Farey series and the Riemann hypothesis

Masami Yoshimoto (吉元 昌己 · 九州大学大学院)

The aim of this paper is to consider the equivalent conditions to the Riemann hypothesis in terms of Farey series.

Farey series $F_x$ of order $x$ is the sequence of all irreducible fractions in $(0, 1]$ with denominator not bigger than $x$, arranged in increasing order of magnitude;

$$F_x = F[x] = \left\{ \frac{b_\nu}{c_\nu} \mid (b_\nu, c_\nu) = 1, 0 < b_\nu \leq c_\nu \leq x \right\}$$

and the cardinality of $F_x$ is the summatory function of Euler’s function

$$\# F_x = \Phi(x) = \sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x) \quad \text{(Mertens)}.$$ 

This asymptotic formula is due to Mertens.

For example, from $F_2$ we form $F_3$:

$$F_2 = \left\{ \frac{1}{2}, \frac{1}{1} \right\} \rightarrow F_3 = \left\{ \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\},$$

and so on.

And the Riemann hypothesis RH states that the Riemann zeta function does not vanish for real part $\sigma$ of $s$ bigger than $\frac{1}{2}$. It is well known that the RH is equivalent to each of the following asymptotic formulas, forms of the prime number theorem:

$$\text{RH} \iff \zeta(s) \neq 0 \text{ for } \sigma := \Re s > \frac{1}{2}$$

$$\iff M(x) := \sum_{n \leq x} \mu(n) = O \left( x^{\frac{1}{2} + \epsilon} \right)$$

$\mu(n)$: Möbius' function

$$\iff \psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \log p$$

$$= x + O \left( x^{\frac{1}{2} + \epsilon} \right),$$

$\Lambda(n)$: von Mangoldt's function

$\psi(x)$: Chebyshev's function
Here the weak Riemann hypothesis RH(\(\alpha\)) states that \(\zeta(s)\) does not vanish for \(\sigma > \alpha\):

\[
\text{RH}(\alpha) \iff \zeta(s) \neq 0 \text{ for } \sigma > \alpha.
\]

In this paper I'd like to state main results of Part V [9] and Part VI [6] of this series of papers. In Part V we aim at the implications of the RH(\(\alpha\)) on the estimates of error terms associated to Farey series, with occasional acquisition of equivalent conditions.

**Principle.**
Suppose \(f\) has a bounded derivative and consider the error term \(E_f(x)\) defined by

\[
E_f(x) := \sum_{\nu=1}^{\Phi(x)} f(\rho\nu) - \Phi(x) \int_{0}^{1} f(u) du.
\]

Suppose the RH implies the estimate:

\[
\text{RH} \implies E_f(x) = O \left( x^{\frac{1}{2} + \epsilon} \right),
\]

and that the Mellin transform \(F(s)\) defined by

\[
F(s) = s \zeta(s) \int_{1}^{\infty} E_f(x) x^{-s-1} dx \quad \text{for } \sigma > 1
\]

satisfies following conditions:

(i) \(F(s)\) is regular for \(\sigma > \frac{1}{2}, \ s \neq 1\),

(ii) \(F(s) \neq 0\) for \(\frac{1}{2} < \sigma < 1\).

Then

\[
\text{RH} \iff E_f(x) = O \left( x^{\frac{1}{2} + \epsilon} \right).
\]

We note that if we define the arithmetic function \(a(n)\) by

\[
a(n) = \sum_{k=1}^{n} f \left( \frac{k}{n} \right) - n \int_{0}^{1} f(u) du,
\]

then \(E_f(x)\) can be written as

\[
E_f(x) = \sum_{n \leq x} (\mu \ast a)(n) = \sum_{n \leq x} M \left( \frac{x}{n} \right) a(n),
\]

where \(\ast\) denotes the Dirichlet convolution, and \(F(s)\) becomes the generating function of \(a(n)\):

\[
F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.
\]
Theorem 1 (Part V, Theorem 2). Suppose $f$ is an integrable function on $(0,1)$. Let $a(n)$ be the arithmetic function defined by (1), let $F(s)$ be the generating function (2) of $a(n)$, and we suppose that $F(s)$ and $a(n)$ satisfy following conditions (i)–(v):

