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Some Examples of the Penrose Transform

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Let $\mathbb{F}(\mathbb{C}^3)$ denote the space of <u>flags</u> in $\mathbb{C}^3$:

$$\left\{\begin{array}{l}(L,P) \text{ where } L \text{ is a 1-dimensional linear subspace of } \mathbb{C}^3,\\ \qquad\qquad\quad P \text{ is a 2-dimensional linear subspace of } \mathbb{C}^3,\\ \qquad\qquad\quad \text{and } L\subset P\end{array}\right\}.$$

Equip $\mathbb{C}^3$ with its standard inner product and define $\tau : \mathbb{F}(\mathbb{C}^3) \to \mathbb{CP}_2$ by

$$(L,P) \longmapsto L^\perp \cap P, \quad \text{the orthogonal complement of L in P.}$$

Though $\mathbb{F}(\mathbb{C}^3)$ and $\mathbb{CP}_2$ are complex manifolds and the fibres of τ in $\mathbb{F}(\mathbb{C}^3)$ are complex submanifolds, τ itself is not holomorphic. In fact, $\mathbb{F}(\mathbb{C}^3)$ is the <u>twistor space</u> of $\mathbb{CP}_2$ (as described, for example, in [2]). The <u>Penrose transform</u> interprets analytic cohomology on $\mathbb{F}(\mathbb{C}^3)$ in terms of differential equations on $\mathbb{CP}_2$. The aim of this lecture is to explain this transform and how it may be used to derive results concerning the integral geometry of geodesics in $\mathbb{CP}_2$ with respect to the Fubini-Study metric. This is joint work with Toby Bailey at the University of Edinburgh.

A Penrose transform may be constructed in the following circumstances. Suppose Z is a complex manifold, X is a smooth manifold, and $\tau : Z \to X$ is a smooth mapping of maximal rank with compact complex fibres. (In fact, Z need only had a formally integrable or <u>involutive</u> structure, namely a complex subbundle $T^{0,1}$ of its complexified tangent bundle, closed under Lie bracket. Again the fibres of τ would be required to be compact and we would insist that $T^{0,1}$ restrict to a complex structure in the usual sense on each of these fibres.) Roughly speaking, the construction is as follows. Suppose V is a holomorphic vector bundle on Z, and $\omega \in H^r(Z, \mathcal{O}(V))$. Then we may restrict may consider $\omega|_{\tau^{-1}(x)}$ as an element of the finite-dimensional vector space $H^r(\tau^{-1}(x), \mathcal{O}(V|_{\tau^{-1}(x)}))$. As $x \in X$ varies, these vector spaces

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typically define a smooth vector bundle $\tau_* V$ on X and $\omega|_{\tau^{-1}(x)}$ is then a smooth section of this bundle. This would be the Penrose transform $\mathcal{P}\omega$ of ω. Usually, $\mathcal{P}\omega$ is subject to certain differential equations as a result of arising in this way. The machinery for identifying these equations is as follows.

Let $\Lambda^{0,1}$ denote the vector bundle of $(0,1)$-forms on Z and $\Lambda^{0,1}_\tau$ the $(0,1)$-forms along the fibres of τ. The short exact sequence of vector bundles

$$0 \to B^1 \longrightarrow \Lambda^{0,1} \longrightarrow \Lambda^{0,1}_\tau \to 0$$

defines B^1. It induces a filtration of the Dolbeault complex $\Lambda^{0,\bullet}$ which the $\overline{\partial}$-operator respects. In particular, there is a differential operator

$$\overline{\partial}_\tau : B^1 \longrightarrow \Lambda^{0,1}_\tau \otimes B^1$$

which endows B^1 with a partially holomorphic structure—it is a holomorphic vector bundle along each fibres of τ. The exterior powers B^p of B^1 are also partially holomorphic and the spectral sequence of this filtered complex reads

