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An Approximation Scheme for Gauss Curvature Flow

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§1 Introduction. We propose here an approximation scheme for Gauss curvature flow of a convex hypersurface in $\mathbb{R}^{n+1}$ and explain how to prove the convergence of the scheme to the Gauss curvature flow.

The Gauss curvature flow of a convex hypersurface in $\mathbb{R}^{n+1}$ is described as follows. Let $\Gamma_0$ be a convex hypersurface of $\mathbb{R}^{n+1}$ and $F_0 : S^n \rightarrow \mathbb{R}^{n+1}$ be a parametric representation of $\Gamma_0$. The Gauss curvature flow of this hypersurface $\Gamma_0$ is a collection $F(\cdot, t) : S^n \rightarrow \mathbb{R}^{n+1}$ of closed hypersurfaces with parameter $t \in [0, T)$ which is a solution of the initial value problem

$$
\begin{align*}
\frac{\partial F}{\partial t}(s, t) &= -K^{\beta}(s, t)n(s, t) \\
F(s, 0) &= F_0(s)
\end{align*}
(1.1)
$$

where $\beta > 0$ is a constant and $K(s, t)$ and $n(s, t)$ denote the Gauss curvature and the outward unit normal vector, respectively, at $F(s, t)$ of the hypersurface $F(\cdot, t)$.

W. J. Firey [F] proposed problem (1.1) with $\beta = 1$ as a mathematical model of the wearing process of stones on beach by waves and studied some basic properties of the solution $F$ of this problem. Afterwards, K. Tso [T] and then B. Chow [C] studied problem (1.1) and established the following existence theorem.

Theorem 1. If $F_0$ represents a smooth, strictly convex hypersurface, then there exist a positive $T > 0$ and a unique smooth solution $F : S^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$ of (1.1)
such that \( F(\cdot, t) \) represents a strictly convex set \( \Gamma_t \equiv F(S^n, t) \subset \mathbb{R}^{n+1} \) for all \( 0 < t < T \) and \( \Gamma_t \) converge to a point as \( t \to T \).

Another way of describing Gauss curvature flow is the so-called level-set approach, in which the evolving convex hypersurfaces \( \Gamma_t \) are regarded as the 0-level set of a function \( u \) defined on \( \mathbb{R}^{n+1} \times [0, \infty) \). More precisely, the idea is explained as follows. Given a compact convex hypersurface \( \Gamma_0 \), we choose a function \( g \in BUC(\mathbb{R}^{n+1}) \) so that

\[
\Gamma_0 = \{ z \in \mathbb{R}^{n+1} \mid g(z) = 0 \} \quad \text{and} \quad \{ z \in \mathbb{R}^{n+1} \mid g(z) \leq 0 \} \quad \text{is convex,}
\]

and consider the initial value problem

\[
\begin{aligned}
(1.3) \quad & \begin{cases}
(i) \ & u_t + G(Du, D^2u) = 0 \quad \text{in} \quad \mathbb{R}^{n+1} \times (0, \infty), \\
(ii) \ & u(z, 0) = g(z) \quad (z \in \mathbb{R}^{n+1}).
\end{cases}
\end{aligned}
\]

Here the function \( G : (\mathbb{R}^{n+1} \setminus \{0\}) \times S(n+1) \to \mathbb{R} \) is defined by

\[
G(p, X) = -|p| \left\{ \det_+ \left( \frac{1}{|p|} (I - \overline{p} \otimes \overline{p}) X (I - \overline{p} \otimes \overline{p}) + \overline{p} \otimes \overline{p} \right) \right\}^\beta
\]

where \( \overline{p} = p/|p| \) and

\[
\det_+ A = \prod_{i=1}^{n+1} \max\{\lambda_i, 0\} \quad \text{for} \quad A \in S(n+1),
\]

with \( \lambda_i \) \((i = 1, ..., n+1)\) denoting the eigenvalues of \( A \in S(n+1) \). Now, the (generalized) Gauss curvature flow of \( \Gamma_0 \) is defined as the collection \( \{ \Gamma_t \}_{t \geq 0} \) of the closed subsets

\[
(1.4) \quad \Gamma_t = \{ z \in \mathbb{R}^{n+1} \mid u(z, t) = 0 \} \subset \mathbb{R}^{n+1}.
\]