(i) $F(s)$ is absolutely convergent for $\sigma > \sigma_a$ with $\sigma_a \leq 1$,
(ii) $F(s)$ is continued to an analytic function with finitely many singularities in the half-plane $\sigma > \alpha$,
(iii) $F(s) \ll |t|^\kappa + \epsilon$ for some $\kappa \geq 0$ and every $\epsilon > 0$ uniformly in the region $\alpha < \sigma \leq 1$, $|t| \geq t_0 > 0$,
(iv) $(\mu * a)(n) \ll n^\beta + \epsilon$ for some $\beta$, $0 \leq \beta \leq \sigma_a$,
(v) there exists a non-negative number $\theta$ satisfying

$$\sum_{n=1}^{\infty} \frac{|(\mu * a)(n)|}{n^\sigma} \ll (\sigma - 1)^{-\theta} \quad \text{as} \quad \sigma \to 1.$$

Then, on the RH($\alpha$), we have the (asymptotic) formula:

$$\sum_{n \leq x} (\mu * a)(n) = \frac{1}{2\pi i} \int_C \frac{F(s)}{s\zeta(s)} x^s ds + O \left( x^{\omega+\epsilon} \right),$$

where

$$\omega = \min_{0 \leq \xi \leq 1} \{ \max \{ \beta + 1 - \xi, 1 + (\kappa - 1)\xi, \alpha + \kappa\xi \} \},$$

and the contour $C$ encircles all singularities of $F(s)/\zeta(s)$ in the strip $\alpha < \sigma < 1$.

In particular, in the special cases of $\kappa = 0$ and $\beta = 0$ we have $\omega = \max \{ \alpha, \beta \}$, and $\omega = \alpha + \kappa(1 - \alpha)$, respectively.

Corollary 1 (Codecà-Perelli [2], Theorem 1). (i) Let $f(u)$ be absolutely continuous and let $f' \in L^p[0,1]$ for some $p \in (1,2]$. Then, on the RH($\eta$), we have

$$E_f(x) = O \left( x^{\max \{ \eta, \frac{1}{p} \} + \epsilon} \right).$$

(This covers the Main result of Codecà-Perelli, Theorem 1.)

(ii) Moreover, if $F(s)$ satisfies the conditions (i)–(iii) of Theorem 1 and $0 \leq \kappa \leq \frac{2}{p} - 1$. Then, on the RH($\alpha$),

$$E_f(x) = O \left( x^{\alpha + \kappa(1 - \alpha) + \epsilon} \right).$$

Corollary 2. For any rational number $\frac{r}{q} \in (0,1)$ other than $\frac{1}{2}$, the RH implies

$$E \left( \frac{r}{q}; x \right) := \sum_{\nu \leq \frac{r}{q}} 1 - \frac{r}{q} \Phi(x) = O \left( x^{\frac{1}{2} + \frac{31}{32} + \epsilon} \right),$$

by the result of Kolesnik.

Moreover, if we assume the GRH (on some Dirichlet $L$-functions mod $q$) and the RH, we have the estimate:

$$E \left( \frac{r}{q}; x \right) = O \left( x^{\frac{1}{2} + \epsilon} \right). \quad \text{(Codecà [1])}$$
We can not only cover the strong result of Codecà-Perelli [2] (Corollary 1), some results of Codecà [1] and their developments as above, but also we can widen the width of validity of the parameter by $\frac{1}{2}$ of some theorems proved earlier.

In particular, on the GRH, the RH is equivalent to each of estimates

$$E\left(\frac{1}{3}; x\right) = O\left(x^{\frac{1}{2}+\epsilon}\right)$$

and

$$E\left(\frac{1}{4}; x\right) = O\left(x^{\frac{1}{2}+\epsilon}\right).$$

**Theorem 2** (Part VI). If $f(u)$ is the gap-Fourier series;

$$f(u) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^{mk_{1}}(2n+1)^{k_{2}}} \cos 2\pi 2^{m_{1}}(2n+1)^{l_{2}} u,$$

