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The Cauchy problem for nonlinear wave equations in the homogeneous Sobolev space

M.Nakamura 中村 誠*

Department of Mathematics
Hokkaido University
Sapporo 060, Japan

1 Introduction

In this note I describe some recent work on nonlinear wave equations, done jointly with T.Ozawa (Hokkaido university). We study the Cauchy problem for nonlinear wave equations of the form

$$\partial_t^2 u - \Delta u = f(u) \tag{1.1}$$

in the homogeneous Sobolev space $\dot{H}^{\mu}(\mathbf{R}^n)$ with $n \geq 2$ and $0 \leq \mu < n/2$, where Δ denotes the Laplacian in $\mathbf{R}^n$ and the typical form of $f(u)$ is the single power interaction $\lambda|u|^{p-1}u$ with $\lambda \in \mathbf{R}$ and $1 < p < \infty$. As usually done, with data $u(0) = \phi, \partial_t u(0) = \psi$ we regard (1.1) as the following integral equation.

$$u(t) = \Phi(u(t)) \equiv \dot{K}(t)\phi + K(t)\psi + \int_0^t K(t-\tau)f(u(\tau))d\tau, \tag{1.2}$$

where $\dot{K}(t) = \cos t\sqrt{-\Delta}$, $K(t) = (\sin t\sqrt{-\Delta})/\sqrt{-\Delta}$.

There are many papers on the Cauchy problem for (1.1) and large time behavior of global solutions, see [2, 4, 5, 7-15, 17-23]. Recently, in [15] Lindblad and Sogge studied (1.1) in the Sobolev space with minimal regularity assumptions on the data. One of the key ingredients in [15] is generalized Strichartz estimates on the free wave equation. Those estimates are

*JSPS Fellow

described exclusively in terms of the homogeneous Sobolev space, and accordingly, the associated estimates on the nonlinear term are required to take a form in the framework of the homogeneous Sobolev spaces.

Unfortunately, however, when it comes to the Leibniz rule for fractional derivatives, it sometimes happens that additional regularity assumptions on f would be necessary more than one needed.

Meanwhile, we have recently found that the problem could be efficiently dealt in the framework of the homogeneous Besov spaces [16], see also [3, 7]. Moreover, the Strichartz estimate are now available in the fully extended version, especially in the homogeneous Besov setting [10].

The purpose of this paper is to reexamine the results of [15] on the Cauchy problem for (1.1) in the homogeneous Sobolev spaces by means of a number of sharp estimates described in terms of the homogeneous Besov spaces. As a result of the homogeneous Besov technique, we have refined and generalized the previous results in some directions. To state our theorem, we make a series of definitions.

Definition 1.1 For $s \geq -1$ and $p \geq 1$, we define a class of functions $G(s,p)$ in $C(\mathbf{C},\mathbf{C})$ as following. We say $f \in G(s,p)$ if f satisfies either of the following conditions

1. For some nonnegative integers a, b with $p = a+b$, $f(z) = C_1 + C_2 z^a \bar{z}^b$, where C_1 and C_2 are constants and C_1 is disregarded if $s \leq 0$.

2. $[s]+1 < p$, $f \in C^{[s]+1}(\mathbf{C},\mathbf{C})$. $f(0) = \cdots = f^{([s]+1)}(0) = 0$, where $f(0) = 0$ may be disregarded if $s > 0$ or p satisfies $[s]+2 \leq p$. Moreover, f satisfies the estimates for all $z, w \in \mathbf{C}$

$$\begin{aligned} &|f^{([s]+1)}(z) - f^{([s]+1)}(w)| \\ &\leq \begin{cases} C(|z|^{p-[s]-2} + |w|^{p-[s]-2})|z-w| & \text{if} \quad [s]+2 \leq p, \\ C|z-w|^{p-[s]-1} & \text{if} \quad [s]+1 < p < [s]+2, \end{cases} \end{aligned} \tag{1.3}$$

   where $[s]$ denotes the largest integer less than or equal to s, but $[0] = -1$. We call s the first index of $G(s,p)$.

Definition 1.2 Let $\epsilon > 0$. Let Ω_ϵ be

$$\Omega_\epsilon \equiv \{(1/q, 1/r) \mid 0 \leq 1/q, 1/r \leq 1/2, \quad \epsilon \leq 1/r \leq 1/2 - 2/((n-1)q), \\ (1/q, 1/r) \notin B_\epsilon(1/2, 1/2 - 1/(n-1))\},$$

where $B_\epsilon(1/2, 1/2-1/(n-1))$ denotes an open ball with radius ϵ and center at $(1/2, 1/2-1/(n-1))$. Let $0 \le \mu < n/2$. Let $\Omega_{\epsilon,\mu}$ be

$$\begin{aligned}\Omega_{\epsilon,\mu} \equiv \{(1/q, 1/r, \rho) \quad | \quad & (1/q, 1/r) \in \Omega_\epsilon, 0 \le \rho \le \mu, \\ & \mu = \rho + n(1/2 - 1/r) - 1/q, 0 \le 1/q \le n/2 - \mu - n\epsilon\}.\end{aligned}$$