One of main results in [IS] (see [IS, Theorems 1.8 and 1.9]) states that if \( g \in BUC(\mathbb{R}^{n+1}) \), then there is a unique viscosity solution \( u \in BUC(\mathbb{R}^{n+1} \times [0, \infty)) \) of (1.3) and that the collection \( \{ \Gamma_t \}_{t \geq 0} \) defined by (1.4) is independent of the choice of \( g \). See [IS] as well for the correct definition of viscosity solution for (1.3), (i). This assertion is,
of course, a generalization of a well-known, similar observation due to Chen-Giga-Goto [CGG] and Evans-Spruck [ES] for mean curvature flow and alike.

An argument in [CEI] guarantees that under assumption (1.2), if we set

\[(1.5) \quad V_t = \{ z \in \mathbb{R}^{n+1} \mid u(z, t) \leq 0 \}, \]

then \( V_t \) is a convex set and \( \Gamma_t = \partial V_t \) for all \( t \geq 0 \). We shall also call the collection \( \{ V_t \}_{t \geq 0} \) the generalized Gauss curvature flow of the convex body \( V_0 \).

See [CEI] for a discussion on the consistency of this level-set approach and the parametric representation approach based on (1.1).

In what follows we discuss only on the generalized Gauss curvature flow defined via the level-set approach as above and hence suppress the word "generalized" in the argument below.

\[\S 2 \quad \text{An approximation scheme and the main result.} \quad \]

Now, we introduce an approximation scheme for Gauss curvature flow. We need notation. We denote by \( C(m) \) the collection of all closed subsets of \( \mathbb{R}^m \). Let \( A \in C(n+1) \) and \( p \in S^n \). Define

\[(2.1) \quad \ell_0(A, p) = \sup \{ \langle z, p \rangle \mid z \in A \}. \]

Of course, \( \ell_0(\emptyset, p) = -\infty \) and, if the set \( \{ \langle z, p \rangle \mid z \in A \} \) is not bounded above, \( \ell_0(A, p) = \infty \).

For \( t > 0 \) define \( S(A, p, t) \) and \( C(A, p, t) \), subsets of \( \mathbb{R}^{n+1} \), by

\[(2.2) \quad S(A, p, t) = \{ z \in A \mid \langle z, p \rangle \leq \ell_0(A, p) - t \} \]

and

\[(2.3) \quad C(A, p, t) = \{ z \in A \mid \langle z, p \rangle > \ell_0(A, p) - t \} \quad (= A \setminus S(A, p, t)). \]
Moreover, we define $d(A, p, t) \in [0, \infty]$ by

\begin{equation}
(2.4) \quad d(A, p, t) = \inf\{s > 0 \mid \mathcal{L}^{n+1}(C(A, p, s)) \geq t\},
\end{equation}

where $\mathcal{L}^{n+1}(B)$ denotes the $(n+1)$-dimensional Lebesgue measure of the set $B$, and for $\mu > 0$ set

\begin{equation}
(2.5) \quad d_\mu(A, p, t) = \min\{d(A, p, t), \mu\}
\end{equation}

Finally, for any $A \in C(n+1)$, $h > 0$, and $\mu > 0$ we define

\begin{equation}
(2.6) \quad T_h^\mu(A) = \bigcap_{p \in S^n} S\left(A, p, d_\mu(A, p, \alpha_n h^{\frac{1}{2\beta}})^{\beta(n+2)}\right),
\end{equation}

where

$$\alpha_n = \frac{2^{n+2} \omega_n}{n + 2}, \text{ with } \omega_n = \text{the volume of the unit ball } \subset \mathbb{R}^n.$$

It is clear that for all $A \in C(n+1)$, $T_h^\mu(A) \subset A$ and $T_h^\mu(A) \in C(n+1)$ and if $A$ is convex then so is $T_h^\mu(A)$.

