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Transcendence of Rogers–Ramanujan continued fraction and reciprocal sums of Fibonacci numbers

Daniel Duverney
Keiji Nishioka
Kumiko Nishioka
Iekata Shiokawa

This is a report on the recent work of Duverney, Ke. Nishioka, Ku. Nishioka, and the author [11] concerning the title of this paper. Let $P(q)$, $Q(q)$, $R(q)$ be the Ramanujan’s functions defined by

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n},$$

$$Q(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1 - q^n},$$

$$R(q) = 1 - 540 \sum_{n=1}^{\infty} \sigma_5(n)q^n = 1 - 540 \sum_{n=1}^{\infty} \frac{n^5q^n}{1 - q^n},$$

which are the classical Eisenstein series $E_2(q), E_4(q), E_6(q)$ respectively, where $\sigma_i(n) = \sum_{d|n}d^i$. Mahler [17] proved the algebraically independency of the functions $P(q), Q(q), R(q)$ over $\mathbb{C}(q)$. Letting $'$ denote the derivation $q^d\frac{d}{dq}$, we have

$$P' = \frac{1}{12}(P^2 - Q), \quad Q' = \frac{1}{3}(PQ - R), \quad R' = \frac{1}{2}(PR - Q^2)$$

( cf. [15; Theorem 5.3]). We put

$$\Delta = \frac{1}{1728}(Q^3 - R^2), \quad J = \frac{Q^3}{\Delta}.$$

The modular function $j(\tau)$ is described as $j(\tau) = J(q)$, where $q = e^{2\pi i \tau}, \text{Im}\tau > 0$. Barré–Serieix, Diaz, Gramain, and Philibert [3] proved the transcendency of the value $J(\alpha)$ for any $\alpha \in \overline{\mathbb{Q}}, 0 < |\alpha| < 1$. By the equalities

$$\frac{J'}{J} = -\frac{R}{Q}, \quad \frac{J''}{J'} = \frac{1}{6}P - \frac{2}{3} \frac{R}{Q} - \frac{1}{2} \frac{Q^2}{R},$$
we have $Q \in \mathbb{Q}(J, J', J'')$, and hence

$$\mathbb{Q}(P, Q, R) = \mathbb{Q}(J, J', J'') = K,$$ say.

We note that $K$ is a differential field, i.e., closed under the derivation 't'. Now we state

**Nesterenko's theorem** ([19], [20]). If $\alpha \in \mathbb{C}$, $0 < |\alpha| < 1$, then

$$\text{trans.deg}_\mathbb{Q} \mathbb{Q}(\alpha, P(\alpha), Q(\alpha), R(\alpha)) \geq 3.$$

**Corollary 1.** If $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, then each of the following set

1) $P(\alpha), Q(\alpha), R(\alpha)$, 2) $J(\alpha), J'(\alpha), J''(\alpha)$ are algebraically independent.

**Corollary 2.** The numbers $\pi, e^\pi$, and $\Gamma(1/4)$ are algebraically independent.

Let $\eta(q)$ be the eta function defined by

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

which is known to satisfy

$$\eta(q)^{24} = \Delta(q).$$

**Corollary 3.** If $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, then

$$\text{trans.deg}_\mathbb{Q} \mathbb{Q}(\alpha, \eta(\alpha), \eta'(\alpha), \eta''(\alpha)) \geq 3.$$

In particular, the infinite product $\prod_{n=1}^{\infty} (1-\alpha^n)$ is transcendental for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$.

Let $\vartheta_3, \vartheta = \vartheta_4, \vartheta_2$ be Jacobi's theta series defined by

$$\vartheta_3 = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}, \quad \vartheta = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}, \quad \vartheta_2 = 2q^{1/4} \sum_{n=1}^{\infty} q^{n(n-1)}.$$

**Corollary 4** (Bertrand [5]). Let $y = y(q)$ be one of $\vartheta_3, \vartheta, \vartheta_2$. If $\alpha \in \overline{\mathbb{C}}$, $0 < |\alpha| < 1$, then

$$\text{trans.deg}_\mathbb{Q} \mathbb{Q}(\alpha, y(\alpha), y'(\alpha), y''(\alpha)) \geq 3.$$

In particular, the number $\sum_{n=1}^{\infty} \alpha^{n^2}$ is transcendental for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$. 
We note that Corollary 4 provides the best possible results as $y$ is known to satisfy an algebraic differential equations of the third order defined over $\mathbb{Q}$ (cf. Jacobi [13]). A survey on Nesterenko’s theorem can be found in Waldschmidt [23].

