EXISTENCE OF PERIODIC SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS IN BANACH SPACES

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1. INTRODUCTION

Let $X$ be a Banach space, let $A$ be a closed, convex subset of $X \times X$, and let $f : \mathbb{R} \times \overline{D(A)} \rightarrow X$ be a Carathéodory mapping which is $T$-periodic in its first variable, and let $h \in L^1(0, T; X)$.

In this paper, we study the existence of $T$-periodic solutions to a class of a nonlinear evolution equations of the form

$$u'(t) + Au(t) \ni f(t, u(t)) + h(t) \quad \text{for } t \in \mathbb{R}. \tag{1.1}$$

This problem has been studied by many authors; cf. [1, 3, 5, 12, 14, 15, 19, 20]. In the case when $A$ is the subdifferential of a proper, lower semicontinuous convex function on a Hilbert space, Ōtani [14] obtained a nice result. Vrabie [20] considered the case that $A$ is a strongly continuous operator. He considered the case that $X$ is a Banach space, $A$ is an $m$-accretive operator, and $f : \mathbb{R} \times \overline{D(A)} \rightarrow X$ is a Carathéodory mapping such that $\overline{D(A)}$ is convex, $-A$ generates a compact semigroup, $f$ is $T$-periodic in its first variable and there exists $a > 0$ such that $A - aI$ is $m$-accretive, and

$$\lim_{r \to \infty} \frac{1}{r} \sup \{ ||f(t, v)|| : t \in \mathbb{R}, v \in \overline{D(A)}, ||v|| \leq r \} < a,$$

and he showed that (1.1) has a $T$-periodic, integral solution in the case of $h = 0$. Caşcaval and Vrabie [5] partially extended his result to the case that $X$ is a Hilbert space, $-A$ generates a compact semigroup, and $f : \mathbb{R} \times \overline{D(A)} \rightarrow X$ is a Carathéodory mapping such that $f$ is $T$-periodic in its first variable and bounded on every bounded subset in $\mathbb{R} \times \overline{D(A)}$, and there exists $r > 0$ such that $B_r \cap \overline{D(A)}$ is nonempty and

$$\langle y - f(t, x), x \rangle \geq 0 \quad \text{for every } (x, y) \in A \text{ with } ||x|| = r \text{ and } t \in [0, T],$$

and they showed that (1.1) has a $T$-periodic, strong solution in the case of $h = 0$.

The objects of this paper are to obtain a generalization of Caşcaval and Vrabie's result by relaxing the conditions that $X$ is a Hilbert space and $f$ is a continuous mapping, and an existence result on the $T$-periodic problem (1.1) for every $h \in L^1(0, T; X)$ in the case when $X$ is a Banach space and $f$ is a Carathéodory mapping. The idea is inspired by [10] in which Górniewicz and Plaskacz studied the existence of periodic solution of an ordinary differential equation. Our results are the following:

**Theorem 1.** Let $X$ be a separable Banach space and let $A$ be a closed, convex subset of $X \times X$ such that $\overline{D(A)}$ is convex and $-A$ generates a compact semigroup. Let $T > 0$ and let $f$ be a Carathéodory mapping from $[0, T] \times \overline{D(A)}$ into $X$. Assume that there exist $r > 0$ and $\epsilon > 0$ such that $B_r \cap \overline{D(A)}$ is nonempty,

$$\int_0^T \sup_{x \in \overline{D(A)} \cap B_{r+\epsilon}} ||f(t, x)|| dt < \infty,$$
and for every \((x, y) \in A\) with \(r - \varepsilon \leq \|x\| \leq r + \varepsilon\), there exists \(z \in Jx\) such that
\[
\langle y - f(t, x), z \rangle \geq 0 \quad \text{for almost every } t \in (0, T),
\]
where \(B_r = \{u \in X : \|u\| \leq r\}\) and \(J\) is the duality mapping from \(X\) into its topological dual. Then there exists at least one \(T\)-periodic, integral solution of
\[
u'(t) + Au(t) \ni f(t, u(t)) \quad \text{for } 0 \leq t \leq T.\]

**Theorem 2.** Let \(X, A, T\) and \(f\) be as in Theorem 1. Assume that for every \(\rho > 0\) there exists \(a_{\rho} \in L^1(0, T)\) such that \(\|f(t, x)\| \leq a_{\rho}(t)\) for almost every \(t \in [0, T]\) and for every \(x \in D(A)\) with \(\|x\| \leq \rho\). Assume also that there exist \(r > 0\), \(c > 0\) and \(b \in L^1(0, T)\) such that for every \((x, y) \in A\) with \(\|x\| \geq r\), there exists \(z \in Jx\) such that
\[
\langle y - f(t, x), z \rangle \geq c\|x\|^2 - b(t)\|x\| \quad \text{for almost every } t \in (0, T).
\]
Then for every \(h \in L^1(0, T; X)\), there exists at least one \(T\)-periodic, integral solution \(u\) of
\[
u'(t) + Au(t) \ni f(t, u(t)) + h(t) \quad \text{for } 0 \leq t \leq T.
\]

2. Preliminaries

Throughout this paper, all vector spaces are real, we denote by \(\mathbb{N}\) and \(\mathbb{R}\), the set of all positive integers and the set of all real numbers, respectively, and by homology, we understand the Čech homology with rational coefficients; see [8, 9].

