = \Phi_t(\delta_2) = (13), \ M_3 = \Phi_t(\delta_3) = (123). \) (\( \Phi_t = \Phi_{f_t} \).)
See Fig. 13 and Fig. 14.

As $t \to 0$, the curves $1 \to 2$, whose initial points are 1 and end points are 2, converge to the point 1=2, and we get the following picture of $X_0$:

In fact the equation $X_t: w^3 - 3tw - z = 0$ converges to $X_0: w^3 - z = 0$ as $t \to 0$.

Example 6. $f = \{f_t\}$, $f_t: X_t \to \mathbb{P}^1$, $(z, w) \mapsto z$, where $X_t: w^2 - z(z-t)(z-1) = 0$. $M_j = \Phi_t(\delta_j) = (12)$ for $1 \leq j \leq 4$. See Fig. 16 and Fig. 17.
(In Fig. 17, ② is the upper backside of the torus.)

As \( t \to 0 \), the curves \( 1 \to 2 \) converges to the point \( 1 = 2 \), and we get the following picture of \( X_0 \):

![Diagram 1](image1)

(② is the upper backside.)

Fig. 18

**Example 7.** \( f = \{ f_t \}, f_t : X_t \to \mathbb{P}^1, (z, w) \mapsto z \), where \( X_t : w^2 - z^4 + t^4 = 0 \). \( M_j = \Phi_t(\delta_j) = (12) \) for \( 1 \leq j \leq 4 \). See Fig. 19 and Fig. 20.

![Diagram 2](image2)

(In Fig. 20, ② is the upper backside of the torus.)

As \( t \to 0 \), the curves \( 1 \to 2, 2 \to 3, 3 \to 4 \) converge to the point \( 1 = 2 = 3 = 4 \) and we get the following picture of \( X_0 \):

![Diagram 3](image3)
**Assertion.** Topologically, the degenerating curve \( X_0 = f^{-1}(0 \times \mathbb{R}^l) \) can be described by \( \Phi_t = \Phi_{f_t} \), where \( t \in \Delta^* \) is a fixed point.

We explain this assertion as follows: For a family \( \{f_t\} \), we assume for simplicity that
\[ g_i(0) = \cdots = g_k(0) \quad \text{and other } g_j(0) \quad (k+1 \leq j \leq n) \quad \text{are different from } g_i(0) \quad \text{and are mutually distinct.} \]

Put \( M_1 = \Phi_t(g_1), \cdots, M_n = \Phi_t(g_n) \). Let \( H \) be the subgroup of \( S_d \) which is generated by \( M_1, \cdots, M_k \). \( H \) may not be a transitive subgroup. We denote
\[ \Omega_1, \cdots, \Omega_N \]
the orbits of \( H \) on \( \{1, 2, \cdots, d\} \).

**Definition 3.** For a permutation \( A \in S_d \), if \( A \) is written as \( A = A_1 \cdots A_w \), the product of mutually prime cyclic
permutations, then we call the number $w = w(A)$ the weight of $A$. ($w(A)$ depends on $d$ also. For example, if $d = 4$ and $A = (123)$, then $w(A) = w((123)(4)) = 2$.) \]

Let $\chi(X_t)$ denote the Euler characteristic of $X_t$. Then we can easily show

**Theorem 3.** For $t \neq 0$,
\[
\begin{align*}
(1) \quad & \chi(X_t) = 2 - 2g = 2d - nd + \sum_{j=1}^{n} w(M_j), \\
(2) \quad & \chi(X_0) = 2d - (n-k+1)d + \nu + \sum_{j=k+1}^{n} w(M_j), \\
(3) \quad & \chi(X_0) - \chi(X_t) = d(k-1) + \nu - \sum_{j=1}^{k} w(M_j), \\
(4) \quad & \chi(X_0) \geq \chi(X_t). \]
\]

From Theorem 3 and some consideration, we get the following theorems:

**Theorem 4.**
\[
\begin{align*}
(1) \quad & f_0^{-1}(\Omega_t(0)) \text{ consists of } \nu+(n-k) \text{ points, among which } \nu \text{ points can be identified with } \Omega_1, \ldots, \Omega_{n}. \\
(2) \quad & \text{Put } M_0 = M_k \ldots M_1. \text{ Then } M_0 \text{ induces a permutation } M_0j : \Omega_j \mapsto \Omega_j. \text{ Under this notation, } X_0 \text{ has } w(M_0j) \text{ irreducible components at the point corresponding to } \Omega_j. \]
\]

**Theorem 5.** The following 4 conditions are mutually
equivalent:
(1) \( X_0 \) is homeomorphic to \( X_t \) for \( t \neq 0 \).
(2) \( \chi(X_0) = \chi(X_t) \) for \( t \neq 0 \).
(3) \( d(k-1) = \sum_{j=1}^{k} w(A_j) - v \).
(4) \( d(k-1) = \sum_{j=1}^{k} w(A_j) - w(A_0) \).