The following lemmas are useful to prove the transcendency of some numbers related to modular functions.

**Lemma 1**([10]). Let $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$. If a nonconstant function $f(q)$ is algebraic over $K$ and defined at $\alpha$, then $f(\alpha)$ is transcendental.

**Lemma 2**([10]). Let $y = y(q)$ be one of the functions $\eta, \vartheta_3, \vartheta, \vartheta_2$. Then $y(q^k)$, $y'(q^k)$, $y''(q^k)$, ... are algebraic over $K$ for any positive integer $k$.

The Rogers–Ramanujan continued fraction $RR(q)$ is defined by

$$RR(q) = 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}$$

which is known to have the expressions

$$RR(q) = \frac{\sum_{k=0}^{\infty} q^{k^2}}{\sum_{k=0}^{\infty} (1-q)(1-q^2)\cdots(1-q^k)} = \prod_{k=0}^{\infty} \frac{(1-q^{5k+2})(1-q^{5k+3})}{(1-q^{5k+1})(1-q^{5k+4})}$$

(cf. [4; Chap. 16, Entry 15, 38(iii)]). Irrationality measures for some values of this continued fraction were given by Osgood [21] and Shiokawa [22]. The latter proved that for any integer $d \geq 2$ there is a constant $C = C(d) > 0$ such that

$$|RR\left(\frac{1}{d}\right) - \frac{p}{q}| > Cq^{-2B/\sqrt{\log q}}$$

for all integers $p, q (\geq 2)$, where $B = \sqrt{\log d}$. Matala–Aho [18] obtained some higher degree irrationality results. For example, $RR((\sqrt{5} - 1)/2) \notin \mathbb{Q}(\sqrt{5})$.

**Theorem 1**([11]). The Rogers–Ramanujan continued fraction $RR(\alpha)$ is transcendental for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$. 
Proof. Let
\[ F(q) = \frac{q^{1/5}}{1 + 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \frac{q^4}{1 + \cdots}}}}}, \]
then
\[ \frac{1}{F(q)} - F(q) - 1 = q^{-1/5} \prod_{n=1}^{\infty} \frac{1 - q^{n/5}}{1 - q^{5n}} = \frac{\eta(q^{1/5})}{\eta(q^5)} \]
(see [4; p.85]). Applying Lemma 1 and 2 to the function \( f(q) = \eta(q)/\eta(q^{25}) \), we see that \( f(\alpha) \) is transcendental for any \( \alpha \in \overline{\mathbb{Q}} \), \( 0 < |\alpha| < 1 \), and so is \( F(\alpha) \) from the formula above.

We give here further examples of continued fractions whose transcendence can be easily deduced from Lemma 1 and 2. For any \( \alpha \in \overline{\mathbb{Q}} \), \( 0 < |\alpha| < 1 \), the following continued fractions (i), (ii), (iii) are transcendental:

(i) \[ \frac{1}{1 + \frac{\alpha}{1 + \frac{\alpha^2}{1 + \frac{\alpha^3}{1 + \frac{\alpha^4}{1 + \frac{\alpha^5}{1 + \cdots}}}}} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n. \]

Theorem 2([11]). If \( \alpha \beta = \pm 1 \), then the numbers
\[ \sum_{n=1}^{\infty} \frac{1}{U_{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{V_{2s}} \]
are transcendental for any positive integer \( s \).

Theorem 3([11]). If \( \alpha \beta = 1 \), then the number
\[ \sum_{n=1}^{\infty} \frac{1}{V_{2s}^n} \]
is transcendental for any positive integer \( s \).
**Theorem 4** ([11]). If $\alpha \beta = -1$, then the number
\[
\sum_{n=1}^{\infty} \frac{1}{U_{2n-1}^{s}}
\]
is transcendental for any positive integer $s$.