Let \(Y\) and \(Z\) be topological spaces. Let \(T\) be a subset of \(Y \times Z\). We identify the set \(T\) with a multivalued mapping \(T\) from \(Y\) into \(Z\) by \(Ty = \{z \in Z : (y, z) \in T\}\) for every \(y \in Y\). We denote by \(D(T)\) and \(R(T)\), the sets \(\{y \in Y : Ty \neq \emptyset\}\) and \(\bigcup\{Ty : y \in D(T)\}\), respectively. We say that \(T\) is upper semicontinuous if for every \(y_0 \in Y\) and open set \(V\) in \(Z\) with \(Ty_0 \subset V\), there exists an open neighborhood \(U\) of \(y_0\) such that \(Ty \subset V\) for every \(y \in U\).

The following fixed point theorem was obtained in [9, 17]. Since [17] is written in Japanese, we give the proof in Appendix.

**Proposition 1** (Górničewicz, Shioji). Let \(Y\) be a convex subset of a locally convex, Hausdorff topological vector space \(E\) and let \(K\) be a compact subset of \(Y\). Let \(T\) be an upper semicontinuous mapping from \(Y\) into \(K\) such that for every \(y \in Y\), \(Ty\) is a nonempty, acyclic, compact subset of \(K\). Then there is an element \(y \in Y\) such that \(y \in Ty\).

Let \(X\) be a Banach space, let \(D\) be a subset of \(X\) and let \(r > 0\). We denote by \(\overline{D}\), the closure of \(D\) and we denote by \(B_r\), the closed ball in \(X\) with center 0 and radius \(r\). Let \(X^*\) be the topological dual of \(X\). The value of \(x^* \in X^*\) at \(x \in X\) will be denoted by \(\langle x, x^* \rangle\). Let \(J\) be the multivalued mapping from \(X\) into \(X^*\) defined by \(Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}\) for every \(x \in X\). We call \(J\) the duality mapping from \(X\) into \(X^*\). For every \((x, y) \in X \times X\), we define
\[
[x, y]_+ = \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}.
\]
We know that \((x, y) \mapsto [x, y]_+\) is an upper semicontinuous function from \(X \times X\) into \(\mathbb{R}\). We say a subset \(A \subset X \times X\) is accretive if \([x_1 - x_2, y_1 - y_2],_+ \geq 0\) for every \((x_1, y_1), (x_2, y_2) \in A\). We know that \(A\) is accretive if and only if for every \((x_1, y_1), (x_2, y_2) \in A\), there exists \(w^* \in J(x_1 - x_2)\) such that \(\langle y_1 - y_2, w^* \rangle \geq 0\). We say an accretive set \(A\) is \(m\)-accretive if \(R(I + \lambda A) = X\) for every \(\lambda > 0\). Let \(a, b \in \mathbb{R}\) with \(a < b\). We denote by \(C(a, b; X)\), the space of all continuous functions from \([a, b]\) into \(X\). For \(1 \leq p < \infty\), we also denote by \(L^p(a, b; X)\), the space of all strongly measurable, \(p\)-integrable, \(X\)-valued functions defined almost everywhere on \([a, b]\).
Let $A \subseteq X \times X$ be an $m$-accretive set, let $f \in L^1(a, b; X)$ and let $x \in \overline{D(A)}$. We say a function $u : [a, b] \rightarrow X$ is a strong solution of the initial value problem:

$$u(a) = x, \quad u'(t) + Au(t) \ni f(t) \quad \text{for} \quad a \leq t \leq b,$$

if $u$ is differentiable almost everywhere on $[a, b]$, $u$ is absolutely continuous, $u(a) = x$ and $u'(t) + Au(t) \ni f(t)$ almost everywhere on $[a, b]$. We say a function $u : [a, b] \rightarrow X$ is an integral solution of the initial value problem (2.1), if $u$ is continuous on $[a, b]$, $u(a) = x$, $u(t) \in \overline{D(A)}$ for every $a \leq t \leq b$ and

$$\|u(t) - y\| \leq \|u(s) - y\| + \int_s^t [u(\tau) - y, f(\tau) - z]_+ d\tau$$

for every $(y, z) \in A$ and $s, t$ with $a \leq s \leq t \leq b$. If $u$ is a strong solution of (2.1), then $u$ is an integral solution of (2.1). We know from [2, 4] that the initial value problem (2.1) has a unique integral solution. If $u$ and $v$ are the integral solutions of (2.1) corresponding to $(x, f), (y, g) \in \overline{D(A)} \times L^1(a, b; X)$ respectively, then

$$\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t [u(\tau) - v(\tau), f(\tau) - g(\tau)]_+ d\tau$$

for $a \leq s \leq t \leq b$.

If $A \subseteq X \times X$ is $m$-accretive, then

$$S(t)x = \lim_{n \to \infty} \left( I + \frac{t}{n}A \right)^{-n} x$$

exists for every $x \in \overline{D(A)}$ and uniformly for $t$ on every bounded interval in the set of nonnegative real numbers; see [2, 6]. We say the family $\{S(t) : \overline{D(A)} \rightarrow \overline{D(A)}, t \geq 0\}$ is the nonlinear semigroup generated by $-A$. We remark that for every $x \in \overline{D(A)}$, $t \mapsto S(t)x$ is the unique integral solution of $u(0) = x$ and $u'(t) + Au(t) \ni 0$ for $t \geq 0$. We say $\{S(t) : \overline{D(A)} \rightarrow \overline{D(A)}, t \geq 0\}$ is compact if $S(t)$ is compact for every $t > 0$.

To prove our theorems, we need the following propositions; see [20, Theorem 2] and [7, Lemma 2]:

**Proposition 2** (Vrabie). Let $X$ be a Banach space and let $A$ be an $m$-accretive subset of $X \times X$ such that $-A$ generates a compact semigroup. Let $B$ be a bounded subset of $\overline{D(A)}$, let $a, b \in \mathbb{R}$ with $a < b$ and let $G$ be a uniformly integrable subset of $L^1(a, b; X)$. Then the set of all integral solutions of (2.1) corresponding to $(x, f) \in B \times G$ is relatively compact in $C(d, b; X)$ for every $d \in (a, b)$, and if, in addition, $B$ is relatively compact in $X$, the set is relatively compact in $C(a, b; X)$.