Fix a compact convex set $V_0 \subset \mathbb{R}^{n+1}$. Fix $h > 0$ and $\mu > 0$. Define the sequence $\{C_i\}_{i \in \mathbb{N}}$ of subsets of $\mathbb{R}^{n+1}$ by the recursion formula

$$C_1 = V_0 \quad \text{and} \quad C_{i+1} = T_h^\mu(C_i) \quad \text{for } i \in \mathbb{N}.$$
and the collection \( \{V_t^{\mu,h}\}_{t \geq 0} \) of subsets of \( \mathbb{R}^{n+1} \) by

\[
V_t^{\mu,h} = C_i \quad \text{if} \quad (i-1)h \leq t < ih \quad \text{and} \quad i \in \mathbb{N}.
\]

This collection \( \{V_t^{\mu,h}\}_{t \geq 0} \), with \( \mu > 0 \) and \( h > 0 \), is our approximation scheme for the Gauss curvature flow \( \{V_t\}_{t \geq 0} \) defined by (1.5).

The main result in this paper is the following

**Theorem 2.** Assume that \( V_0 \) is compact and convex, \( \beta \geq \frac{1}{n+2} \), and \( \mu \in \left(0, \frac{1}{6}\right) \). For each \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( 0 < h < \delta \), then

\[
\bigcup_{t \geq 0} V_t \times \{t\} \subset \bigcup_{t \geq 0} V_t^{\mu,h} \times \{t\} + B^{n+2}(0, \epsilon)
\]

and

\[
\bigcup_{t \geq 0} V_t^{\mu,h} \times \{t\} \subset \bigcup_{t \geq 0} V_t \times \{t\} + B^{n+2}(0, \epsilon),
\]

where \( B^{n+2}(0, \epsilon) \) denotes the closed ball of radius \( \epsilon \) and center at the origin in \( \mathbb{R}^{n+2} \). That is, as \( h \searrow 0 \), the sets \( \bigcup_{t \geq 0} V_t^{\mu,h} \times \{t\} \) converge to the set \( \bigcup_{t \geq 0} V_t \times \{t\} \) in the Hausdorff distance.

The underlying idea of our definition of \( T_h^\mu \) (or our approximation scheme) can be explained as follows. Let \( A \in C(n+1) \) be a smooth, strictly convex domain and \( B_0 \in \partial A \).

Now, we assume that \( B_0 = 0 \) and \( (0, -1) \in \mathbb{R}^n \times \mathbb{R} \) is the outward unit normal vector of \( A \) at 0. Writing \( z = (x, y) \in \mathbb{R}^n \times \mathbb{R} \) for generic points in \( \mathbb{R}^{n+1} \), we may claim that in a neighborhood of 0, the set \( A \) is almost identical to the paraboloid

\[
P = \{(x, y) \mid y \geq \frac{1}{2}(\kappa_1 x_1^2 + \cdots + \kappa_n x_n^2)\},
\]
where \( \kappa_i \) denotes the principal curvatures of the surface \( \partial A \) at 0.

For \( d > 0 \) we denote
\[
P(d) = P \cap \{(x, y) \mid y \leq d\},
\]
and compute the volume of the set \( P(d) \), to find
\[
\mathcal{L}^{n+1}(P(d)) = \alpha_n \frac{d^{\frac{n+2}{2}}}{\sqrt{K}},
\]
(2.10)

\( K \) denoting the Gauss curvature of \( \partial A \) at 0, i.e., \( K = \prod_{i=1}^{n} \kappa_i \).

If a convex body, starting with \( A \) at time 0, is shrinking with velocity
\[
v = K^\beta,
\]
in the directions of inward normal vectors of \( A \), then the boundary point of the convex body at time \( h > 0 \) with \((0, -1)\) as its outward normal direction must be somewhere near the point \( B_1 \) with coordinates
\[
(0, K^\beta h).
\]

If we set
\[
d = (K^\beta h)^{\frac{1}{\beta(n+2)}},
\]
and plug this into (2.10), then we find the formula
\[
\mathcal{L}^{n+1}(P(d)) = \alpha_n h^{\frac{1}{2\beta}}.
\]
(2.11)

A nice feature in formula (2.11) is that it does not involve the Gauss curvature \( K \) explicitly any more. Moreover, the formula determines \( d \) uniquely as a function of \( h > 0 \). Thus, reversing the above process, i.e., fixing first \( d > 0 \) by formula (2.11) and then setting \( a = d^{\beta(n+2)} \), we can identify the point \( B_1 \) as the point with coordinates \((0, a)\) without knowing the Gauss curvature \( K \).