$$E_1^{p,q} = \Gamma(X, \tau^q_* B^p) \Longrightarrow H^{p+q}(Z, \mathcal{O}).$$

If we suppose that the dimension of the Dolbeault cohomology along the fibres of τ

$$\frac{\ker \overline{\partial}_\tau : \Gamma(\tau^{-1}(x), \Lambda^{0,q}_\tau \otimes B^p) \longrightarrow \Gamma(\tau^{-1}(x), \Lambda^{0,q+1}_\tau \otimes B^p)}{\operatorname{im} \overline{\partial}_\tau : \Gamma(\tau^{-1}(x), \Lambda^{0,q-1}_\tau \otimes B^p) \longrightarrow \Gamma(\tau^{-1}(x), \Lambda^{0,q}_\tau \otimes B^p)}$$

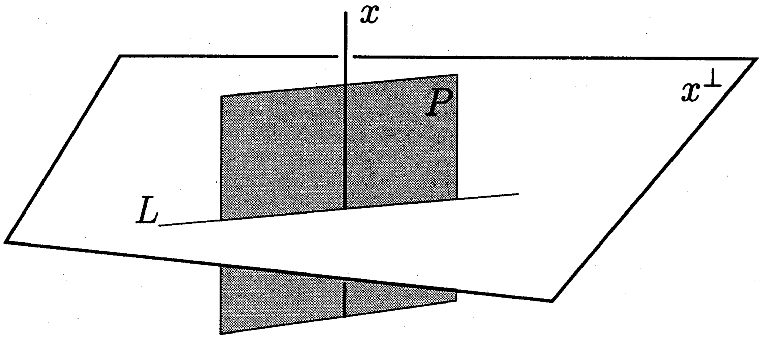
is independent of $x \in X$, then $\tau^q_* B^p$ is simply the smooth vector bundle on X with these spaces as fibres. As a minor variation on this theme, if V is a holomorphic vector bundle on Z, then the bundles $B^p \otimes V$ are partially holomorphic and there is a spectral sequence

$$E_1^{p,q} = \Gamma(X, \tau^q_*(B^p \otimes V)) \Longrightarrow H^{p+q}(Z, \mathcal{O}(V)).$$

The fibres of $\tau : \mathbb{F}(\mathbb{C}^3) \to \mathbb{CP}_2$ are easily identified. Over $x \in \mathbb{CP}_2$, the fibre is

$$\left\{ \begin{array}{r} (L,P) \in \mathbb{F}(\mathbb{C}^3) \text{ s.t. } L \subset x^\perp \text{ or,} \\ \text{equivalently, } P \supset x \end{array} \right\}$$

as in the following diagram.



Either as $\mathbb{P}(x^\perp)$ or $\mathbb{P}(\mathbb{C}^3/x)$, this is just $\mathbb{CP}_1$ whose cohomology is easily computed. In fact, the fibration $\tau : \mathbb{F}(\mathbb{C}^3) \to \mathbb{CP}_2$ is homogeneous under the obvious action of U(3) and all the bundles we will be concerned with are homogeneous under this action. This reduces the computation of these cohomologies to an elementary exercise in representation theory (applying the Bott-Borel-Weil theorem).

It would take us too far astray to describe the general computations. Instead, there follow some typical results, suppressing all details of their derivation. If V is taken to be the canonical bundle Ω^3 on $\mathbb{F}(\mathbb{C}^3)$, then the E_1-level of spectral sequence reads

$$\begin{array}{ccccccc}
| & & & & & & \\
\Gamma(\mathbb{CP}_2, \Lambda^{1,1}_\perp) & \to & \Gamma(\mathbb{CP}_2, \Lambda^3) & \to & \Gamma(\mathbb{CP}_2, \Lambda^4) & & 0 \\
| & & & & & & \\
0 & \text{———} & 0 & \text{———} & 0 & \text{———} & 0 \text{ ——}
\end{array}$$

where Λ^p is the bundle of complex-valued p-forms and $\Lambda^{1,1}_\perp$ the (1,1)-forms orthogonal to the Kähler form. Exterior differentiation provides the differentials of this spectral sequence. These computations hold over any open subset X of $\mathbb{CP}_2$ and, in particular, the Penrose transform gives an isomorphism

$$\mathcal{P} : H^1(\tau^{-1}(X), \Omega^3) \xrightarrow{\cong} \{\omega \in \Gamma(X, \Lambda^{1,1}_\perp) \text{ s.t. } d\omega = 0\}.$$