**Proposition 3** (De Blasi and Myjak). Let $B$ be a subset of a separable Banach space $X$ and let $f$ be a Carathéodory mapping from $[0, 1] \times B$ into $X$ such that $\int_0^1 \sup_{x \in B} \|f(t, x)\| dt < \infty$. Then for every $\varepsilon > 0$, there exists a locally Lipschitz mapping $g$ from $[0, 1] \times B$ into $X$ such that

$$\int_0^1 \sup_{x \in B} \|f(t, x) - g(t, x)\| dt < \varepsilon.$$
3. Proof of Theorem 1

In this section, we give the proof of Theorem 1. Let \( \alpha \) be a continuous function from \([0, \infty)\) into \([0, 1]\) such that \( \alpha(t) = 1 \) for \( t \in [0, r + \varepsilon/2] \) and \( \alpha(t) = 0 \) for \( t \in [r + 3\varepsilon/4, \infty) \). Define a Carathéodory mapping \( \tilde{f} \) from \([0, T] \times \overline{D(A)}\) into \( X \) by \( \tilde{f}(t, x) = \alpha(||x||)f(t, x) \) for \( (t, x) \in [0, T] \times \overline{D(A)} \). Since \( \int_{0}^{T} \sup_{x \in \overline{D(A)}} ||\tilde{f}(t, x)|| dt < \infty \), Proposition 3 yields a sequence of locally Lipschitz functions \( \{\tilde{f}_n\} \) from \([0, T] \times \overline{D(A)}\) into \( X \) such that \( \tilde{f}_n(t, x) = 0 \) for \( (t, x) \in [0, T] \times (\overline{D(A)} \setminus B_{r+\varepsilon}) \) and

\[
\int_{0}^{T} \sup_{x \in \overline{D(A)}} ||\tilde{f}(t, x) - \tilde{f}_n(t, x)|| dt < \frac{1}{n}.
\]

For every \( x \in \overline{D(A)} \cap B_r \), we set

\[ S_x = \{ u : [0, T] \rightarrow \overline{D(A)}, \text{ } u \text{ is an integral solution of } (3.1) \} \]

\[ u(0) = x, \quad u'(t) + Au(t) \ni \tilde{f}(t, u(t)) \text{ for } 0 \leq t \leq T. \]

For every \( n \in \mathbb{N} \) and \( \sigma \in [0, T] \), we denote by \( F_{n, \sigma} \), the function from \([0, T] \times \overline{D(A)}\) into \( X \) defined by

\[ F_{n, \sigma}(t, x) = \begin{cases} \tilde{f}(t, x) & \text{if } (t, x) \in [0, \sigma] \times \overline{D(A)}, \\ \tilde{f}_n(t, x) & \text{if } (t, x) \in (\sigma, T] \times \overline{D(A)}. \end{cases} \]

For every \( n \in \mathbb{N} \) and \( x \in \overline{D(A)} \cap B_r \), we also set

\[ S_n x = \bigcup_{\sigma \in [0, T]} \{ u : [0, T] \rightarrow \overline{D(A)}, \text{ } u \text{ is an integral solution of } (3.3) \} \]

\[ u(0) = x, \quad u'(t) + Au(t) \ni F_{n, \sigma}(t, u(t)) \text{ for } 0 \leq t \leq T. \]

Since \( \int_{0}^{T} \sup_{x \in \overline{D(A)}} ||\tilde{f}(t, x)|| dt < \infty \), \( \int_{0}^{T} \sup_{x \in \overline{D(A)}} ||\tilde{f}_n(t, x)|| dt < \infty \) and \(-A\) generates a compact semigroup, we know from [18, Theorem 2] or [21, Theorem 3.8.1] that there exist integral solutions for (3.2) and (3.3) on a small interval \([0, \delta]\), and we also know from [21, Theorem 3.8.2] that such integral solutions are continuably on \([0, T]\). So \( S_x \) and \( S_n x \) are nonempty for every \( x \in \overline{D(A)} \cap B_r \) and \( n \in \mathbb{N} \).

**Lemma 1.** For every \( x \in \overline{D(A)} \cap B_r \) and \( u \in S_x \), \( ||u(t)|| \leq r \) for every \( t \in [0, T] \).

**Proof.** Let \( x \in \overline{D(A)} \cap B_r \) and let \( u \in S_x \). Let \([T_0, T_1]\) be an interval contained in \([0, T]\) such that \( u(T_0) = r \) and \( r - \varepsilon/4 \leq ||u(t)|| \leq r + \varepsilon/4 \) for every \( t \in [T_0, T_1] \). Since \( u \) is an integral solution of \( u'(t) + Au(t) \ni \tilde{f}(t, u(t)) \) on the interval \([T_0, T_1]\), for every \( \delta \in (0, \varepsilon/4) \), there exist \( t_0, \cdots, t_N \in [0, T] \), \( x_0, \cdots, x_N \in \overline{D(A)} \), \( f_0, \cdots, f_N \in X \) such that

\[
T_0 = t_0 < t_1 < \cdots < t_{N-1} < T_1 \leq t_N, \quad \max(t_i - t_{i-1}) \leq \delta,
\]

\[
\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} ||f_i - \tilde{f}(t, u(t))|| dt \leq \delta,
\]