Roughly speaking, the set \( T_h^\beta(A) \) is defined as the convex hull of all the points \( B_1 \) obtained from \( B_0 \in \partial A \) by the process described above.
§3 Some properties of $T_{h}^{\mu}$. We begin with the following proposition, which says that the mapping $T_{h}^{\mu}$ is invariant under translation and orthogonal transformation. Henceforth $h$, $\beta$, and $\mu$ denote fixed positive constants.

**Proposition 1.** For any $A \in C(n+1)$, $U \in O(n+1)$, and $z \in \mathbb{R}^{n+1}$ we have

\[(3.1) \quad T_{h}^{\mu}(U(A)) = U(T_{h}^{\mu}(A)),\]

\[(3.2) \quad T_{h}^{\mu}(z + A) = z + T_{h}^{\mu}(A).\]

Here and later $O(n+1)$ denotes the set of orthogonal matrices of order $n+1$.

The proof is straightforward and left to the reader.

The next proposition asserts the monotonicity of $T_{h}^{\mu}$.

**Proposition 2.** Assume that $\beta \geq \frac{1}{n+2}$ and $0 < \mu < \frac{1}{6}$. Then, for any $A, B \in C(n+1)$, if $A \subset B$ we have

\[(3.3) \quad T_{h}^{\mu}(A) \subset T_{h}^{\mu}(B).\]

**Remark.** The restriction that $\beta \geq \frac{1}{n+2}$ and $\mu < \frac{1}{6}$ in Theorem 2 is due to the above proposition. The condition, $\mu < \frac{1}{6}$, is not optimal in this respect and we will not seek for the optimal one here.

The following property will be needed in the proof of Theorem 2.

**Proposition 3.** Assume that $\beta \geq \frac{1}{n+2}$ and $0 < \mu < \frac{1}{6}$. Let $A_{\varepsilon} \in C(n+1)$, with $0 < \varepsilon \leq 1$, be compact and satisfy

\[A_{\varepsilon} \subset A_{\delta} \quad \text{if} \quad \varepsilon < \delta.\]
Then we have

\[
T^\mu_h \left( \bigcap_{0 < \epsilon \leq 1} A_\epsilon \right) = \bigcap_{0 < \epsilon \leq 1} T^\mu_h (A_\epsilon).
\]

§4 Level-set approach. We shall take the level-set approach to proving Theorem 2 and here introduce the level-set approach to our approximation scheme.

In what follows we fix \( \beta \geq \frac{1}{n+2} \) and \( \mu \in (0, \frac{1}{6}) \). Fix \( h > 0 \) as well. For any function \( \varphi \in C(\mathbb{R}^{n+1}) \), following [E], we set

\[
M_{h\varphi}(z) = \inf \{ \lambda \in \mathbb{R} \mid z \in T^\mu_h(\{\varphi \leq \lambda\}) \} \quad (z \in \mathbb{R}^{n+1}).
\]

Here and henceforth we use the notation \( \{P\} \) for \( \{z \mid P(z)\} \), where \( P \) or \( P(z) \) is a proposition concerning \( z \).

We see immediately from (4.1) that for \( \varphi \in C(\mathbb{R}^{n+1}) \) and \( \lambda \in \mathbb{R} \),

\[
T^\mu_h(\{\varphi \leq \lambda\}) \subset \{M_{h\varphi} \leq \lambda\} \quad \text{and} \quad \{M_{h\varphi} \leq \lambda\} \subset \bigcap_{\gamma > \lambda} T^\mu_h(\{\varphi \leq \gamma\}).
\]

Loosely speaking, these say that \( T^\mu_h \) maps the sub-level set of \( \varphi \) of height \( \lambda \) to the sub-level set of \( M_{h\varphi} \) of height \( \lambda \). In other words, the mapping \( T^\mu_h \) on sets of \( \mathbb{R}^{n+1} \) can be understood by seeing the mapping \( M_{h} \) on functions in \( \mathbb{R}^{n+1} \).