This follows our earlier rough description with the transform itself obtained simply by restriction to the fibres $\tau^{-1}(x)$ as $x \in X$ varies. Though the global isomorphism

$$\mathcal{P} : H^1(\mathbb{F}(\mathbb{C}^3), \Omega^3) \xrightarrow{\cong} \{\omega \in \Gamma(\mathbb{CP}_2, \Lambda^{1,1}_\perp) \text{ s.t. } d\omega = 0\}$$

is valid, both sides vanish, the left hand side by the Bott-Borel-Weil theorem and the right hand side by Hodge theory.

If V is the trivial bundle, the Penrose transform is not simply obtained by restriction to the fibres (since, $H^1(\mathbb{CP}_1, \mathcal{O}) = 0$). Nevertheless, the spectral sequence is still valid. It reads

$$\begin{array}{llll} | & & & \\ 0 & 0 & 0 & 0 \\ | & & & \\ \Gamma(\mathbb{CP}_2, \Lambda^0) \text{——} & \Gamma(\mathbb{CP}_2, \Lambda^1) \text{——} & \Gamma(\mathbb{CP}_2, \Lambda^{1,1}_\perp) \text{——} & 0 \text{——} \end{array}$$

The resulting isomorphism

$$\mathcal{P} : H^1(\mathbb{F}(\mathbb{C}^3), \mathcal{O}) \xrightarrow{\cong} \frac{\ker d^{1,1}_\perp : \Gamma(\mathbb{CP}_2, \Lambda^1) \to \Gamma(\mathbb{CP}_2, \Lambda^{1,1}_\perp)}{\operatorname{im} d : \Gamma(\mathbb{CP}_2, \Lambda^0) \to \Gamma(\mathbb{CP}_2, \Lambda^1)}$$

is, again, simply a confirmation that both sides vanish. (A non-linear version of this isomorphism was used by Buchdahl [2] to classify the <u>instantons</u> on $\mathbb{CP}_2$.)

A less trivial example is obtained by taking V to be Θ, the holomorphic tangent bundle. In this case, the spectral sequence, after some preliminary cancellation, reads

$$\begin{array}{llll} | & & & \\ 0 & 0 & 0 & 0 \\ | & & & \\ \Gamma(\mathbb{CP}_2, \Lambda^1) \text{——} & \Gamma(\mathbb{CP}_2, \bigodot^2_\circ \Lambda^1) \text{——} & \Gamma(\mathbb{CP}_2, \boxplus^{2,2}_{\circ\perp} \Lambda^1) \text{——} & 0 \text{——} \end{array}$$

which needs some further explanation as follows. The bundle $\bigodot^2_\circ \Lambda^1$ is the symmetric trace-free two-tensors and $\boxplus \Lambda^1$ is the bundle of tensors with Riemann symmetries, $\boxplus_\circ \Lambda^1$ the totally trace-free subbundle, $\boxplus^{2,2}_\circ \Lambda^1$ those of type $(2,2)$, and, finally, $\boxplus^{2,2}_{\circ\perp} \Lambda^1$ the (irreducible five-dimensional) subbundle orthogonal to the Kähler form. The differentials are

$$\omega_a \longmapsto \text{the trace-free symmetric part of } \nabla_a \omega_b$$

and

$$\theta_{ab} \longmapsto \text{the } \boxplus^{2,2}_{\circ\perp} \text{ part of } \nabla_a \nabla_b \theta_{cd}$$

where ∇_a is the Levi-Civita connection for the Fubini-Study metric. Though Θ is not an irreducible homogeneous bundle on $\mathbb{F}(\mathbb{C}^3)$, the Bott-Borel-Weil theorem applies to its irreducible subquotients and $H^1(\mathbb{F}(\mathbb{C}^3), \Theta)$ is easily shown to vanish. We have proved the following:

Theorem 1 *Suppose θ_{ab} is a smooth symmetric two-tensor on $\mathbb{CP}_2$ whose $\boxplus^{2,2}_{\circ\perp}$ part of $\nabla_a\nabla_b\theta_{cd}$ vanishes. Then, there is a smooth one-form ω_a on $\mathbb{CP}_2$ such that*

$$\theta_{ab} = \nabla_{(a}\omega_{b)} + \psi g_{ab},$$

for some smooth one-form ω_a and function ψ. The round brackets here mean to take the symmetric part and g_{ab} denotes the Fubini-Study metric.

This is part of a series of results which have consequences in integral geometry, as follows. The X-ray transform on $\mathbb{RP}_n$ is obtained by integrating a smooth function over geodesics on $\mathbb{RP}_n$ to obtain a function on the space of geodesics. (The metric on $\mathbb{RP}_n$ is the standard one in which all geodesics are closed.) It is well-known that this X-ray transform is injective. The standard embedding $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ is totally geodesic and, under the action of $\mathrm{U}(n+1)$, a large family of totally geodesically embedded $\mathbb{RP}_n$'s are obtained. In particular, every geodesic on $\mathbb{CP}_n$ lies on one of these $\mathbb{RP}_n$'s. It is immediate that the X-ray transform is injective on $\mathbb{CP}_n$.

A smooth one-form θ on $\mathbb{RP}_n$ has zero energy if its integral over every geodesic vanishes. Clearly, if θ is exact, then it has zero energy. In [6], Michel proved the converse. In [4], Gasqui and Goldschmidt established the corresponding result for $\mathbb{CP}_n$. They reason from $\mathbb{RP}_n$ much as follows. If θ is a zero-energy one-form on $\mathbb{CP}_n$, then it is zero-energy on each totally geodesic $\mathbb{RP}_n$, therefore exact and, therefore, closed. This is a strong constraint on $d\theta$, namely that its restriction to every such $\mathbb{RP}_n$ vanish. It is a straightforward matter of algebra to check that this constraint is precisely that $d\theta$ be a smooth multiple of κ, the Kähler form. Then

$$*d\theta = d\theta \wedge \underbrace{\kappa \wedge \cdots \wedge \kappa}_{n-2} \quad \text{so} \quad d{*}d\theta = 0$$

and an integration by parts

$$\|d\theta\|^2 = \int_{\mathbb{CP}_n} d\theta \wedge *d\theta = \int_{\mathbb{CP}_n} \theta \wedge d{*}d\theta = 0$$

shows that θ is closed and, hence, exact. (Instead, Gasqui and Goldschmidt use representation theory to decompose the relevant function spaces.) It is interesting to note that, for $\mathbb{CP}_2$, the Penrose transform together with the vanishing of $H^1(\mathbb{F}(\mathbb{C}^3), \mathcal{O})$ provides an alternative way of finishing the argument.

In a similar way, Theorem 1 may be used in proving the infinitesimal Blaschke rigidity of $\mathbb{CP}_2$. This result, due to Tsukamoto [7], says that if a smooth symmetric two-tensor θ_{ab} on $\mathbb{CP}_2$ has zero energy, then it is of the form $\nabla_{(a}\omega_{b)}$ for some smooth one-form ω_a. (His proof is involves a detailed representation theoretic analysis of the relevant function spaces.) In view of Theorem 1, it suffices to prove that