(3.4)

\[
\frac{x_i - x_{i-1}}{t_i - t_{i-1}} + Ax_i \ni f_i \quad \text{for } i = 1, 2, \ldots, N
\]

and

(3.5)

\[
||u(t) - u(t)|| \leq \delta \quad \text{for every } t \in [T_0, T_1],
\]
where
\[ v(t) = \begin{cases} x_0 & \text{if } t = t_0, \\ x_i & \text{if } t \in (t_{i-1}, t_i], \quad i = 1, 2, \ldots, N; \end{cases} \]
see [13]. From the hypothesis of our theorem, (3.4), (3.5) and \(0 < \delta < \epsilon/4\), for every \(i = 1, 2, \ldots, N\), there exists \(z_i^* \in Jx_i\) such that
\[ \left\langle f_i - \frac{x_i - x_{i-1}}{t_i - t_{i-1}} - \tilde{f}(t, x_i), z_i^* \right\rangle \geq 0 \quad \text{for almost every } t \in [0, T]. \]
So we have
\[ \|x_j\| \leq \|x_0\| + \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} \|f_i - \tilde{f}(t, x_i)\| \, dt \leq r + 2\delta + \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} \|\tilde{f}(t, u(t)) - \tilde{f}(t, x_i)\| \, dt \]
for every \(j = 1, 2, \ldots, N\), which implies
\[ \|u(t)\| \leq r + 3\delta + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|\tilde{f}(t, u(t)) - \tilde{f}(t, x_i)\| \, dt \]
for every \(t \in [T_0, T_1]\). Since \(\delta \in (0, \epsilon/4)\) is arbitrary, we obtain \(\|u(t)\| \leq r\) for every \(t \in [T_0, T_1]\). This completes the proof.

**Lemma 2.** For every \(n \in N\) and \(x \in \overline{D(A)} \cap B_r\), \(S_n x\) is compact, where \(S_n x\) is endowed with the \(C(0, T; X)\) topology.

**Proof.** Since \(S_n x\) is relatively compact from Proposition 2, we only need to show that \(S_n x\) is closed. Let \(n \in N\) and let \(x \in \overline{D(A)} \cap B_r\). Let \(\{u_m\}\) be a sequence in \(S_n x\) which converges to \(u\). We shall show \(u \in S_n x\). For every \(m \in N\), there exists \(\sigma_m \in [0, T]\) such that \(u_m\) is an integral solution of
\[ (3.6) \quad u_m(0) = x, \quad u_m'(t) + Au_m(t) = F_{n, \sigma_m}(t, u_m(t)) \quad \text{for } 0 \leq t \leq T. \]
We may assume that \(\{\sigma_m\}\) converges to \(\sigma \in [0, T]\). Then \(\{F_{n, \sigma_m}(t, u_m(t))\}\) converges to \(F_{n, \sigma}(t, u(t))\) in \(L^1(0, T; X)\). Since \(u_m\) is an integral solution of (3.6), we have
\[ \|u_m(t) - y\| \leq \|u_m(s) - y\| + \int_{s}^{t} [u_m(\tau) - y, F_{n, \sigma_m}(\tau, u_m(\tau)) - z] \, d\tau \]
for every \((y, z) \in A, s, t\) with \(0 \leq s \leq t \leq T\) and \(m \in N\). Tending \(m\) to infinity, we have \(u \in S_n x\). Hence, \(S_n x\) is closed.

The following is crucial to prove our theorem. In the proof, we use the method employed in [11, Proposition 3] and [22].

**Lemma 3.** For every \(n \in N\) and \(x \in \overline{D(A)} \cap B_r\), \(S_n x\) is contractible.

**Proof.** Let \(n \in N\) and let \(x \in \overline{D(A)} \cap B_r\). For every \(s \in [0, 1]\) and \(v \in S_n x\), we denote by \(w_{s,v}\), the integral solution \(w_{s,v} : [sT, T] \rightarrow \overline{D(A)}\) of
\[ w_{s,v}(sT) = v(sT), \quad w_{s,v}'(\tau) + Aw_{s,v}(\tau) = \tilde{f}_n(\tau, w_{s,v}(\tau)) \quad \text{for } sT \leq t \leq T. \]
Define a function \(H\) from \([0, 1] \times S_n x\) into \(S_n x\) by
\[ H(s, v)(t) = \begin{cases} v(t) & \text{if } 0 \leq t \leq sT, \\ w_{s,v}(t) & \text{if } sT \leq t \leq T \end{cases} \quad \text{for every } (s, v) \in [0, 1] \times S_n x. \]
We shall show that $H$ is continuous. Let $(s_0, v_0) \in [0, 1] \times S_n x$. Since $\tilde{f}_n$ is locally Lipschitz, for every $\tau \in [0, T]$, there exist $\delta_\tau > 0$ and $K_\tau > 0$ such that $y \in \overline{D(A)}$, $t \in [s_0 T, T]$ with $|t - \tau| < \delta_\tau$ and $\|y - w_{s_0, v_0}(t)\| < \delta_\tau$ imply $\|\tilde{f}_n(t, y) - \tilde{f}_n(t, w_{s_0, v_0}(t))\| \leq K_\tau \|y - w_{s_0, v_0}(t)\|$. From the compactness of $[s_0 T, T]$, there exists $\{\tau_1, \cdots, \tau_m\} \subset [s_0 T, T]$ such that $[s_0 T, T] \subset \bigcup_{i=1}^{m} (\tau_i - \delta_{\tau_i}, \tau_i + \delta_{\tau_i})$. Set $\delta = \min\{\delta_{\tau_1}, \cdots, \delta_{\tau_m}\}$ and $K = \max\{K_{\tau_1}, \cdots, K_{\tau_m}\}$. Then we have
\[(3.7) \quad \|\tilde{f}_n(t, y) - \tilde{f}_n(t, w_{s_0, v_0}(t))\| \leq K \|y - w_{s_0, v_0}(t)\|\]
for every $(t, y) \in [s_0 T, T] \times \overline{D(A)}$ with $\|y - w_{s_0, v_0}(t)\| < \delta$. Fix $\eta \in (0, \delta)$. Choose $\rho > 0$ satisfying $\rho < \delta$ and $\rho < \eta/(4e^{KT})$. We can also choose $\zeta \in (0, \rho]$ such that $\int_{t}^{t+\zeta T} ||\tilde{f}_n(\tau, y)|| d\tau < \rho$ and $\int_{t}^{t+\zeta T} ||\tilde{f}_n(\tau, y)|| d\tau < \rho$ for every $t \in [0, (1 - \zeta)T]$.