Fix \( \varphi \in C(\mathbb{R}^{n+1}) \). Since \( T^\mu_h(\{\varphi \leq \lambda\}) \subset \{\varphi \leq \lambda\} \) for \( \lambda \in \mathbb{R} \), we see that for all \( \varphi \in C(\mathbb{R}^{n+1}) \),

\[
M_{h\varphi} \geq \varphi \quad \text{in} \ \mathbb{R}^{n+1}.
\]
Also, it follows from Proposition 2 that if \( \varphi, \psi \in C(\mathbb{R}^{n+1}) \) and \( \varphi \leq \psi \) in \( \mathbb{R}^{n+1} \), then

\[
M_h \varphi \leq M_h \psi \quad \text{in} \quad \mathbb{R}^{n+1}.
\]

It follows that \( M_h \varphi \) is a real-valued function on \( \mathbb{R}^{n+1} \).

Proposition 1 has direct consequences for \( M_h \). Indeed, for any \( \varphi \in C(\mathbb{R}^{n+1}) \) we have

\[
(M_h \varphi) \circ U = M_h (\varphi \circ U) \quad \text{for all} \quad U \in O(n+1),
\]

where \( U \in O(n+1) \) is regarded as a mapping, and

\[
M_h \circ \tau_y = \tau_y \circ M_h \quad \text{for all} \quad y \in \mathbb{R}^{n+1},
\]

where \( \tau_y \) denotes the translation by \( y \), i.e., \( \tau_y \varphi(z) = \varphi(z-y) \). The proof of these claims are again left to the reader.

Next, we observe that \( M_h \varphi \in UC(\mathbb{R}^{n+1}) \) for all \( \varphi \in UC(\mathbb{R}^{n+1}) \). This will be proved as a consequence of (4.3), (4.5), and the following claim.

Let \( \theta \in C(\mathbb{R}) \) be any nondecreasing function. The claim is:

\[
M_h (\theta \circ \varphi) = \theta \circ (M_h \varphi) \quad \text{for} \quad \varphi \in C(\mathbb{R}^{n+1}).
\]

The proof of this claim, which is again easy, is left to the reader. (See (2.10) in [I1] for a similar observation.)

To conclude the uniform continuity, let \( \varphi \in UC(\mathbb{R}^{n+1}) \) and \( \omega \) denote the modulus of continuity of \( \varphi \). If \( y \in \mathbb{R}^{n+1} \), then

\[
\tau_y \varphi \leq \varphi + \omega(|y|),
\]

and so, using (4.5), (4.3), and (4.6) with \( \theta(r) = r + \omega(|y|) \), we see that

\[
M_h \varphi(z - y) = M_h (\tau_y \varphi)(z) \leq M_h \varphi(z) + \omega(|y|) \quad (z \in \mathbb{R}^{n+1}),
\]
from which follows the uniform continuity of $M_h \varphi$, i.e.,

\[(4.7) \quad |M_h \varphi(z) - M_h \varphi(y)| \leq \omega(|z - y|) \quad \text{for all } z, y \in \mathbb{R}^{n+1}.\]

Similarly, we have

\[(4.8) \quad \|M_h \varphi - M_h \psi\| \leq \|\varphi - \psi\| \quad \text{for } \varphi, \psi \in C(\mathbb{R}^{n+1}),\]

where $\|\varphi\| = \sup_{\mathbb{R}^{n+1}} |\varphi| \in [0, \infty]$.

Also, we easily see that if $c$ is a constant function on $\mathbb{R}^{n+1}$ then

$M_h c = c$.

