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<th>Title</th>
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<tbody>
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Transcendence of Rogers–Ramanujan continued fraction and reciprocal sums of Fibonacci numbers

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This is a report on the recent work of Duverney, Ke. Nishioka, Ku. Nishioka, and the author [11] concerning the title of this paper. Let $P(q), Q(q), R(q)$ be the Ramanujan's functions defined by

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$
$$Q(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n},$$
$$R(q) = 1 - 540 \sum_{n=1}^{\infty} \sigma_5(n) q^n = 1 - 540 \sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n},$$

which are the classical Eisenstein series $E_2(q), E_4(q), E_6(q)$ respectively, where $\sigma_i(n) = \sum_{d|n} d^i$. Mahler [17] proved the algebraically independency of the functions $P(q), Q(q), R(q)$ over $\mathbb{C}(q)$. Letting $'$ denote the derivation $q \frac{d}{dq}$, we have

$$P' = \frac{1}{12} (P^2 - Q), \quad Q' = \frac{1}{3} (PQ - R), \quad R' = \frac{1}{2} (PR - Q^2)$$

(cf. [15; Theorem 5.3]). We put

$$\Delta = \frac{1}{1728} (Q^3 - R^2), \quad J = \frac{Q^3}{\Delta}.$$  

The modular function $j(\tau)$ is described as $j(\tau) = J(q)$, where $q = e^{2\pi i \tau}, \text{Im} \tau > 0$. Barré–Serieix, Diaz, Gramain, and Philibert [3] proved the transcendency of the value $J(\alpha)$ for any $\alpha \in \overline{\mathbb{Q}}, 0 < |\alpha| < 1$. By the equalities

$$\frac{J'}{J} = -\frac{R}{Q}, \quad \frac{J''}{J'} = \frac{1}{6} P - \frac{2}{3} \frac{R}{Q} - \frac{1}{2} \frac{Q^2}{R},$$
we have \( Q \in \mathbb{Q}(J, J', J'') \), and hence
\[
\mathbb{Q}(P, Q, R) = \mathbb{Q}(J, J', J'') = K, \text{ say.}
\]
We note that \( K \) is a differential field, i.e., closed under the derivation \( ' ' \). Now we state

Nesterenko’s theorem ([19], [20]). If \( \alpha \in \mathbb{C}, \ 0 < |\alpha| < 1 \), then
\[
\text{trans.deg}_Q \mathbb{Q}(\alpha, P(\alpha), Q(\alpha), R(\alpha)) \geq 3.
\]

**Corollary 1.** If \( \alpha \in \overline{\mathbb{Q}}, \ 0 < |\alpha| < 1 \), then each of the following set
1) \( P(\alpha), Q(\alpha), R(\alpha) \), 2) \( J(\alpha), J'(\alpha), J''(\alpha) \) are algebraically independent.

**Corollary 2.** The numbers \( \pi, e^\pi \), and \( \Gamma(1/4) \) are algebraically independent.

Let \( \eta(q) \) be the eta function defined by
\[
\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
\]
which is known to satisfy
\[
\eta(q)^{24} = \Delta(q).
\]

**Corollary 3.** If \( \alpha \in \overline{\mathbb{Q}}, \ 0 < |\alpha| < 1 \), then
\[
\text{trans.deg}_Q \mathbb{Q}(\alpha, \eta(\alpha), \eta'(\alpha), \eta''(\alpha)) \geq 3.
\]

In particular, the infinite product \( \prod_{n=1}^{\infty} (1 - \alpha^n) \) is transcendental for any \( \alpha \in \overline{\mathbb{Q}}, \ 0 < |\alpha| < 1 \).

Let \( \vartheta_3, \vartheta = \vartheta_4, \vartheta_2 \) be Jacobi’s theta series defined by
\[
\vartheta_3 = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}, \quad \vartheta = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}, \quad \vartheta_2 = 2q^{1/4} \sum_{n=1}^{\infty} q^{n(n-1)}.
\]

**Corollary 4** (Bertrand [5]). Let \( y = y(q) \) be one of \( \vartheta_3, \vartheta, \vartheta_2 \). If \( \alpha \in \overline{\mathbb{C}}, \ 0 < |\alpha| < 1 \), then
\[
\text{trans.deg}_Q \mathbb{Q}(\alpha, y(\alpha), y'(\alpha), y''(\alpha)) \geq 3.
\]

In particular, the number \( \sum_{n=1}^{\infty} \alpha^n \) is transcendental for any \( \alpha \in \overline{\mathbb{Q}}, \ 0 < |\alpha| < 1 \).
We note that Corollary 4 provides the best possible results as $y$ is known to satisfy an algebraic differential equations of the third order defined over $\mathbb{Q}$ (cf. Jacobi [13]). A survey on Nesterenko’s theorem can be found in Waldschmidt [23].