Definition 1.3 For any $-\infty \le a \le 0 \le b \le \infty$, we define an interval $I \equiv [a,b] \cap \mathbf{R}$ with length $|a-b|$ and for $R > 0$ a function space $X_\epsilon(I,R)$ with metric d by

$$\begin{aligned} X_\epsilon(I,R) &\equiv \{u \in \cap_{(1/q,1/r,\rho)\in\Omega_{\epsilon,\mu}} L^q(I, \dot{B}^\rho_r) \quad | \\ &\qquad \max_{(1/q,1/r,\rho)\in\Omega_{\epsilon,\mu}} \|u; L^q(I, \dot{B}^\rho_r)\| \le R\}, \\ d(u,v) &\equiv \max_{(1/q,1/r,\rho)\in\Omega_{\epsilon,\mu}} \|u - v; L^q(I, \dot{B}^\rho_r)\|. \end{aligned}$$

In our theorem below, $\|(\phi,\psi)\|_\mu$ denotes $\max(\|\phi; \dot{H}^\mu\|, \|\psi; \dot{H}^{\mu-1}\|)$, α denotes the lower root of the quadratic equation

$$F(x) \equiv x^2 - ((n^2-3)/(2n-2))x + (n^2+n+4)/(4n-4) = 0. \qquad (1.4)$$

It follows that $\min(1,\alpha) = 1$ for $n \le 6$ and $(n+1)/(2n-2) < \alpha < 1$ for $n \ge 7$. Finally, $\beta(\mu)$ is given by

$$\beta(\mu) \equiv \frac{n^2+n+4-2(n-3)\mu}{2(n-1)(n-2\mu)}.$$

It follows that $\beta(\alpha) = \alpha$, $\beta((n-4)/2) = 1$ and that $\beta(\mu)$ is a strictly increasing function in μ.

Theorem 1.1 *Let $n \ge 2$, $0 \le \mu < n/2$ and $2/(n-2\mu) \le p-1$. Let n, f, p satisfy any of the following conditions.*

(*A1*) $n = 2$, $f \in G(0,p)$ *and*

$$p - 1 \le \begin{cases} (2(n+1)/n + 4\mu)/(n+1-4\mu) & \text{for } 0 \le \mu \le 1/4, \\ 4/(n+1-4\mu) & \text{for } 1/4 < \mu < 1/2, \\ 4/(n-2\mu) & \text{for } 1/2 \le \mu < n/2. \end{cases}$$

(*A2*) $n \ge 3$, $0 \le \mu \le \min(1,\alpha)$, $f \in G(0,p)$ *and*

$$\begin{aligned} p-1 &\le (3(n+1) + 2(n-1)\mu)/(n^2+n-4n\mu) \\ &\qquad\qquad \text{for } 0 \le \mu < (n-3)/(2n-2), \\ p-1 &< 4/(n+1-4\mu) \quad \text{for } \mu = (n-3)/(2n-2), \end{aligned}$$

$$p - 1 \le \begin{cases} 4/(n+1-4\mu) & \text{for } (n-3)/(2n-2) < \mu < 1/2, \\ 4/(n-2\mu) & \text{for } 1/2 \le \mu \le \min(1,\alpha). \end{cases}$$

(A3) $n \geq 7$, $\alpha < \mu < (n-4)/2$, $f \in G(\mu - \beta(\mu), p)$ *and* $p - 1 \leq 4/(n-2\mu)$.

(A4) $n \geq 3$, $\max(1, (n-4)/2) \leq \mu < n/2$, $f \in G(\mu - 1, p)$ *and* $p - 1 \leq 4/(n-2\mu)$.

Let $\epsilon > 0$ be sufficiently small. Then for any data $(\phi, \psi) \in \dot{H}^{\mu} \times \dot{H}^{\mu-1}$ there exists a unique local solution of (1.2) in $X_{\epsilon}(I, R)$ with $|I| > 0$ sufficiently small and R sufficiently large. Moreover if $p - 1 = 4/(n-2\mu)$ and $\|(\phi, \psi)\|_{\mu}$ is sufficiently small, there exists a unique global solution in $X_{\epsilon}((-\infty, \infty), R)$ with R sufficiently small.

On the solutions given by above, we have the following results:

(1) $(u, \partial_t u)$ is continuous in time with respect to the norm $\dot{H}^{\mu} \times \dot{H}^{\mu-1}$.

(2) The solution u depends on the data (ϕ, ψ) continuously. Namely let v be the solution of (1.2) with data (ϕ_0, ψ_0) such that $\|(\phi - \phi_0, \psi - \psi_0)\|_{\mu}$ tends to zero, then $d(u, v) \to 0$ for $p \notin J$, $v \to u$ in $\mathcal{D}'(\mathbf{R}^{n+1})$ for $p \in J$, where $\mathcal{D}'(\mathbf{R}^{n+1})$ denotes the space of distribution and J denotes an interval defined only for (A3) and (A4) as $J \equiv ([\mu - \beta(\mu)] + 1, [\mu - \beta(\mu)] + 2)$ for (A3), $J \equiv ([\mu], [\mu] + 1)$ for (A4).