From $v_0 \in S_n x$, there exists $\sigma \in [0, T]$ such that $v_0$ is an integral solution of $v_0(0) = x$, $v_0'(\tau) + Av_0(\tau) \ni F_n,\sigma(\tau, v_0(\tau))$ for $0 \leq \tau \leq T$.

Let $(s, v) \in [0, T] \times S_n x$ such that $|s - s_0| < \zeta$ and $\|v - v_0\| < \zeta$. For every $t \in [s_0 T, sT]$ in the case of $s \geq s_0$ or for every $t \in [sT, s_0 T]$ in the case of $s_0 \geq s$, we have
\[
\|H(s, v)(t) - H(s_0, v_0)(t)\| \leq \begin{cases} 
\|v(t) - v_0(t)\| + \int_{s_0 T}^{t} \|F_n,\sigma(\tau, v_0(\tau)) - \tilde{f}_n(\tau, w_{s_0, v_0}(\tau))\| d\tau & \text{if } s \geq s_0, \\
\|v(sT) - v_0(sT)\| + \int_{sT}^{t} ||\tilde{f}_n(\tau, w_{s, v}(\tau)) - F_n,\sigma(\tau, v_0(\tau))|| d\tau & \text{if } s_0 \geq s, 
\end{cases}
\leq 4\rho.
\]

Then we have
\[(3.8) \quad \|H(s, v)(t) - H(s_0, v_0)(t)\| \leq 4\rho + \int_{T'}^{t} \|\tilde{f}_n(\tau, w_{s, v}(\tau)) - \tilde{f}_n(\tau, w_{s_0, v_0}(\tau))\| d\tau
\]
for every $t \in [T', T]$, where $T' = \max\{sT, s_0 T\}$. We shall show that $\|H(s, v)(t) - H(s_0, v_0)(t)\| < \eta$ for every $t \in [0, T]$. Suppose not. Then there exists $t_0 \in (T', T)$ such that $\|H(s, v)(t_0) - H(s_0, v_0)(t_0)\| = \eta$ and $\|H(s, v)(t) - H(s_0, v_0)(t)\| < \eta$ for every $t \in [T', t_0)$. By (3.7), (3.8) and Gronwall's inequality, we have $\|H(s, v)(t_0) - H(s_0, v_0)(t_0)\| \leq 4\rho e^{KT} < \eta$, which is a contradiction. So, we have $\|H(s, v)(t) - H(s_0, v_0)(t)\| < \eta$ for every $t \in [0, T]$. Hence $H$ is continuous.

On the other hand, for every $v \in S_n x$, $H(1, v) = v$ and $H(0, v) = w$, where $w$ is the integral solution of
\[w(0) = x, \quad w'(t) + Aw(t) \ni \tilde{f}_n(t, w(t)) \quad \text{for } 0 \leq t \leq T.\]
Therefore $S_n x$ is contractible.

\[\square\]

**Lemma 4.** For every $x \in \overline{D(A)} \cap B_r$, $S x$ is compact and acyclic.
Proof. Let $x \in \overline{D(A)} \cap B_r$. Since $F_{n,T} = \tilde{f}$ for every $n \in \mathbb{N}$, we have $Sx \subset \bigcap_{n=1}^{\infty} S_n x$. We shall show the opposite inclusion. Let $u \in \bigcap_{n=1}^{\infty} S_n x$. Then for every $n \in \mathbb{N}$, there exists $\sigma_n \in [0,T]$ such that $u$ is an integral solution of

$$u(0) = x, \quad u'(t) + Au(t) \ni F_{n,\sigma_n}(t, u(t)) \quad \text{for } 0 \leq t \leq T.$$ 

Then, from (3.1), we have

$$||u(t) - y|| \leq ||u(s) - y|| + \int_{s}^{t} [u(\tau) - y, \tilde{f}(\tau, u(\tau)) - z] + d\tau + \frac{1}{n}$$

for every $(y, z) \in A, s, t$ with $0 \leq s \leq t \leq T$ and $n \in \mathbb{N}$. Tending $n$ to infinity, we obtain $u \in Sx$. So we have $Sx = \bigcap_{n=1}^{\infty} S_n x$. From Lemma 2, Lemma 3 and the continuity property of the Čech homology, we have that $Sx$ is compact and acyclic.

Now, we can give the proof of our theorem.