Our proof of Theorem 2 will be carried out via the following

**Theorem 3.** Let $g \in \text{BUC}(\mathbb{R}^{n+1})$ be such that for any $\lambda < \sup_{\mathbb{R}^{n+1}} g$, the set $\{g \leq \lambda\}$ is compact and convex. Let $u \in \text{BUC}(\mathbb{R}^{n+1} \times [0, \infty))$ be the viscosity solution of (1.3). Define $v_h : \mathbb{R}^{n+1} \times [0, \infty) \to \mathbb{R}$ by

\[(4.9) \quad v_h(z, t) = M_h^i g(z) \quad \text{if } (i-1)h \leq t < ih \text{ and } i \in \mathbb{N},\]

where $M_h^i$ denotes the $i$ times iterates of the mapping $M_h$. Then for each $0 < T < \infty$, as $h \searrow 0$,

\[(4.10) \quad v_h(z, t) \to u(z, t) \quad \text{uniformly on } \mathbb{R}^{n+1} \times [0, T].\]

The above definition (4.9) is a reformulation of (2.7) in terms of the level-set approach. (See the next Proposition.)

Let us state here a corollary of Proposition 3, which gives a better connection between (4.9) and (2.7).

**Proposition 4.** Let $\gamma \in \mathbb{R}$ and $\varphi \in C(\mathbb{R}^{n+1})$ be such that $\{\varphi \leq \gamma\}$ is a compact set. Then, for $z \in \mathbb{R}^{n+1}$, if $M_h \varphi(z) < \gamma$, then we have

\[(4.11) \quad M_h \varphi(z) = \min \{\lambda \in \mathbb{R} \mid z \in T_h^\mu(\{\varphi \leq \lambda\})\}.\]
Note that under the above hypothesis, if \( \lambda < \gamma \) then we have

\[
\{ M_h \varphi \leq \lambda \} \subset T_h^\mu(\{ \varphi \leq \lambda \}).
\]

**Proof.** Assume that \( z \in \mathbb{R}^{n+1} \) satisfies \( \lambda \equiv M_h \varphi(z) < \gamma \).

It follows that if \( t > \lambda \) then

\[
(4.12) \quad z \in M_h(\{ \varphi \leq t \}).
\]

Fix any \( \eta \in (\lambda, \gamma) \). Note that

\[
\{ \varphi \leq \lambda \} = \bigcap_{\lambda < t \leq \eta} \{ \varphi \leq t \}.
\]

Now, from Proposition 3 and (4.12), we have

\[
T_h^\mu(\{ \varphi \leq \lambda \}) = \bigcap_{\lambda < t \leq \eta} T_h^\mu(\{ \varphi \leq t \}) \ni z,
\]

whence follows (4.11). \( \square \)

§5 Approximate derivative of \( M_h \) at \( h = 0 \). In this section we assume that \( \beta \geq \frac{1}{n+2} \) and \( \mu \in (0, \frac{1}{6}) \).

The key observation in the proof of Theorem 3 will be stated in this section, which roughly says that the generator of Gauss curvature flow in terms of the level-set approach, i.e., \(-G\) in (1.3), (i) "approximates the derivative" of \( M_h \) at \( h = 0 \).

Indeed, we have the following two theorems. The reader who is interested in the proof of these theorems should consult [I2].

**Theorem 4.** Let \( \varphi \in C^2(\mathbb{R}^{n+1}) \) satisfy \( D \varphi(\hat{z}) \neq 0 \), with \( \hat{z} \in \mathbb{R}^{n+1} \). Then for each \( \varepsilon > 0 \) there is a constant \( \delta > 0 \) such that

\[
M_h \varphi(z) \leq \varphi(z) + (-G(D \varphi(\hat{z}), D^2 \varphi(\hat{z})) + \varepsilon) h \quad (z \in B^{n+1}(\hat{z}, \delta), \ h \in (0, \delta)).
\]
Theorem 5. Let $\varphi \in C^2(\mathbb{R}^{n+1})$ satisfy $D\varphi(\hat{z}) \neq 0$, with $\hat{z} \in \mathbb{R}^{n+1}$. Assume that

\[
\begin{cases}
\varphi(z) > \varphi(\hat{z}) & \text{if } (z, D\varphi(\hat{z})) \geq 0 \text{ and } z \neq 0, \\
\liminf_{|z| \to \infty} \varphi(z) > \varphi(\hat{z}), \\
-G(D\varphi(\hat{z}), D^2\varphi(\hat{z})) > 0.
\end{cases}
\]