(3) Let $p - 1 = 4/(n-2\mu)$. There exists a pair (ϕ_+, ψ_+) in $\dot{H}^{\mu} \times \dot{H}^{\mu-1}$ such that

$$\|u(t) - \dot{K}(t)\phi_+ - K(t)\psi_+; \dot{H}^{\mu}\| \to 0 \quad as \quad t \to \infty.$$

(4) Let $p - 1 = 4/(n-2\mu)$. Let $\gamma > 0$ be sufficiently small. Then for any data (ϕ_-, ψ_-) which satisfies $\|(\phi_-, \psi_-)\|_{\mu} < \gamma$, there exists a global solution u and a pair (ϕ_+, ψ_+) in $\dot{H}^{\mu} \times \dot{H}^{\mu-1}$ such that

$$\|u(t) - \dot{K}(t)\phi_{\pm} - K(t)\psi_{\pm}; \dot{H}^{\mu}\| \to 0 \quad as \quad t \to \pm\infty.$$

Moreover if $p \notin J$, then the map $(\phi_-, \psi_-) \mapsto (\phi_+, \psi_+)$ is continuous in $\dot{H}^{\mu} \times \dot{H}^{\mu-1}$.

Remark 1. By dilation argument, it is natural to call $p = 1 + 4/(n-2\mu)$ the critical exponent for the well-posedness of the Cauchy problem for (1.2) in $\dot{H}^{\mu} \times \dot{H}^{\mu-1}$. On the other hand, H.Lindblad and C.D.Sogge ([14, 15]) showed the ill-posedness in the following three cases: (a) $p > 1 + 4/(n + 1 - 4\mu)$ with $n = 2$ and $1/4 < \mu \leq 1/2$. (b) $p > 1 + 4/(n + 1 - 4\mu)$ with $n \geq 3$ and $(n-3)/(2n-2) < \mu \leq 1/2$. (c) $p = 2$ with $n = 3$ and $\mu = 0$.

Remark 2. We use the homogeneous Besov space for the linear and nonlinear estimates for (1.2), by which it becomes easy to deal with the fractional derivative of nonlinear term (see Propositions 2.1 and 2.2). For the definition of the homogeneous Besov space and its properties, we refer to [1, 8, 10, 24]. Our results for the local and global solvability of (1.2) in the homogeneous Sobolev space $\dot{H}^\mu$ with $3/2 < \mu < n/2$ and the corresponding results on scattering are new.

2 Estimates for nonlinear terms

Proposition 2.1 *Let $s > 0$, $1 \leq p$ and $f \in G(s,p)$. Let $1 \leq \ell < \infty, 2 \leq q < \infty, 2 \leq r \leq \infty$ with $1/\ell = (p-1)/q + 1/r$. Then*

$$\|f(u); \dot{B}^s_\ell\| \leq C\|u; \dot{B}^0_q\|^{p-1}\|u; \dot{B}^s_r\|, \tag{2.5}$$

$$\begin{aligned} &\|f(u) - f(v); \dot{B}^s_\ell\| \\ \leq\ & C\max(\|u; \dot{B}^0_q\|, \|v; \dot{B}^0_q\|)^{p-1}\|u-v; \dot{B}^s_r\| \\ &+C\max(\|u; \dot{B}^0_q\|, \|v; \dot{B}^0_q\|)^{p-2}\|u-v; \dot{B}^0_q\| \max(\|u; \dot{B}^s_r\|, \|v; \dot{B}^s_r\|) \\ &+C\max(\|u; \dot{B}^0_q\|, \|v; \dot{B}^0_q\|)^{[s]}\|u-v; \dot{B}^0_q\|^{p-[s]-1} \max(\|u; \dot{B}^s_r\|, \|v; \dot{B}^s_r\|), \end{aligned}$$

where the second and third terms on the right hand side of the last inequality are disregarded for $p < 2$ and $p \notin ([s]+1, [s]+2)$ respectively.

Proof) We have already shown the first inequality in [16]. The second inequality would be proved analogously and we omit the proof. □

For the proof of the next proposition, we describe fundamental relations between $1/q$ and $1/r$ with $(1/q, 1/r) \in \cup_{\epsilon>0}\Omega_\epsilon$.

Lemma 2.1 *Let $\mu, \rho \in \mathbf{R}$. Let $1/q, 1/r$ satisfy $\mu = \rho - n(1/2 - 1/r) - 1/q$. If ρ, q satisfy any of the following conditions, then the above $1/q, 1/r$ satisfy $(1/q, 1/r) \in \cup_{\epsilon>0}\Omega_\epsilon$.*

(1) $n = 2$, $0 \leq 1/q < n/2 - (\mu - \rho)$ for $\mu - 1 < p \leq \mu - 3/4$, $0 \leq 1/q \leq (n-1)(\mu - \rho)/(n+1)$ for $\mu - 3/4 < p \leq \mu$.