In the special case of $s = 1$, these theorems are proved in [10] by direct calculation without using Lemma 3 below.

For the proof, we need another lemma. Let
\[
k = \vartheta_{3}^{2}(q)/\vartheta_{3}^{2}(q)
\]
be the modulus of the complete elliptic integrals
\[
K = \int_{0}^{1} \frac{dt}{\sqrt{(1 - t^{2})(1 - k^{2}t^{2})}}, \quad E = \int_{0}^{1} \frac{\sqrt{1 - k^{2}t^{2}}}{\sqrt{1 - t^{2}}} dt,
\]
of the first and the second kind, respectively. Then we have
\[
\frac{K}{\pi} = \frac{1}{2} \vartheta_{3}^{2}(q), \quad \frac{E}{\pi} = \frac{K}{\pi} + \frac{\pi}{K} \vartheta'(q),
\]
where $\vartheta' = q \frac{d\vartheta}{dq}$ (cf. [6; (2.1.13), (2.3.17)]).

**Lemma 3** ([11]). Let $s$ be any positive integer and let
\[
f_{2s}(q) = \sum_{n=1}^{\infty} \frac{1}{(q^{-n} - q^{n})^{2s}}, \quad g_{s}(q) = \sum_{n=1}^{\infty} \frac{1}{(q^{-n} + q^{n})^{s}}.
\]
Then $f_{2s}(q), f_{2s}(q^{2}), g_{s}(q)$, and $g_{s}(q^{2})$ are algebraic over the field $\mathbb{Q}(P(q), Q(q), R(q))$.

**Proof.** Let $s$ be a positive integer. We put
\[
I_{2s} = \sum_{n=1}^{\infty} \cosh^{2s}(n\pi c) = \sum_{n=1}^{\infty} \left( \frac{2}{q^{-n} - q^{n}} \right)^{2s}, \quad q = e^{-\pi c},
\]
\[
\Pi_{s} = \sum_{n=1}^{\infty} \sech^{s}(n\pi c) = \sum_{n=1}^{\infty} \left( \frac{2}{q^{-n} + q^{n}} \right)^{s},
\]
so that
\[
f_{2s}(q) = 2^{-2s}I_{2s}, \quad g_{s}(q) = 2^{-s}\Pi_{s}.
\]
Then Zucker [26] obtained expansions of $I_{2s}$, $II_{s}$, and $II_{2s+1}$ as polynomials of $k$, $K/\pi$, and $E/\pi$ with rational coefficients, which can be found in Table 1(i), 1(ii), and 1(vi) in [26], respectively. Hence the lemma follows from Lemma 2.

**Proof of Theorem 2.** If $\alpha\beta = 1$, then we have

$$
(\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_{2n}^{2s}} = \sum_{n=1}^{\infty} \frac{1}{(\beta^{-n} - \beta^{n})^{2s}} = f_{2s}(\beta),
$$

$$
\sum_{n=1}^{\infty} \frac{1}{V_{2n}^{2s}} = \sum_{n=1}^{\infty} \frac{1}{(\beta^{-n} + \beta^{n})^{2s}} = g_{2s}(\beta),
$$

and the results follow from Lemma 3 and 1. If $\alpha\beta = -1$, then we have

$$
(\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_{2n}^{2s}} = \sum_{n=1}^{\infty} \frac{1}{((-\beta)^{-n} - \beta^{n})^{2s}}
$$

$$
= \sum_{n=1}^{\infty} \frac{1}{((-\beta)^{-2n} - \beta^{2n})^{2s}} + \sum_{n=1}^{\infty} \frac{1}{((-\beta)^{-2n+1} - \beta^{2n+1})^{2s}}
$$

$$
= f_{2s}(\beta^2) + g_{2s}(\beta) - g_{2s}(\beta^2),
$$

$$
\sum_{n=1}^{\infty} \frac{1}{V_{2n}^{2s}} = \sum_{n=1}^{\infty} \frac{1}{((-\beta)^{-n} + \beta^{n})^{2s}}
$$

$$
= \sum_{n=1}^{\infty} \frac{1}{(\beta^{-2n} + \beta^{2n})^{2s}} + \sum_{n=1}^{\infty} \frac{1}{((-\beta)^{-2n+1} + \beta^{2n+1})^{2s}}
$$

$$
= g_{2s}(\beta^2) + f_{2s}(\beta) - f_{2s}(\beta^2).
$$

**Proof of Theorem 3.**

$$
\sum_{n=1}^{\infty} \frac{1}{V_{n}^{s}} = \sum_{n=1}^{\infty} \frac{1}{(\beta^{-n} + \beta^{n})^{s}} = g_{s}(\beta).
$$