$$\boxplus^{2,2}_{\circ\perp}(\nabla_a\nabla_b\theta_{cd}) = 0$$

for then, $\theta_{ab} = \nabla_{(a}\omega_{b)} + \psi g_{ab}$ and taking the energy of both sides shows that $\psi = 0$. The infinitesimal Blaschke rigidity of $\mathbb{RP}_n$ is due to Michel [5] (or, see [1] for a Penrose transform proof). It follows that a zero-energy θ_{ab} on $\mathbb{CP}_n$ is subject to differential constraints when restricted to any totally geodesic $\mathbb{RP}_2$. These turn out to be second order constraints, part of a resolution due to Calabi [3] (a special case of the Bernstein-Gelfand-Gelfand resolution). They are of the form

$$\boxplus(\nabla_a\nabla_b\theta_{cd} + Cg_{ab}\theta_{cd})$$

for a suitable constant C where ∇_a is the Levi-Civita connection on $\mathbb{RP}_2$ for g_{ab}, the standard metric. We may define a differential operator on $\mathbb{CP}_2$ by exactly the same formula but using the Fubini-Study metric and connection. We conclude that the resulting tensor vanishes upon restriction to any totally geodesic $\mathbb{RP}_2$. This places further algebraic restrictions on a tensor which, in the first instance, has Riemann symmetries. It turns out that the only tensors to have this property are obtained by applying the Young symmetrizer $\boxplus$ to $\kappa \otimes \lambda$ where κ is the Kähler form and λ an arbitrary two-form. Explicitly,

$$\kappa_{ab}\lambda_{cd} \stackrel{\boxplus}{\longmapsto} \tfrac{1}{2}\kappa_{ab}\lambda_{cd} + \tfrac{1}{2}\kappa_{cd}\lambda_{ab} + \tfrac{1}{4}\kappa_{ac}\lambda_{bd} + \tfrac{1}{4}\kappa_{bd}\lambda_{ac} - \tfrac{1}{4}\kappa_{bc}\lambda_{ad} - \tfrac{1}{4}\kappa_{ad}\lambda_{bc}.$$

This is more than enough to ensure that the $\boxplus^{2,2}_{\circ\perp}$ piece of $\nabla_a\nabla_b\theta_{cd} + Cg_{ab}\theta_{cd}$ vanishes. However, trace-removal eliminates the $Cg_{ab}\theta_{cd}$ term and we are left with $\boxplus^{2,2}_{\circ\perp}(\nabla_a\nabla_b\theta_{cd}) = 0$, as required.

More generally, it is shown in [1] that a symmetric $(k+1)$-form θ on $\mathbb{RP}_n$ is the symmetrized covariant derivative of a symmetric k-form if and only if it has zero energy. The Penrose transform and the vanishing of $H^1(\mathbb{F}(\mathbb{C}^3), \bigodot^k \Theta)$ then give a proof of the corresponding result for $\mathbb{CP}_2$. As noted in [4] or [7], this is sufficient to obtain the result on $\mathbb{CP}_n$ and also for quaternionic projective spaces and the Cayley plane.

References

[1] T.N. Bailey and M.G. Eastwood, *Zero-energy fields on real projective space*, Geom. Dedicata, to appear♯.

[2] N.P. Buchdahl, *Instantons on* **CP₂**, Jour. Diff. Geom. **24** (1986), 19–52.

[3] E. Calabi, *On compact Riemannian manifolds with constant curvature*, Differential Geometry, Proc. Symp. Pure Math., vol. III, pp. 155–180, Amer. Math. Soc., 1961.

[4] J. Gasqui and H. Goldschmidt, *Une caractérisation des fomes exactes de degré 1 sur les epsaces projectifs*, Comment. Math. Helv. **60** (1985), 46–53.

[5] R. Michel, *Problèmes d'analyse géométriques liés à la conjecture de Blaschke*, Bull. Soc. Math. France **101** (1973), 17–69.

[6] R. Michel, *Sur quelques problèmes de géométrie globale des géodésiques*, Bol. Soc. Bras. Mat. **9** (1978), 19–38.

[7] C. Tsukamoto *Infinitesimal Blaschke conjectures on projective spaces*, Ann. Scient. Éc. Norm. Sup. **14** (1981), 339–356.

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