(2) $n \geq 3$, $0 \leq 1/q < 1/2$ for $\rho = \mu - (n-1)/2$, $0 \leq 1/q \leq 1/2$ for $\mu - (n-1)/2 < \rho < \mu - (n+1)/(2n-2)$, $0 \leq 1/q < 1/2$ for $\rho = \mu - (n+1)/(2n-2)$, $0 \leq 1/q \leq (n-1)(\mu-\rho)/(n+1)$ for $\mu - (n+1)/(2n-2) < \rho \leq \mu$.

Proposition 2.2 *Let n, μ, p, f satisfy any of the assumptions in Theorem 1.1. Let $-\rho_0$ be the first index of G. Then for sufficiently small $\epsilon > 0$, there* *exists a pair* $(1/q_0, 1/r_0) \in \Omega_\epsilon$ *with* $\mu = 1 - (\rho_0 + n(1/2 - 1/r_0) - 1/q_0)$ *and two triplets* $(1/q_i, 1/r_i, \rho_i) \in \Omega_{\epsilon,\mu}$, $i = 1, 2$, *such that*

$$\|f(u); L^{q_0'}(I, \dot{B}^{-\rho_0}_{r_0'})\| \le C|I|^\sigma \|u\|_1^{p-1}\|u\|_2, \tag{2.6}$$

$$\begin{aligned} &\|f(u) - f(v); L^{q_0'}(I, \dot{B}^{-\rho_0}_{r_0'})\| \\ \le\ & C|I|^\sigma \max(\|u\|_1, \|v\|_1)^{p-1}\|u - v\|_2 \\ &+ C|I|^\sigma \max(\|u\|_1, \|v\|_1)^{p-2}\|u - v\|_1 \max(\|u\|_2, \|v\|_2) \\ &+ C|I|^\sigma \max(\|u\|_1, \|v\|_1)^{[-\rho_0]}\|u - v\|_1^{p-[-\rho_0]-1} \max(\|u\|_2, \|v\|_2), \end{aligned}$$

where $\|\cdot\|_i = \|\cdot; L^{q_i}(I, \dot{B}^{\rho_i}_{r_i})\|$ *and* $\sigma = 2-(p-1)(n-2\mu)/2$ *and the constant Cis independent of I. On the right hand side of the last inequality, the second and third terms are disregarded for* $p < 2$ *and* $p \notin ([-\rho_0]+1, [-\rho_0]+2)$ *respectively.*

Proof) Let $1/r^* = 1/r_1 - \rho_1/n$ and $1/r^{**} = 1/r_2 - (\rho_0 + \rho_2)/n$. If $\rho_1 \ge 0$, $0 \le -\rho_0 \le \rho_2$, $0 < 1/r^* \le 1/2$, $0 \le 1/r^{**} \le 1/2$, $1/r_0' = (p-1)/r^* + 1/r^{**}$ and $\sigma = 1/q_0' - (p-1)/q_1 - 1/q_2 \ge 0$, then by Proposition 2.1 and the embeddings $\dot{B}^{\rho_1}_{r_1} \subset \dot{B}^0_{r^*}$, $\dot{B}^{\rho_2}_{r_2} \subset \dot{B}^{-\rho_0}_{r^{**}}$ and the Hölder inequality in time, we obtain the required inequality, where we use the embedding $L^q \subset \dot{B}^0_q$ with $1 < q \le 2$ for $\rho_0 = 0$.

By a simple calculation, we see that the above assumptions are satisfied by a pair $(1/q_0, 1/r_0) \in \Omega_\epsilon$ and ρ_0 with $1 - \mu = \rho_0 + n(1/2 - 1/r_0) - 1/q_0$ and two triplets $(1/q_i, 1/r_i, \rho_i) \in \Omega_{\epsilon,\mu}$, $i = 1, 2$, which satisfy the following conditions.

1. $\rho_1 \ge 0, \quad 0 \le -\rho_0 \le \rho_2.$ (2.7)
2. $1/q_i < n/2 - \mu, \ i = 1, 2.$ (2.8)
3. $1/q_0 + 1/q_2 = \ell(1/q_1) \equiv (p-1)(n/2 - \mu - 1/q_1) - 1.$ (2.9)
4. $\sigma = 2 - (p-1)(n - 2\mu)/2 \ge 0.$ (2.10)

We show the existence of the above triplets $(1/q_i, 1/r_i, \rho_i)$, $i = 0, 1, 2$ using Lemma 2.1. We make some comments here. By the condition 3, we must assume $\mu < n/2$ and $p - 1 \ge 2/(n - 2\mu)$. By 4, we must assume $p - 1 \le 4/(n - 2\mu)$, but this is required for the well-posedness of (1.2) in $\dot{H}^\mu$.

In the following, we consider the case $n \ge 4$ only since the proofs for the case $n = 2, 3$ are analogous. We make a classification on μ. The problem is reduced to the existence of the required $1/q_i$, $i = 1, 2$.