**Proof of Theorem 4.**

$$
(\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_{2n-1}^{2s}} = g_{2s}(\beta) - g_{2s}(\beta^2),
$$

$$
(\alpha - \beta)^{-(2s-1)} \sum_{n=1}^{\infty} \frac{1}{U_{2n-1}^{2s-1}} = - \sum_{n=1}^{\infty} \frac{1}{(\beta^{-(2n-1)} + \beta^{2n-1})^{2s-1}}
$$

$$
= -g_{2s-1}(\beta) + g_{2s-1}(\beta^2).$$
Fibonacci numbers \( \{F_n\}_{n \geq 1} \) and Lucas numbers \( \{L_n\}_{n \geq 1} \) are defined by
\[
F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0),
\]
\[
L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0),
\]
and written as
\[
F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n \quad (n \geq 0),
\]
where
\[
\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.
\]

**Corollary**([11]). The numbers
\[
\sum_{n=1}^{\infty} \frac{1}{F_{n^s}} \quad \sum_{n=1}^{\infty} \frac{1}{L_{n^s}} \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^s} \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^s}
\]
are transcendental for any positive integer \( s \).

André-Jeannin [1] proved the irrationality of the number
\[
\sum_{n=1}^{\infty} \frac{1}{F_n}.
\]
Duverney [8] gave another proof and Kato [14] showed by Duverney’s method that the number
\[
\sum_{n=1}^{\infty} \frac{1}{F_{an}}
\]
is irrational for any positive integer \( a \). It is not known whether these numbers are transcendental or not. Bundschuh and Väänänen [7] gave an irrationality measure for \( \sum_{n=1}^{\infty} \frac{1}{F_{n^{-1}}} \); namely
\[
\left| \sum_{n=1}^{\infty} \frac{1}{F_n} - \frac{p}{q} \right| > \frac{1}{q^{8.621}}
\]
holds for all rationals \( p/q \) with sufficiently large \( q \).

Finally, we state two problems which are interesting in comparison with the arithmetical properties of the values of the Riemann zeta function \( \zeta(s) \) at \( s = 2, 3, 4, \ldots \)

**Problem 1.** Is the number
\[
\sum_{n=1}^{\infty} \frac{1}{F_n^3}
\]
irrational?

**Problem 2.** Are the numbers

\[ \sum_{n=1}^{\infty} \frac{1}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^4}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^6} \]

algebraically independent?

**References**


[13] C. G. J. Jacobi, Über die Differentialalgleichung, welcher die Reihen \( 1 \pm 2q \pm 2q^4 \pm 2q^9 + \text{etc.} \), \( 2\sqrt[4]{q} + 2\sqrt[4]{q^5} + 2\sqrt[4]{q^9} + \text{etc.} \), Genügen leisten. J. Reine Angew. Math. 36 (1847), 97–112.


Daniel Duverney
24 Place du Concert
59800 Lille (France)
duverney@gat.univ-lille1.fr

Keiji Nishioka
Faculty of Environmental Info
Keio University, Endoh 5322
Fujisawa 246 (Japan)
knis@sfc.keio.ac.jp

Kumiko Nishioka
Mathematics, Hiyoshi Campus
Keio University
Hiyoshi 4-1-1, Kohoku-ku,
Yokohama 223 (Japan)
nishioka@math.hc.keio.ac.jp

Iekata Shiokawa
Department of Mathematics
Keio University
Hiyoshi 3-14-1, Kohoku-ku,
Yokohama 223 (Japan)
shiokawa@math.keio.ac.jp