However, the topological structure of \( f = \{ f_t \} \) is not determined by \( \Phi_t \) alone. It depends also on the braid monodromy \( \Theta(\delta) \). Here

\[ \delta : u \mapsto t = t_0 e^{iu} \quad (0 \leq u \leq 2\pi) \]

is the loop around \( t = 0 \). (\( t_0 \in \Delta^x \) is a fixed point.)

We assume that

\[ g_j(t) \neq \infty \text{ for every } t \in \Delta \text{ and } 1 \leq j \leq n \].

Then \( \{ g_1(t_0 e^{iu}), \ldots, g_n(t_0 e^{iu}) \} \)_{0 \leq u \leq 2\pi}
gives an (Artin) braid of \( n \) strings, which is called the braid monodromy of the curve \( B_f \) around \( t = 0 \) and is denoted by \( \Theta(\delta) \).

The braid \( \Theta = \Theta(\delta) \) cannot be arbitrary. It is defined from a complex analytic curve \( B_f \). So, such a braid we call a complex analytic braid.

Now, for a degenerating family \( f = \{ f_t \} : X \to \Delta \times \mathbb{P} \),
we assume as above

\[(\Delta \times \{\infty\}) \cap B_f = \emptyset.\]

We fix a reference point \(t_0 \in \Delta^*\) and put

\[\delta_j = \delta_j(t_0) \quad \text{for} \quad 1 \leq j \leq n.\]

Then the Artin braid group \(B_n\) naturally acts on the fundamental group \(\pi_1(\mathbb{P}^1 - \{\delta_1, \ldots, \delta_n\}, \delta_0)\).

A theorem of Zariski-van Kampen (see e.g. Dimca [1]) asserts

\[\text{Theorem 6 (Zariski-van Kampen).} \quad \pi_1(\Delta \times \mathbb{P}^1 - B_f, \delta_0) = \langle \delta_1, \ldots, \delta_n \mid \delta_1 \ldots \delta_n = 1, \ \theta(\delta) \delta_j = \delta_j (1 \leq j \leq n) \rangle,\]

where \(\delta_j\) are lassos as in Fig. 3 for \(f_{t_0}: X_{t_0} \rightarrow \mathbb{P}^1\).

The monodromy representation \(\Phi_f\) of \(f: X \rightarrow \Delta \times \mathbb{P}^1\) is equal to \(\Phi_{t_0} = \Phi_{f_{t_0}}\). By Theorem 6, \(\Phi_{t_0}\) satisfies

\[\Phi_{t_0} \cdot \theta(\delta) = \Phi_{t_0}.\]

**Definition 4.** \(f = \{f_t\}\) and \(f' = \{f'_t\}\) are said to be topologically equivalent if

\[
\begin{array}{ccc}
X & \xrightarrow{\exists \psi} & X' \\
\downarrow f & & \downarrow f' \\
\Delta \times \mathbb{P}^1 & \xrightarrow{\exists \varphi} & \Delta' \times \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{\exists \eta} & \Delta'
\end{array}
\]
where $\psi, \varphi, \eta$ are orientation preserving homeomorphisms.

Using fundamental results in the theory of fiber bundles (see Steenrod [6]), we get the following theorem:

**Theorem 7.** There exists a one to one correspondence between \{ topological equivalence class of $f = \{ f_t \}$, where $f_{t_0}$ ($t \neq 0$) has the degree $n$ and has prescribed $n$ branch points \} and \{ $(\Phi, \theta)$ | $\Phi$ is the representation class of $\Phi : \pi_1(\mathbb{P}^1 - \{ \xi_1, \ldots, \xi_n, \xi_0 \}) \to S_d$ such that $\text{Im} \Phi$ is transitive, $\theta \in B_n$ a complex analytic braid such that $\Phi \cdot \theta = \Phi \} / B_n$.

Here $\sigma \in B_n$ acts on $(\Phi, \theta)$ as follows:

$$\sigma \cdot (\Phi, \theta) = (\Phi \cdot \sigma, \sigma^{-1} \theta \sigma).$$

Considering a trivial family, we get the following corollary which seems a known result.

**Corollary.** There exists a one to one correspondence between \{ topological equivalence class of $f : X \to \mathbb{P}^1$ of degree $d$ with prescribed $n$ branch points \} and \{ $\Phi$ | $\Phi$ is the representation class of $\Phi : \pi_1(\mathbb{P}^1 - \{ \xi_1, \ldots, \xi_n, \xi_0 \}) \to S_d$ such that $\text{Im} \Phi$ is transitive \} / $B_n$.

**References**

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