Case 1. $0 \le \mu < (n-3)/(2n-2)$

Let $\rho_i = 0$, $i = 0,1,2$. Let $0 \le 1/q_0 \le 1/2$, $0 \le 1/q_i \le (n-1)\mu/(n+1)$, $i = 1,2$. Then by Lemma 2.1, we have $(1/q_0, 1/r_0) \in \Omega_\epsilon$ and $(1/q_i, 1/r_i, \rho_i) \in \Omega_{\epsilon,\mu}$, $i = 1,2$, for sufficiently small $\epsilon > 0$. Now $1/q_i$, $i = 0,1,2$, must satisfy (2.9), but the existence of such $1/q_i$, $i = 0,1,2$, is guaranteed if p satisfies

$$2/(n-2\mu) \le p-1 \le (3(n+1)+2(n-1)\mu)/(n^2+n-4n\mu). \qquad (2.11)$$

Case 2. $\mu = (n-3)(2n-2)$

In Case 1, with $1/q_0 \le 1/2$ replaced with $1/q_0 < 1/2$, we conclude the existence of the required $1/q_i$, $i = 0,1,2$, if p satisfies

$$2/(n-2\mu) \le p-1 < 4/(n+1-4\mu). \qquad (2.12)$$

In the following cases, the argument after setting ρ_i, $1/q_i$, $i = 0,1,2$, is similar to that of Case 1, so that we omit it and write the assumption on p only.

Case 3. $(n-3)/(2n-2) < \mu \le (n+1)/(2n-2)$

Let $\rho_i = 0$, $i = 0,1,2$. Let $0 \le 1/q_0 \le (n-1)(1-\mu)/(n+1)$, $0 \le 1/q_i \le (n-1)\mu/(n+1)$, $i = 1,2$, for $\mu < (n+1)/(2n-2)$, $0 \le 1/q_i < 1/2$, $i = 1,2$, for $\mu = (n+1)/(2n-2)$. The required assumption on p is

$$2/(n-2\mu) \le p-1 \le \begin{cases} 4/(n+1-4\mu) & \text{if } \mu < 1/2, \\ 4/(n-2\mu) & \text{if } \mu \ge 1/2. \end{cases} \qquad (2.13)$$

Case 4. $(n+1)/(2n-2) < \mu \le \min(1,\alpha)$

Let $\rho_i = 0$, $i = 0,1,2$. Let $0 \le 1/q_0 \le (n-1)(1-\mu)/(n+1)$, $0 \le 1/q_i \le 1/2$, $i = 1,2$. The assumption on p is $2/(n-2\mu) \le p-1 \le 4/(n-2\mu)$.

We refer to the constant α which depends on the spatial dimension. By the condition (2.9), we must assume at least $\ell(1/2) \le (n-1)(1-\mu)/(n+1)+1/2$, which is equivalent to

$$p-1 \le (2(n-1)(1-\mu)/(n+1)+3)/(n-1-2\mu). \qquad (2.14)$$

To enlarge the right hand side than $4/(n-2\mu)$, μ must satisfy $F(\mu) \ge 0$. But this is guaranteed if $\mu \le \alpha$ since α is the lower root of $F(x) = 0$.

Case 5. $n \ge 7$, $\alpha < \mu < (n-4)/2$

Let $-\rho_0 = \rho_1 = \rho_2 = \mu - \beta(\mu)$. Let $0 \le 1/q_0 \le (n-1)(1-\beta(\mu))/(n+1)$, $0 \le 1/q_i \le 1/2$, $i = 1,2$. The assumption on p is $2/(n-2\mu) \le p-1 \le 4/(n-2\mu)$.

We refer to $\beta(\mu)$ which depends on the spatial dimension and μ. By the condition (2.9), we must assume at least

$$p-1 \le (2(n-1)(1-\beta(\mu))/(n+1)+3)/(n-1-2\mu), \tag{2.15}$$

but the right hand side is equal to $4/(n-2\mu)$ by the definition of $\beta(\mu)$.

Case 6. $\max(1,(n-4)/2) \le \mu < n/2$

Let $-\rho_0 = \rho_1 = \rho_2 = \mu - 1$. Let $1/q_0 = 0$, $0 \le 1/q_i \le 1/2$, $i = 1, 2$, and $1/q_i < n/2 - \mu$, $i = 1, 2$. The assumption on p is $2/(n-2\mu) \le p-1 \le 4/(n-2\mu)$. □

3 Proof of Theorem 1.1

We prove Theorem 1.1 in this section.

Proof of Theorem 1.1) First of all, we recall the following inequalities by Proposition 3.1 in [10].

$$\|\dot{K}(t)\phi; L^q(I, \dot{B}^\rho_r)\| \le C\|\phi; \dot{H}^\mu\|, \tag{3.16}$$

$$\|K(t)\psi; L^q(I, \dot{B}^\rho_r)\| \le C\|\psi; \dot{H}^{\mu-1}\|, \tag{3.17}$$

$$\|\int_0^t K(t-\tau)h(\tau)d\tau; L^q(I, \dot{B}^\rho_r)\| \le C\|h; L^{q_0'}(I, \dot{B}^{-\rho_0}_{r_0'})\|, \tag{3.18}$$

for any $\mu, \rho, \rho_0 \in \mathbf{R}$ and $(1/q, 1/r), (1/q_0, 1/r_0) \in \Omega_\epsilon$ with $\mu = \rho + n(1/2 - 1/r) - 1/q = 1 - (\rho_0 + n(1/2 - 1/r_0) - 1/q_0)$, where C is a constant independent of I.