The following lemmas are useful to prove the transcendency of some numbers related to modular functions.

**Lemma 1**([10]). Let $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$. If a nonconstant function $f(q)$ is algebraic over $K$ and defined at $\alpha$, then $f(\alpha)$ is transcendental.

**Lemma 2**([10]). Let $y = y(q)$ be one of the functions $\eta, \vartheta_3, \vartheta, \vartheta_2$. Then $y(q^k), y'(q^k), y''(q^k), \ldots$ are algebraic over $K$ for any positive integer $k$.

The Rogers–Ramanujan continued fraction $RR(q)$ is defined by

$$RR(q) = 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}$$

which is known to have the expressions

$$RR(q) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \cdot \prod_{k=0}^{\infty} \frac{(1-q^{5k+2})(1-q^{5k+3})}{(1-q^{5k+1})(1-q^{5k+4})}$$

(cf. [4; Chap. 16, Entry 15, 38(iii)]). Irrationality measures for some values of this continued fraction were given by Osgood [21] and Shiokawa [22]. The latter proved that for any integer $d \geq 2$ there is a constant $C = C(d) > 0$ such that

$$\left| RR\left( \frac{1}{d} \right) - \frac{p}{q} \right| > Cq^{-2-B/\sqrt{\log q}}$$

for all integers $p, q (\geq 2)$, where $B = \sqrt{\log d}$. Matala–Aho [18] obtained some higher degree irrationality results. For example, $RR((\sqrt{5} - 1)/2) \notin \mathbb{Q}(\sqrt{5})$.

**Theorem 1**([11]). The Rogers–Ramanujan continued fraction $RR(\alpha)$ is transcendental for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$. 

Proof. Let
\[ F(q) = \frac{q^{1/5}}{1} \frac{q}{1} \frac{q^2}{1} \frac{q^3}{1} \frac{q^4}{1} \cdots, \]
then
\[ \frac{1}{F(q)} - F(q) - 1 = q^{-1/5} \prod_{n=1}^{\infty} \frac{(1-q^{n/5})}{(1-q^{5n})} = \frac{\eta(q^{1/5})}{\eta(q^5)} \]
(see [4; p.85]). Applying Lemma 1 and 2 to the function \( f(q) = \eta(q)/\eta(q^{25}) \), we see that \( f(\alpha) \) is transcendental for any \( \alpha \in \overline{\mathbb{Q}} \), \( 0 < |\alpha| < 1 \), and so is \( F(\alpha) \) from the formula above.

We give here further examples of continued fractions whose transcendence can be easily deduced from Lemma 1 and 2. For any \( \alpha \in \overline{\mathbb{Q}} \), \( 0 < |\alpha| < 1 \), the following continued fractions (i), (ii), (iii) are transcendental:

(i) \[ \frac{1}{1 + \frac{\alpha}{1 + \alpha + \frac{\alpha^2}{1 + \alpha^2 + \frac{\alpha^3}{1 + \alpha^3}} + \cdots} } \] (see [4; Chap.19, Entry 1 (i)]).

(ii) \[ \frac{1}{1 + \frac{\alpha^2}{1 + \alpha + \frac{\alpha^4}{1 + \alpha^4 + \frac{\alpha^6}{1 + \alpha^6}} + \cdots} } \] (see [4; Chap.19, Entry 1 (ii)]).

(iii) \[ \frac{1}{1 + \frac{\alpha + \alpha^2}{1 + \frac{\alpha^2 + \alpha^4}{1 + \frac{\alpha^3 + \alpha^6}{1 + \cdots} }} } \] (see [4; Chap.20, Entry 1]).

Let \( \alpha \) and \( \beta \) be algebraic numbers with \( \alpha \neq \beta \) and \( |\beta| < 1 \). Put
\[ U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n. \]

**Theorem 2**([11]). If \( \alpha \beta = \pm 1 \), then the numbers
\[ \sum_{n=1}^{\infty} \frac{1}{U_{2n}^s}, \quad \sum_{n=1}^{\infty} \frac{1}{V_{2n}^s} \]
are transcendental for any positive integer \( s \).

**Theorem 3**([11]). If \( \alpha \beta = 1 \), then the number
\[ \sum_{n=1}^{\infty} \frac{1}{V_{2n}^s} \]
is transcendental for any positive integer \( s \).
**Theorem 4** ([11]). If \( \alpha \beta = -1 \), then the number

\[
\sum_{n=1}^{\infty} \frac{1}{U_{2n-1}^{s}}
\]

is transcendental for any positive integer \( s \).

In the special case of \( s = 1 \), these theorems are proved in [10] by direct calculation without using Lemma 3 below.