with $k_{1}, k_{2} \in \mathbb{C}, l_{1}, l_{2} \in \mathbb{N}, 2\Re k_{1} \geq l_{1}+1$ and $2\Re k_{2} \geq l_{2}+2$, then we have the equivalence:

$$\text{RH} \iff E_{f}(x) = O\left(x^{\frac{1}{2}+\epsilon}\right).$$

**Corollary 3.** $k_{1} = l_{1} = 2 = 1, k_{2} = 2 \implies f(u)$ is Takagi's function, and

$$F(s) = \frac{3}{2} \frac{1-2^{-s-1}}{1-2^{-s}} \zeta(s) \zeta(s+1) \neq 0 \text{ for } \sigma > \frac{1}{2}.$$

Hence

$$\text{RH} \iff E_{f}(x) = O\left(x^{\frac{1}{2}+\epsilon}\right).$$

($k_{1} = k_{2} = l_{1} = l_{2} = 2 \implies f: \text{Riemann's function}$)

Recall that if $E_{f}(x) = (M \ast a)(x)$ with suitable $a(n)$, then

$$F(s) = s\zeta(s) \int_{1}^{\infty} E_{f}(x)x^{-s-1}dx = \sum_{n=1}^{\infty} \frac{a(n)}{n^{s}},$$

and vice versa.

Hence, when $f(u)$ has a Fourier expansion

$$f(u) = \sum_{n=1}^{\infty} c(n) \cos 2\pi nu$$

satisfying the condition;

$$\sum_{n=1}^{\infty} |c(n)|d(n) < \infty,$$
then, with $a(n) = n \sum_{m=1}^{\infty} c(mn)$, we have the Ramanujan expansion:

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \sum_{n=1}^{\infty} c(n) \sigma_{1-s}(n), \quad \sigma > 1$$

($E_f = M \ast a$ also holds).

Conversely, if $F(s)$ is the generating Dirichlet series:

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \sigma \geq 1,$$

the Fourier coefficient of the corresponding $f$ is given by

$$c(n) = \frac{1}{n} \sum_{k=1}^{\infty} \frac{\mu(k)}{k} a(kn).$$

This is a Hecke-like correspondence.

Example.

\[
\begin{array}{ccc}
\sum_{n=1}^{\infty} c(n)e^{2\pi in\tau} & \rightarrow & \sum_{n=1}^{\infty} c(n)\sigma_{1-s}(n) \\
\cos 2\pi \tau & \rightarrow & \zeta(s) \\
\log 2|\sin \pi x| & \rightarrow & \psi(x) \\
B_{2n}(\tau) & \rightarrow & \sum_{m \leq x} M\left(\frac{x}{m}\right) \frac{1}{m^{2n-1}}
\end{array}
\]

Here $B_k(x)$ is the $k$-th Bernoulli polynomial.

Theorem 3 (Part VI). (i) Let $f_{k,l}(u)$ be a gap Fourier series

$$f_{k,l}(u) := \sum_{n=1}^{\infty} \frac{1}{n^k} \cos 2\pi n^l u \quad \text{for } \Re k > 1, l \in \mathbb{N}.$$

Then $F_{k,l}$ can be decomposed with $G_{k,l}$ having an Euler product as follows:

$$F_{k,l}(s) = \zeta(k)\zeta(ls+k-l)G_{k,l}(s),$$

$$G_{k,l}(s) = \prod_{p} \left(1 + p^{-k} \sum_{n=1}^{l-1} p^{n(1-s)}\right).$$
(ii) If \(2\Re k \geq l + 2\), we have

\[ \text{RH} \iff E_{f_{k,l}}(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right). \]

(iii) For \(k = 2, l = 3\)

\[ \text{RH} \iff E_{f_{2,3}}(x) = \frac{\zeta(2) G_{2,3}\left(\frac{2}{3}\right)}{2\zeta\left(\frac{2}{3}\right)} x^{\frac{2}{3}} + O\left(x^{\frac{1}{2}+\varepsilon}\right), \]

\[ G_{2,3}\left(\frac{2}{3}\right) = \prod_p \left(1 + p^{-\frac{2}{3}} + p^{-\frac{1}{3}}\right). \]

References


Graduate School of Mathematics
Kyushu University
Fukuoka 812-8581
Japan