Let n, μ, p, f satisfy any of the assumptions in Theorem 1.1. Let ϵ, $1/q_i$, $1/r_i$, ρ_i, $i = 0, 1, 2$, be those in Proposition 2.2. By the above inequalities and Proposition 2.2, we have

$$\begin{aligned} &\|\Phi(u); L^q(I, \dot{B}^\rho_r)\| \\ \le\ & C\|(\phi,\psi)\|_\mu + C\|f(u); L^{q_0'}(I, \dot{B}^{-\rho_0}_{r_0'})\| \\ \le\ & C\|(\phi,\psi)\|_\mu + C|I|^\sigma \|u; L^{q_1}(I, \dot{B}^{\rho_1}_{r_1})\|^{p-1}\|u; L^{q_2}(I, \dot{B}^{\rho_2}_{r_2})\| \end{aligned} \tag{3.19}$$

for any $(1/q, 1/r, \rho) \in \Omega_{\epsilon,\mu}$, where C is independent of I. Therefore we obtain

$$\max_{(1/q,1/r,\rho)\in\Omega_{\epsilon,\mu}} \|\Phi(u); L^q(I, \dot{B}^\rho_r)\| \le C\|(\phi,\psi)\|_\mu + C|I|^\sigma R^p, \tag{3.20}$$

for any $u \in X_\epsilon(I, R)$. Similarly we have

$$d(\Phi(u), \Phi(v)) \le C|I|^\sigma R^{p-1} d(u,v) + C|I|^\sigma R^{[-\rho_0]+1} d(u,v)^{p-[-\rho_0]-1}, \tag{3.21}$$

for any $u, v \in X_\epsilon(I, R)$, where C is independent of I and the second term on the right hand side of (3.21) is disregarded for $p \notin J$. If $p \notin J$, then the unique solution of (1.2) is given by the standard contraction argument on $(X_\epsilon(I, R), d)$ with R sufficiently large and $|I| > 0$ sufficiently small for the local solution, with R and $\|(\phi, \psi)\|_\mu$ sufficiently small for the global solution. If $p \in J$, then we have only to consider the case $(n+1)/(2n-2) < \mu < n/2$. Let $|I|$ and R satisfy

$$C\|(\phi, \psi)\|_\mu + C|I|^\sigma R^p \leq R, \quad C|I|^\sigma R^{p-1} < 1, \tag{3.22}$$

and let $\|(\phi, \psi)\|_\mu$ be sufficiently small for $\sigma = 0$. Let $u_0 = 0$ and $u_{i+1} \equiv \Phi(u_i)$ for $i = 1, 2, \cdots$. Then there is a subsequence $\{u_{i_k}\}_k \subset \{u_i\}_i$ and $u \in X_\epsilon(I, R)$ such that u_{i_k} converges to u in the distribution sense as $k \to \infty$. On the other hand, let $(n-3)/(2n-2) < \mu_0 < (n+1)/(2n-2)$ and let $\lambda > 0$ and $\Lambda(\lambda) \equiv \{(t, x) \in \mathbf{R}^{n+1} \mid |x| < \lambda - |t|\}$, then we have for sufficiently small $\epsilon > 0$,

$$\begin{aligned}
&\max_{(1/q, 1/r, 0) \in \Omega_{\epsilon, \mu_0}} \|u_{i+2} - u_{i+1}; L^q L^r(\Lambda(\lambda))\| \\
\leq\ & C|I|^\sigma R^{p-1} \max_{(1/q, 1/r, 0) \in \Omega_{\epsilon, \mu_0}} \|u_{i+1} - u_i; L^q L^r(\Lambda(\lambda))\|.
\end{aligned} \tag{3.23}$$

Indeed, let w and w_λ satisfy $(\partial_t^2 - \Delta)w = h$, $w(0) = \partial_t w(0) = 0$ and $(\partial_t^2 - \Delta)w_\lambda = h\chi_{\Lambda(\lambda)}$, $w_\lambda(0) = \partial_t w_\lambda(0) = 0$, then $w = w_\lambda$ on $\Lambda(\lambda)$, where $\chi_{\Lambda(\lambda)}$ is a characteristic function on $\Lambda(\lambda)$. By this fact and (3.18) and the argument as described in the proof of Proposition 2.2, we obtain the above inequality.