For the proof, we need another lemma. Let

\[
k = \frac{\theta_2^2(q)}{\theta_3^2(q)}
\]

be the modulus of the complete elliptic integrals

\[
K = \int_{0}^{1} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}, \quad E = \int_{0}^{1} \frac{\sqrt{1 - k^2t^2}}{\sqrt{1 - t^2}} dt,
\]

of the first and the second kind, respectively. Then we have

\[
\frac{K}{\pi} = \frac{1}{2} \theta_3^2(q), \quad \frac{E}{\pi} = \frac{K}{\pi} + \frac{\pi}{K} \frac{\vartheta'(q)}{\vartheta(q)},
\]

where \( \vartheta' = q \frac{d\vartheta}{dq} \) (cf. [6; (2.1.13), (2.3.17)]).

**Lemma 3** ([11]). Let \( s \) be any positive integer and let

\[
f_{2s}(q) = \sum_{n=1}^{\infty} \frac{1}{(q^{-n} - q^n)^{2s}}, \quad g_{s}(q) = \sum_{n=1}^{\infty} \frac{1}{(q^{-n} + q^n)^s}.
\]

Then \( f_{2s}(q), f_{2s}(q^2), g_{s}(q) \), and \( g_{s}(q^2) \) are algebraic over the field \( \mathbb{Q}(P(q), Q(q), R(q)) \).

**Proof.** Let \( s \) be a positive integer. We put

\[
I_{2s} = \sum_{n=1}^{\infty} \text{cosech}^{2s}(n\pi c) = \sum_{n=1}^{\infty} \left( \frac{2}{q^{-n} - q^n} \right)^{2s}, \quad q = e^{-\pi c},
\]

\[
\Pi_s = \sum_{n=1}^{\infty} \text{sech}^{s}(n\pi c) = \sum_{n=1}^{\infty} \left( \frac{2}{q^{-n} + q^n} \right)^{s},
\]

so that

\[
f_{2s}(q) = 2^{-2s}I_{2s}, \quad g_s(q) = 2^{-s}\Pi_s.
\]
Then Zucker [26] obtained expansions of $I_{2s}$, $\Pi_{s}$, and $\Pi_{2s+1}$ as polynomials of $k$, $K/\pi$, and $E/\pi$ with rational coefficients, which can be found in Table 1(i), 1(ii), and 1(vi) in [26], respectively. Hence the lemma follows from Lemma 2.

**Proof of Theorem 2.** If $\alpha \beta = 1$, then we have

$$(\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_{2n}^{2s}} = \sum_{n=1}^{\infty} \frac{1}{(\beta^{-n} - \beta^{n})^{2s}} = f_{2s}(\beta),$$

and the results follow from Lemma 3 and 1. If $\alpha \beta = -1$, then we have

$$(\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_{2n}^{2s}} = \sum_{n=1}^{\infty} \frac{1}{((-\beta)^{-n} - \beta^{n})^{2s}}$$

and the results follow from Lemma 3 and 1.
Fibonacci numbers \( \{F_n\}_{n \geq 1} \) and Lucas numbers \( \{L_n\}_{n \geq 1} \) are defined by

\[
F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0),
\]

\[
L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0),
\]

and written as

\[
F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n \quad (n \geq 0),
\]

where

\[
\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.
\]

**Corollary**([11]). The numbers

\[
\sum_{n=1}^{\infty} \frac{1}{F_{n^s}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n}^s}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^s}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^s}
\]

are transcendental for any positive integer \( s \).

André-Jeannin [1] proved the irrationality of the number

\[
\sum_{n=1}^{\infty} \frac{1}{F_n}.
\]

Duverney [8] gave another proof and Kato [14] showed by Duverney’s method that the number

\[
\sum_{n=1}^{\infty} \frac{1}{F_{an}}
\]

is irrational for any positive integer \( a \). It is not known whether these numbers are transcendental or not. Bundschuh and Väänänen [7] gave an irrationality measure for \( \sum_{n=1}^{\infty} F_{n^{-1}} \); namely

\[
\left| \sum_{n=1}^{\infty} \frac{1}{F_n} - \frac{p}{q} \right| > \frac{1}{q^{8.621}}
\]

holds for all rationals \( p/q \) with sufficiently large \( q \).

Finally, we state two problems which are interesting in comparison with the arithmetical properties of the values of the Riemann zeta function \( \zeta(s) \) at \( s = 2, 3, 4, \ldots \).

**Problem 1.** Is the number

\[
\sum_{n=1}^{\infty} \frac{1}{F_n^3}
\]
irrational?

**Problem 2.** Are the numbers

\[ \sum_{n=1}^{\infty} \frac{1}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^4}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^6} \]

algebraically independent?

**References**


[13] C. G. J. Jacobi, Über die Differentialalgleichung, welcher die Reihen \( 1 \pm 2q \pm 2q^4 \pm 2q^9 \pm \text{etc.}, 2\sqrt{q} + 2\sqrt{q^5} + 2\sqrt{q^{25}} + \text{etc.} \), Genüge leisten. J. Reine Angew. Math. 36 (1847), 97–112.
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