Proof of Theorem 1. Let $Z_0 = \bigcup\{v(T) : y \in \overline{D(A)} \cap B_r, v \in Sy\}$. From Lemma 1, $Z_0$ is a nonempty subset of $\overline{D(A)} \cap B_r$. Let $Z$ be the closed, convex hull of $Z_0$. From Proposition 2, $Z_0$ is relatively compact and hence $Z$ is compact. Let $Y$ be the set $\{u \in C(0, T; X) : u(t) \in D(A) \cap B_r \text{ for every } t \in [0,T] \text{ and } u(T) \in Z\}$ and let $T$ be a multivalued mapping from $Y$ into $C(0, T; X)$ defined by

$$Tu = Su(T) \quad \text{for every } u \in Y,$$

i.e., $T$ is the composition of $u \mapsto u(T)$ and $S$. From the compactness of $Z$, Proposition 2, Lemma 1 and Lemma 4, $T(Y)$ is contained in a compact subset of $Y$ and $Tu$ is a nonempty, acyclic, compact subset of $Y$ for every $u \in Y$. We shall show that $T$ is upper semicontinuous. Suppose not. Then there exist $u \in Y$, a open neighborhood $V$ of $Tu$, $\{u_n\} \subset Y$ and $\{v_n\} \subset Y$ such that $\{u_n\}$ converges to $u$ and $v_n \in Tu_n \setminus V$ for every $n \in \mathbb{N}$. From Proposition 2, we may assume $\{v_n\}$ converges to $v$, and hence $v \not\in V$. Since $v_n \in Tu_n$, we have $v_n(0) = u_n(T)$ and

$$||v_n(t) - y|| \leq ||v_n(s) - y|| + \int_{s}^{t} [v_n(\tau) - y, \tilde{f}(\tau, v_n(\tau)) - z] + d\tau$$

for every $(y, z) \in A, s, t$ with $0 \leq s \leq t \leq T$ and $n \in \mathbb{N}$. Tending $n$ to infinity, we obtain $v \in Tu$ which contradicts $Tu \subset V \not\not\subset V$. So, $T$ is upper semicontinuous. Hence, by Proposition 1, there exists a point $y \in Y$ such that $u \in Tu$. By the definition of $T$, $u(0) = u(T)$ and $u$ is an integral solution of $u'(t) + Au(t) \ni f(t, u(t))$ for $0 \leq t \leq T$. From Lemma 1, $u$ is also an integral solution of $u'(t) + Au(t) \ni f(t, u(t))$ for $0 \leq t \leq T$. This completes the proof. \square

4. Proof of Theorem 2

In this section, we give the proof of Theorem 2. Let $h \in L^1(0, T; X)$. Let $M, R$ and $\rho$ be real numbers such that $M = \int_{0}^{T} |b(s)| ds + \int_{0}^{T} ||h(s)|| ds, R = \max\{r + M + 2, (1 + 1/(CT))(M + 1)\}$ and $\rho = R + M + 4$. From the hypothesis of Theorem 2, there exists $a_\rho \in L^1(0, T)$ such that $||f(t, x)|| \leq a_\rho(t)$ for almost every $t \in [0, T]$ and for every $x \in \overline{D(A)}$ with $||x|| \leq \rho$. Let $\alpha$ be a continuous function from $[0, \infty)$ into $[0, 1]$ which satisfies

$$\alpha(\tau) = \begin{cases} 1 & \text{if } \tau \leq \rho - 1, \\ 0 & \text{if } \tau \geq \rho \end{cases} \quad \text{for } \tau \geq 0.$$
Define a function $\tilde{f} : [0, T] \times \overline{D(A)} \to X$ by $\tilde{f}(t, x) = \alpha(||x||) f(t, x)$ for every $(t, x) \in [0, T] \times \overline{D(A)}$. Since $\int_0^T \sup_{x \in \overline{D(A)}} ||\tilde{f}(t, x)|| \, d\tau \leq \int_0^T a_\rho(\tau) \, d\tau < \infty$, Proposition 3 yields a sequence of locally Lipschitz functions $\{\tilde{f}_n\}$ from $[0, T] \times \overline{D(A)}$ into $X$ such that

\begin{equation}
\int_0^T \sup_{x \in \overline{D(A)}} ||\tilde{f}(t, x) - \tilde{f}_n(t, x)|| \, dt < \frac{1}{n} \quad \text{for every } n \in \mathbb{N}.
\end{equation}

For every $n \in \mathbb{N}$ and $x \in \overline{D(A)} \cap B_R$, we set $F_n x = u(T)$, where $u$ is the unique integral solution of

\begin{equation}
u(0) = x, \quad u'(t) + Au(t) \ni \tilde{f}_n(t, u(t)) + h(t) \quad \text{for } 0 \leq t \leq T.
\end{equation}

From [21, Theorem 3.2.2], and [18, Theorem 2] or [21, Theorem 3.8.1], we know that $F_n$ is well defined.

**Lemma 5.** Let $n \in \mathbb{N}$, let $x \in \overline{D(A)} \cap B_R$ and let $u$ be the integral solution of (4.2). Then $\|u(t)\| \leq R + M + 1$ for every $t \in [0, T]$.

**Proof.** Suppose not. Then there exist $T_0, T_1 \in [0, T]$ such that $T_0 < T_1$, $\|u(T_0)\| = R$, $R \leq \|u(t)\| \leq R + M + 2$ for every $t \in [T_0, T_1]$ and $\|u(T_1)\| > R + M + 1$. Since $u$ is the integral solution of $u'(t) + Au(t) \ni \tilde{f}_n(t, u(t)) + h(t)$ on the interval $[T_0, T_1]$, by the same method as in the proof of Lemma 1, we have

\begin{align*}
\|u(T_1)\| & \leq \|u(T_0)\| + \int_0^{T_1} \|h(s)\| \, ds + 1 - c \int_0^{T_1} \|u(s)\| \, ds + \int_0^{T_1} |b(s)| \, ds \\
& \leq R + M + 1.
\end{align*}