By (3.23), we conclude that $\{u_i\}$ converges to some v_λ strongly in $L^q L^r(\Lambda(\lambda))$ for any $(1/q, 1/r, 0) \in \Omega_{\epsilon, \mu_0}$, so that $u = v_\lambda$ on $\Lambda(\lambda)$. Therefore we have for any $\lambda > 0$

$$\begin{aligned}
&\max_{(1/q, 1/r, 0) \in \Omega_{\epsilon, \mu_0}} \|\Phi(u) - u; L^q L^r(\Lambda(\lambda))\| \\
\leq\ & \max_{(1/q, 1/r, 0) \in \Omega_{\epsilon, \mu_0}} \|\Phi(u) - \Phi(u_i) + u_{i+1} - u; L^q L^r(\Lambda(\lambda))\| \\
\leq\ & C|I|^\sigma R^{p-1} \max_{(1/q, 1/r, 0) \in \Omega_{\epsilon, \mu_0}} \|u - u_i; L^q L^r(\Lambda(\lambda))\| \\
\to\ & 0 \quad \text{as } i \to \infty,
\end{aligned}$$

by which we conclude that $u = \Phi(u)$ $a.e$ $(t, x) \in I \times \mathbf{R}^n$, namely $u = \Phi(u)$ in $(X_\epsilon(I, R), d)$.

The uniqueness of the solution also follows from (3.23).

(1) The continuity of the solution $(u, \partial_t u)$ in time with respect to the $\dot{H}^\mu \times \dot{H}^{\mu-1}$-norm follows from the Lebesgue convergence theorem. The proof is standard and we omit it.

(2) For the continuous dependence on the initial data of its solution, we consider the case $p \in J$ only since for $p \notin J$ the last term of (3.21) is disregarded, so that we can use the standard argument [3]. By (3.23), we have

$$\begin{aligned} &\max_{(1/q,1/r,0)\in\Omega_{\epsilon,\mu_0}} \|u-v; L^q L^r(\Lambda(\lambda))\| \\ \leq\ & C_\lambda \|(\phi-\phi_0,\psi-\psi_0)\|_\mu + C|I|^\sigma R^{p-1} \max_{(1/q,1/r,0)\in\Omega_{\epsilon,\mu_0}} \|u-v; L^q L^r(\Lambda(\lambda))\|, \end{aligned}$$

where C_λ is a constant dependent on λ, but not on I. So that we conclude $v \to u$ in $\bigcap_{(1/q,1/r,0)\in\Omega_{\epsilon,\mu_0}} L^q L^r(\Lambda(\lambda))$ as (ϕ_0,ψ_0) tends to (ϕ,ψ), by which we conclude that v converges to u in $\mathcal{D}'(\mathbf{R}^{n+1})$, as required.

(3) Let (ϕ_+,ψ_+) be

$$\phi_+ \equiv \phi + \int_0^\infty K(-\tau)f(u(\tau))d\tau, \quad \psi_+ \equiv \psi + \int_0^\infty \dot{K}(-\tau)f(u(\tau))d\tau.$$

Then we have

$$\begin{aligned} &\|u(t) - \dot{K}(t)\phi_+ - K(t)\psi_+; \dot{H}^\mu\| \\ \leq\ & \|\int_t^\infty K(t-\tau)f(u(\tau))d\tau; \dot{H}^\mu\| \\ \leq\ & C\|u; L^{q_1}((t,\infty),\dot{B}^{\rho_1}_{r_1})\|^{p-1}\|u; L^{q_2}((t,\infty),\dot{B}^{\rho_2}_{r_2})\|, \end{aligned}$$

where we have used a similar result to (3.18) and Proposition 2.2, and we can take $1/q_i$, $i=1,2$, for $1/q_i \neq \infty$ since $p-1 = 4/(n-2\mu)$. Therefore we have

$$\|u(t) - \dot{K}(t)\phi_+ - K(t)\psi_+; \dot{H}^\mu\| \to 0 \ \text{ as } \ t \to \infty.$$

(4) For $(\phi_-,\psi_-) \in \dot{H}^\mu \times \dot{H}^{\mu-1}$, let Φ_- be an operator defined by

$$\Phi_-(u) \equiv \dot{K}(t)\phi_- + K(t)\psi_- + \int_{-\infty}^t K(t-\tau)f(u(\tau))d\tau, \tag{3.24}$$

Similarly to Φ, we have

$$\max_{(1/q,1/r,\rho)\in\Omega_{\epsilon,\mu}} \|\Phi_-(u); L^q(I,\dot{B}^\rho_r)\| \leq C\|(\phi_-,\psi_-)\|_\mu + CR^p, \tag{3.25}$$

$$d(\Phi_-(u),\Phi_-(v)) \leq CR^{p-1}d(u,v) + C|I|^\sigma R^{[-\rho_0]+1}d(u,v)^{p-[-\rho_0]-1}, \tag{3.26}$$

for any $u, v \in X_\epsilon(I,R)$, where the second term on the right hand side of (3.26) is disregarded for $p \notin J$. Therefore for $p \notin J$ we have the unique fixed point of Φ_- in $X_\epsilon(I,R)$ by a contraction argument with $\|(\phi_-,\psi_-)\|_\mu$ and R sufficiently small. We show that for $p \in J$ we also have a fixed point of Φ_- in $X_\epsilon(I,R)$ with $\|(\phi_-,\psi_-)\|_\mu$ and R sufficiently small in the following.