Then for each $\epsilon > 0$ there is a constant $\delta > 0$ such that

\[
M_h\varphi(z) \geq \varphi(z) + (-G(D\varphi(\hat{z}), D^2\varphi(\hat{z})) - \epsilon) h \quad (0 < h \leq \delta, z \in B^{n+1}(\hat{z}, \delta)).
\]

§6 Convergence of the approximation scheme. We shall complete the proof of Theorems 2 and 3 in this section.

We will again assume throughout that $\beta \geq \frac{1}{n+2}$ and $0 < \mu < \frac{1}{6}$.

Proof of Theorem 3. Let $g \in BUC(\mathbb{R}^{n+1})$, $u \in BUC(\mathbb{R}^{n+1} \times [0, \infty))$, and $v_h : \mathbb{R}^{n+1} \times [0, \infty) \to \mathbb{R}$ be as in Theorem 3.

Define

\[
\overline{v}(z,t) = \lim_{r \to 0} \sup \{v_h(y,s) \mid (y,s) \in \mathbb{R}^{n+1} \times [0, \infty) \cap B^{n+2}(z,t;r), 0 < h < r \}
\]

\[
((z,t) \in \mathbb{R}^{n+1} \times [0, \infty)),
\]
\[ v(z, t) = \lim_{r \searrow 0} \inf \{ v_h(y, s) \mid (y, s) \in \mathbb{R}^{n+1} \times [0, \infty) \cap B^{n+2}(z, t; r), 0 < h < r \} \\
((z, t) \in \mathbb{R}^{n+1} \times [0, \infty)). \]

We shall first prove that \( u = \overline{v} = \underline{v} \) in \( \mathbb{R}^{n+1} \times [0, \infty) \), which guarantees that \( v_h(z, t) \to u(z, t) \) uniformly on compact subsets of \( \mathbb{R}^{n+1} \times [0, \infty) \) as \( h \searrow 0 \).

To see this, we prove that \( \overline{v} \) and \( \underline{v} \) are a viscosity subsolution of (1.3), (i) and a viscosity supersolution of (1.3), (i), and that

\[ \lim_{r \searrow 0} \sup \{ |\overline{v}(x, t) - \underline{v}(y, s)| \mid |x - y| < r, 0 < t, s < r \} \leq 0. \]

Then we use a comparison theorem, to find that \( \overline{v} \leq \underline{v} \) in \( \mathbb{R}^{n+1} \times [0, \infty) \). And we see that

\[ u = \overline{v} = \underline{v} \quad \text{on} \quad \mathbb{R}^{n+1} \times [0, \infty). \]

This shows that

\[ v_h(x, t) \to u(x, t) \quad \text{as} \quad h \searrow 0. \]

Here the convergence holds uniformly on any compact subsets of \( \mathbb{R}^{n+1} \times [0, \infty) \).

A simple modification of the above arguments yields the uniform convergence in the whole space \( \mathbb{R}^{n+1} \times [0, \infty) \). \( \square \)

**Proof of Theorem 2.** Fix any compact convex \( V_0 \subset \mathbb{R}^{n+1} \) and choose \( g \in BUC(\mathbb{R}^{n+1}) \) so that \( \{ g \leq 0 \} = V_0 \) and so that for any \( \lambda < \sup_{\mathbb{R}^{n+1}} g \) the set \( \{ g \leq \lambda \} \) is compact and convex. Let \( u \in BUC(\mathbb{R}^{n+1} \times [0, \infty)) \) be the viscosity solution of (1.3) and \( v_h : \mathbb{R}^{n+1} \times [0, \infty) \to \mathbb{R} \) be defined by (4.9).