So we obtain a contradiction. This completes the proof. \qed

**Lemma 6.** For every $n \in \mathbb{N}$, $F_n$ is a mapping from $\overline{D(A)} \cap B_R$ into itself.

**Proof.** Let $n \in \mathbb{N}$, let $x \in \overline{D(A)} \cap B_R$ and let $u$ be the integral solution of (4.2). First, we consider the case that there exists $T_2 \in [0, T]$ such that $\|u(T_2)\| < R - M - 1$. Suppose $\|u(T)\| > R$. Then there exists $T_3 \in (T_2, T)$ such that $\|u(T_3)\| = R - M - 1$ and $\|u(t)\| \geq R - M - 1$ for every $t \in [T_3, T]$. By the same method as in the proof of Lemma 1, we have

\begin{align*}
\|u(T)\| & \leq \|u(T_3)\| + \int_0^{T} \|h(s)\| \, ds + 1 - c \int_0^{T} \|u(s)\| \, ds + \int_0^{T} |b(s)| \, ds \\
& \leq R,
\end{align*}

which is a contradiction. So we have $\|u(T)\| \leq R$. Next, we consider the case that $\|u(t)\| \geq R - M - 1$ for every $t \in [0, T]$. Then, we have

\begin{align*}
\|u(T)\| & \leq \|u(0)\| + \int_0^{T} \|h(t)\| \, dt + 1 - c \int_0^{T} \|u(t)\| \, dt + \int_0^{T} |b(t)| \, dt \\
& \leq \|u(0)\| + M + 1 - c T (R - M - 1) \leq R.
\end{align*}

Hence $F_n$ is a mapping from $\overline{D(A)} \cap B_R$ into itself. \qed

**Lemma 7.** For every $n \in \mathbb{N}$, $F_n$ is continuous.

**Proof.** Let $n \in \mathbb{N}$. Let $x \in \overline{D(A)} \cap B_R$ and let $u$ be the integral solution of (4.2). Since $\tilde{f}_n$ is locally Lipschitz, by the same method to prove (3.7), there exist $K > 0$ and $\eta > 0$ such that

\begin{equation}
\|\tilde{f}_n(t, y) - \tilde{f}_n(t, u(t))\| \leq K\|y - u(t)\|
\end{equation}

\begin{align*}
\|u(T)\| & \leq \|u(0)\| + \int_0^{T} \|h(t)\| \, dt + 1 - c \int_0^{T} \|u(t)\| \, dt + \int_0^{T} |b(t)| \, dt \\
& \leq \|u(0)\| + M + 1 - c T (R - M - 1) \leq R.
\end{align*}

Hence $F_n$ is a mapping from $\overline{D(A)} \cap B_R$ into itself. \qed
for every \((t, y) \in [0, T] \times \overline{D(A)}\) with \(\|y - u(t)\| < \eta\). Let \(\varepsilon \in (0, \eta)\) and let \(\delta > 0\) satisfying \(\delta e^{KT} < \varepsilon\). Let \(y \in \overline{D(A)} \cap B_R\) be a point satisfying \(\|x - y\| < \delta\) and let \(v\) be the integral solution of \(v(0) = y\) and \(v'(t) + Av(t) \ni \tilde{f}_n(t, v(t)) + h(t)\) for \(0 \leq t \leq T\). We shall show \(\|u(t) - v(t)\| < \varepsilon\) for every \(t \in [0, T]\). Suppose not. Then there exists \(t_0 \in (0, T]\) such that \(\|u(t_0) - v(t_0)\| = \varepsilon\) and \(\|u(t) - v(t)\| < \varepsilon\) for every \(t \in [0, t_0)\). By (4.3), Gronwall’s inequality implies \(\|u(t_0) - v(t_0)\| \leq e^{\delta KT} < \varepsilon\), which is a contradiction. Hence \(F_n\) is continuous.

**Proof of Theorem 2.** From Lemma 6, Lemma 7 and Proposition 2, \(F_n\) is a compact mapping from \(\overline{D(A)} \cap B_R\) into itself for every \(n \in \mathbb{N}\). Hence, for every \(n \in \mathbb{N}\), there exists a fixed point of \(F_n\) by Schauder’s fixed point theorem, i.e., there exists an integral solution \(u_n\) of \(\dot{u}_n(t) + Au_n(t) \ni \tilde{f}_n(t, u_n(t)) + h(t)\) for \(0 \leq t \leq T\) such that \(u_n(0) = u_n(T)\) and \(\|u_n(0)\| \leq R\). From Proposition 2, we may assume \(\{u_n\}\) converges to \(u\) in \(C(0, T; X)\). Let \((x, y) \in A\) and let \(s, t \in \mathbb{R}\) with \(0 \leq s \leq t \leq T\). From (3.7), we have

\[
\|u_n(t) - x\| \leq \|u_n(s) - x\| + \int_s^t [u_n(\tau) - x, \tilde{f}(\tau, u_n(\tau)) + h(\tau) - y] + d_{\mathcal{T}+} \frac{1}{n}
\]

for every \(n \in \mathbb{N}\). Tending \(n\) to infinity, we obtain that \(u\) is an integral solution of \(u'(t) + Au(t) \ni f(t, u(t)) + h(t)\) for \(0 \leq t \leq T\) such that \(u(0) = u(T)\) and \(\|u(0)\| \leq R\). From Lemma 5, \(u\) is also an integral solution of \(u'(t) + Au(t) \ni f(t, u(t)) + h(t)\) for \(0 \leq t \leq T\). Hence we obtain the desired result. \(\square\)

### 5. An Example

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n (n \geq 2)\) with smooth boundary \(\Gamma\). We consider the following nonlinear differential equation:

\[
(5.1) \quad \frac{\partial u}{\partial t} - \Delta \rho(u) = g(t, x, u(t, x)) + h(t, x) \quad \text{on } \mathbb{R} \times \Omega
\]

with a boundary condition

\[
(5.2) \quad \rho(u) = 0 \quad \text{on } \mathbb{R} \times \Gamma.
\]