We may assume $(n+1)/(2n-2) < \mu < n/2$. Let $(n-3)/(2n-2) < \mu_0 < (n+1)/(2n-2)$. Let $R_0 > 0$. Let $X_\epsilon(I,R,R_0)$ and d_0 be

$$\begin{aligned} X_\epsilon(I,R,R_0) &\equiv \{u \in X_\epsilon(I,R) \mid \max_{(1/q,1/r,0)\in\Omega_{\epsilon,\mu_0}} \|u; L^q(I,L^r)\| \le R_0\}, \\ d_0(u,v) &\equiv \max_{(1/q,1/r,0)\in\Omega_{\epsilon,\mu_0}} \|u-v; L^q(I,L^r)\|, \end{aligned}$$

for any $u, v \in X_\epsilon(I,R,R_0)$. Then similarly to (3.25) and (3.26), we have

$$\max_{(1/q,1/r,0)\in\Omega_{\epsilon,\mu_0}} \|\Phi_-(u); L^q(I,L^r)\| \le C\|(\phi_-,\psi_-)\|_{\mu_0} + CR^{p-1}R_0, \quad (3.27)$$

$$\max_{(1/q,1/r,\rho)\in\Omega_{\epsilon,\mu}} \|\Phi_-(u); L^q(I,\dot{B}^\rho_r)\| \le C\|(\phi_-,\psi_-)\|_\mu + CR^p, \quad (3.28)$$

$$d_0(\Phi_-(u),\Phi_-(v)) \le CR^{p-1}d_0(u,v). \quad (3.29)$$

So that if $(\phi_-,\psi_-) \in \dot{H}^{\mu_0} \times \dot{H}^{\mu_0-1}$ and if $\|(\phi_-,\psi_-)\|_\mu$ and R are sufficiently small and R_0 sufficiently large, then Φ_- becomes a contraction map on $X_\epsilon(I,R,R_0)$ with the metric d_0. Therefore we obtain the unique fixed point of Φ_-. Let $\|(\phi_-,\psi_-)\|_\mu$ be sufficiently small. Let $\{(\phi_i,\psi_i)\}_{i=1}^\infty$ be a sequence such that $(\phi_i,\psi_i) \to (\phi_-,\psi_-)$ in $\dot{H}^\mu \times \dot{H}^{\mu-1}$ as $i \to \infty$ and $(\phi_i,\psi_i) \in \dot{H}^{\mu_0} \times \dot{H}^{\mu_0-1}$. Then by the above argument, there exists $u_i \in X_\epsilon(I,R,R_0)$ which satisfies

$$u_i = \dot{K}(t)\phi_i + K(t)\psi_i + \int_{-\infty}^t K(t-\tau)f(u_i(\tau))d\tau, \quad (3.30)$$

for i sufficiently large. We can take a subsequence of $\{u_i\}$ which converges to some u in the distribution sense. This u is the required fixed point of Φ_- in $X_\epsilon(I,R)$. For details, we refer to the discussion before Lemma 7.1 and itself in [15]. The result $\|u(t) - \dot{K}(t)\phi_- - K(t)\psi_-; \dot{H}^\mu\| \to 0$ as $t \to -\infty$ now follows similarly to the proof of (3).

Next we show that the scattering map $(\phi_-,\psi_-) \to (\phi_+,\psi_+)$ is continuous in the neighborhood at the origin in $\dot{H}^\mu \times \dot{H}^{\mu-1}$ for $p \notin J$. By the proof of (3) and (4), we have the following relation between (ϕ_-,ψ_-) and (ϕ_+,ψ_+) as

$$\phi_+ = \phi_- + \int_{-\infty}^\infty K(-\tau)f(u(\tau))d\tau, \quad \psi_+ = \psi_- + \int_{-\infty}^\infty \dot{K}(-\tau)f(u(\tau))d\tau, \quad (3.31)$$

where u is the solution of $u = \Phi_-(u)$. Let $((\tilde{\phi}_-,\tilde{\psi}_-),\ \tilde{u},\ (\tilde{\phi}_+,\tilde{\psi}_+))$ be another triplet. It suffices to show that

$$\|(\phi_+ - \tilde{\phi}_+, \psi_+ - \tilde{\psi}_+)\|_\mu \to 0 \ \text{ as } \ \|(\phi_- - \tilde{\phi}_-, \psi_- - \tilde{\psi}_-)\|_\mu \to 0. \quad (3.32)$$

Similarly to the proof of (3.21), we have

$$\|\phi_+ - \tilde{\phi}_+; \dot{H}^\mu\| \le \|\phi_- - \tilde{\phi}_-; \dot{H}^\mu\| + CR^{p-1}d(u,\tilde{u}), \tag{3.33}$$

and

$$d(u,\tilde{u}) \le C\|(\phi_- - \tilde{\phi}_-, \psi_- - \tilde{\psi}_-)\|_\mu + CR^{p-1}d(u,\tilde{u}). \tag{3.34}$$

Since $CR^{p-1} < 1$, we conclude that $\|\phi_+ - \tilde{\phi}_+; \dot{H}^\mu\| \to 0$ as $\|(\phi_- - \tilde{\phi}_-, \psi_- - \tilde{\psi}_-)\|_\mu$ tends to zero. For $\|\psi_+ - \tilde{\psi}_+; \dot{H}^{\mu-1}\|$, the proof is analogous. □

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