First we observe in view of Proposition 4 that

\[ V_t^{\mu, h} = \{ v_h(\cdot, t) \leq 0 \} \quad (h > 0, \ t \geq 0). \]

Next, since \( G \leq 0 \), we infer that for each \( z \in \mathbb{R}^{n+1} \) the function \( u(z, t) \) of \( t \geq 0 \) is nondecreasing. By comparison between \( u \) and the constant function \( \sup_{\mathbb{R}^{n+1}} g \), we
see that \( u \leq \sup_{\mathbb{R}^{n+1}} g \) in \( \mathbb{R}^{n+1} \times [0, \infty) \). By comparing each Gauss curvature flow \( \{u(\cdot, t) \leq \lambda \}_{t \geq 0} \) with the Gauss curvature flow of a ball containing the set \( \{g \leq \lambda\} \), we then conclude that

\[
\lim_{t \to \infty} u(z, t) = \sup_{\mathbb{R}^{n+1}} g
\]

uniformly in \( \mathbb{R}^{n+1} \).

Fix \( \gamma > 0 \) so that \( \gamma < \sup_{\mathbb{R}^{n+1}} g \), and then \( T > 0 \) so that

\[
u(z, t) > \gamma \quad (z \in \mathbb{R}^{n+1}, \ t \geq T).
\]

Fix \( \epsilon > 0 \) and, in view of the compactness of the set \( \{u \leq \gamma\} \) and the continuity of \( u \), choose \( \delta \in (0, \gamma) \) so that

\[
\{u \leq \delta\} \subset \{u \leq 0\} + B^{n+2}(0, \epsilon).
\]

By Theorem 3, we can choose \( \eta > 0 \) so that for all \( 0 < h \leq \eta \),

\[
(6.1) \quad |u(z, t) - v_h(z, t)| \leq \delta \quad (z \in \mathbb{R}^{n+1}, \ t \in [0, T]).
\]

Then, if \( 0 < h \leq \eta, (z, t) \in \mathbb{R}^{n+1} \times [0, T] \), and \( v_h(z, t) \leq 0 \), we have

\[
(z, t) \in \{u \leq \delta\} \subset \{u \leq 0\} + B^{n+2}(0, \epsilon).
\]

If \( 0 < h \leq \eta \) and \( z \in \mathbb{R}^{n+1} \), then, since \( u(z, T) > \gamma > \delta \), we see that \( v_h(z, T) > 0 \). Noting that for each \( z \in \mathbb{R}^{n+1} \) the function \( v_h(z, t) \) of \( t \geq 0 \) is nondecreasing, we find that if \( 0 < h \leq \eta \), then \( v_h(z, t) > 0 \) for all \( (z, t) \in \mathbb{R}^{n+1} \times [T, \infty) \).

Thus we see that

\[
\{v_h \leq 0\} \subset \{u \leq 0\} + B^{n+2}(0, \epsilon),
\]

proving (2.9).

The other inclusion is observed with a little more care. Indeed, we need to divide our considerations into two cases.

Case 1: \( \text{Int} V_0 = \emptyset \). The extinction time in this case is zero (see [CEI]), i.e., \( V_t = \emptyset \) for \( t > 0 \). Therefore, it is obvious that \( \{u \leq 0\} \subset \{v_h \leq 0\} \) for all \( h > 0 \).
Case 2: \( \text{Int} \, V_0 \neq \emptyset \). We may now assume that \( g(x) < 0 \) for all \( x \in \text{Int} \, V_0 \). The convexity of \( V_0 \) guarantees (see [CEI]) that there does not happen "fattening" in the Gauss curvature flow \( \{V_t\}_{t \geq 0} \), from which we can conclude that
\[
\{u \leq 0\} = \bigcup_{s>0} \{u \leq -s\}.
\]

Fix \( \varepsilon > 0 \), and we see from the above that for some \( \delta > 0 \),
\[
\{u \leq 0\} \subset \{u \leq -\delta\} + B^{n+2}(0, \varepsilon).
\]

We may assume that (6.1) holds with current \( \delta > 0 \) and some \( \eta > 0 \). Now it is immediate to conclude that for all \( 0 < h \leq \eta \),
\[
\{u \leq 0\} \subset \{v_h \leq 0\} + B^{n+2}(0, \varepsilon),
\]
which proves (2.8). \( \square \)
References