**Theorem 3.** Let \(\rho \in C(\mathbb{R}) \cap C^{1}(\mathbb{R} \setminus \{0\})\) such that \(\rho(0) = 0\) and there exist \(C > 0\) and \(a > \frac{n-2}{n}\) with

\[
\rho'(r) \geq C|r|^{a-1} \quad \text{for every } r \in \mathbb{R} \setminus \{0\}.
\]

Let \(g : \mathbb{R} \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) such that \(g\) is \(T\)-periodic in its first variable, \(g(t, x, \cdot)\) is continuous for almost every \((t, x) \in \mathbb{R} \times \Omega\) and \(g(\cdot, \cdot, u)\) is measurable for every \(u \in \mathbb{R}\). Assume that there exist \(a \in L^1(0, T)\) and \(b \in L^1((0, T) \times \Omega)\) such that \(|g(t, x, u)| \leq a(t)|u| + b(t, x)\) for \((t, x, u) \in [0, T] \times \Omega \times \mathbb{R}\) and that

\[
\lim_{|u| \to \infty} \text{ess sup}_{(t, x) \in [0, T] \times \Omega} \frac{g(t, x, u)}{u} < 0.
\]

Then for every \(h \in L^1((0, T) \times \Omega)\), (5.1) and (5.2) have at least one \(T\)-periodic integral solution \(u \in C(\mathbb{R}, L^1(\Omega))\).

**Proof.** Let \(A\) be the set defined by \(\{(u, -\Delta \rho(u)) \in L^1(\Omega) \times L^1(\Omega) : \rho(u) \in W^{1,1}_{0}(\Omega)\}\) and let \(f\) be the function from \(\mathbb{R} \times L^1(\Omega)\) into \(L^1(\Omega)\) defined by \(f(t, u)(x) = g(t, x, u(x))\) for every \((t, u, x) \in \mathbb{R} \times L^1(\Omega) \times \Omega\). We know that \(-A\) generates a compact semigroup; see [21, Lemma 2.7.2]. From
the assumption, there exist $\delta, M > 0$ such that $g(t, x, u)/u \leq -\delta$ for $(t, x, u) \in \mathbb{R} \times \Omega \times \mathbb{R}$ with $|u| \geq M$. Then for every $u \in L^1(\Omega)$ and $z \in J^u$, we have
\[
\int_\Omega g(t, x, u(x))z(x)\,dx \leq -\delta\|u\|_{L^1(\Omega)}^2 + \left((\delta + a(t))M|\Omega| + \int_\Omega b(t, x)\,dx\right)\|u\|_{L^1(\Omega)}.
\]
So, from Theorem 2, for every $h \in L^1((0, T) \times \Omega)$, there exists a $T$-periodic integral solution for (5.1) and (5.2).

APPENDIX

In this appendix, we give the proof of Proposition 1. The following is obtained in [16].

Proposition 4. Let $Y$ be a subset of a Hausdorff topological vector space and let $K$ be a Hausdorff topological space. Let $T$ be an upper semicontinuous multivalued mapping from $\text{co}Y$ into $K$ such that for every $y \in \text{co}Y$, $Ty$ is a nonempty, acyclic, compact subset of $K$, and let $G$ be a multivalued mapping from $Y$ into $K$ such that for every $y \in Y$, $Gy$ is a closed subset of $K$, and

\[ T(\text{co}\{y_1, \cdots, y_n\}) \subset \bigcup_{i=1}^n Gy_i \quad \text{for every finite subset } \{y_1, \cdots, y_n\} \text{ of } Y. \tag{5.3} \]

Then $\{Gy : y \in Y\}$ has the finite intersection property.

From Proposition 4, we have the following, which is obtained in [17]. In the following, we can get a coincidence point of $A$ and $T$, though there is no relationship between them.

Proposition 5. Let $Y$ be convex subset of a Hausdorff topological vector space and let $K$ be a compact, Hausdorff topological space. Let $T$ and $A$ be multivalued mappings from $Y$ into $K$ such that $T$ is upper semicontinuous, for every $y \in Y$, $Ty$ is a nonempty, acyclic, compact subset of $K$ and $Ay$ is an open subset of $K$, and for every $z \in K$, $A^{-1}z$ is a nonempty, convex subset of $Y$. Then there is an element $y$ of $Y$ such that $Ay \cap Ty \neq \emptyset$.

Proof. Assume that the conclusion does not hold. Define a multivalued mapping $G$ from $Y$ into $K$ by $Gy = K \setminus Ay$ for every $y \in Y$. We shall show (5.3). Suppose not. Then there exist a finite subset $\{y_1, \cdots, y_n\}$ of $Y$, $y \in \text{co}\{y_1, \cdots, y_n\}$ and $z \in Ty$ such that $z \notin \bigcup_{i=1}^n Gy_i$. So we have $z \in Ay_i$; i.e., $y_i \in A^{-1}z$ for every $i = 1, \ldots, n$. Since $A^{-1}z$ is convex, we have $y \in A^{-1}z$. So we obtain $z \in Ty \cap Ay$, which is a contradiction. Hence by Proposition 4 and the compactness of $K$, there exists $w \in K$ such that $w \in \bigcap_{y \in Y} Gy$. So we get $w \notin Ay$ for all $y \in Y$, which implies $A^{-1}w = \emptyset$, and we get a contradiction. This completes the proof.

Proof of Proposition 1. Let $U$ be an arbitrary, symmetric, convex, open neighborhood of 0 in $E$. Define a multivalued mapping $A$ from $Y$ into $K$ by $Ay = y + U$ for every $y \in Y$. By Proposition 5, there is a point $y_U \in Y$ such that $(y_U + U) \cap Ty_U \neq \emptyset$. From the standard compactness argument, we obtain the conclusion.

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