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On a density of the set of primes dividing the generalized Fibonacci numbers

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Abstract J. C. Lagarias showed the set of prime numbers which divide some Lucas number $L_n$ has positive density using Hasse's method [H]. In his paper he found several results for certain other special second-order linear recurrences [L], [W]. So we will research similar phenomena for slightly generalized second-order linear recurrences.

1 Introduction

In this note we will try to generalize a result of Lagalias on some second-order linear recurrences. Our method will be controlled by Hasse’s one. Then we have to check whether these recurrences satisfy Hasse’s conditions or not.

Now, any irreducible second-order recurrence $\{U_n\}$ whose terms $U_n$ are rational numbers can be expressed in the form

$$U_n = \alpha \theta^n + \bar{\alpha} \bar{\theta}^n,$$

where $\alpha$ and $\theta$ are in the quadratic field $K$ generated by the roots of the characteristic polinomial of $\{U_n\}$, and $\bar{\alpha}, \bar{\theta}$ are the algebraic conjugates of $\alpha, \theta$ in $K$.

Hasse’s conditions are as follows:

1. $\theta/\bar{\theta} = \pm \phi^k$, where $k = 1$ or $2$ for some $\phi$ in $K$,
2. $\bar{\alpha}/\alpha = \zeta^j$, where $\zeta$ is a root of unity in $K$ and $j$ is an integer.

We put $S_U = \{ p : p \text{ is a prime and } p | U_n \text{ for some } n \}$. These particular recurrences $\{U_n\}$, which satifiy the above conditions (1) and (2), have a special property.

Definition 1 A set $\Sigma$ of primes is a Chebotarev set if and only if there is some finite normal extension $L$ of the rationals $Q$ such that a prime $p$ is in $\Sigma$ iff the Artin symbol $\left[ \frac{L}{Q} \right] (p)$ is in specified conjugacy classes of the Galois group $Gal(L/Q)$.

Definition 2 Density $d(S_U)$ is defined

$$\lim_{X \to \infty} \frac{\#S_{U,X}}{\#P_X} = d(S_U),$$

where $\#S_{U,X} = \#\{ p ; p \in S_U, p < X \}$ and $\#P_X = \#\{ p ; p \text{ is a prime, } p < X \} \sim \frac{X}{\log X}$.

Property 1 Both the set $S$ of primes and its complement

$$\bar{S} = \{ p : p \text{ is a prime and } p \notin S \}$$

have a decomposition into disjoint countable unions of Chebotarev sets of primes. That is

$$S = \bigcup_{j=1}^{\infty} S^{(j)}, \quad \bar{S} = \bigcup_{j=1}^{\infty} \bar{S}^{(j)},$$

where $S^{(j)}$ and $\bar{S}^{(j)}$ are Chebotarev sets. Then the densities of the sets satisfy

$$\sum_{j=1}^{\infty} d(S^{(j)}) + \sum_{j=1}^{\infty} d(\bar{S}^{(j)}) = 1.$$ 

If $S$ is any set of primes having Property 1, then $S$ has a natural density $d(S)$ given by

$$d(S) = \sum_{j=1}^{\infty} d(S^{(j)}).$$

2 Known results

Hasse and Lagarias obtained the following prime densities for several types of sequences:

Theorem 1 (H. Hasse [H]) For the sequence $\{V_n\} = \{2^n + 1\}$, the set of primes

$$S_V = \{ p : p \text{ is a prime and } p \text{ divides } 2^n + 1 \text{ for some } n \geq 0 \}$$

$$= \{ p \in \mathbb{P} ; p | V_n \text{ for some } n \}.$$ 

has density $d(S_V) = \frac{17}{24}$.

Hasse’s result actually covers all the sequences

$$\{ A_n \} = \{ a^n + 1 \mid n \geq 0 \},$$

where $a$ is an integer $\geq 3$, and the density of the associated set $S_A = \{ p \in \mathbb{P} : p | A_n \text{ for some } n \}$ is

$$d(S_A) = \frac{2}{3}.$$ 

Theorem 2 (J. C. Lagarias [L]) For the sequence $\{L_n\} (L_{n+1} = L_n + L_{n-1}, \ L_1 = 2, \ L_2 = 1)$, the set of primes

$$S_L = \{ p \in \mathbb{P} ; p | L_n \text{ for some } n \}$$

has density $d(S_L) = \frac{2}{3}$. 

**Theorem 3** (J. C. Lagarias [L2]) For the sequence \( \{W_n\} \) \((W_n = 5W_{n-1} - 7W_{n-2}, W_0 = 1, W_1 = 2)\), the set of primes

\[ S_W = \{ p \in P : p | W_n \text{ for some } n \} \]

has density \( d(S_W) = \frac{3}{4} \).

Lagarias considered

\( \{A_n(m)\}, \ {B_n(m)\} \) \((m \text{ : fixed})\)

where both series admit the condition:

\[ U_n = mU_{n-1} - U_{n-2} \]

with \( A_0(m) = B_0(m) = 1, A_1(m) = m + 1, B_1(m) = m - 1 \), to which Hasse's method is applicable. In the cases of fields \( K = Q(\sqrt{m^2 - 4}) \), for the following sets of primes:

\[ S_A(m) = \{ p \in P : p | A_n(m) \text{ for some } n \}, \]
\[ S_B(m) = \{ p \in P : p | B_n(m) \text{ for some } n \}, \]

it is known that \( d(S_A(m)) = d(S_B(m)) = \frac{1}{3} \).

### 3 Theorem

Let

\( \{U_n\} \) \((U_n = mU_{n-1} + U_{n-2}, U_0 = 2, U_1 = m)\),

be a second-order linear recurrence, where we assume that \( D = m^2 + 4 \) is a prime discriminant of \( K = Q(\sqrt{D}) \). Then we have

**Theorem 4** For the sequence \( \{U_n\} \) \((U_n = mU_{n-1} + U_{n-2}, U_0 = 2, U_1 = m)\), the set of primes

\[ S_U = \{ p \in P : p | U_n \text{ for some } n \} \]

has density \( d(S_U) = \frac{2}{3} \).

**Remark 1** In the case of \( m = 1 \), the theorem above coincides with Theorem 2. We can prove Theorem 4 by a similar way to Theorem 2.

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References


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