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<th>$C^1$ APPROXIMATIONS OF INERTIAL MANIFOLDS VIA FINITE DIFFERENCES AND APPLICATIONS</th>
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1. Introduction

We shall present a method for the construction of approximate inertial manifolds by means of finite differences. The theory of inertial manifolds (IM for short) is a useful tool for reducing the long-time behavior of PDEs to that of finite-dimensional dynamical systems. (See [1-7] and [13]). To compute the reduced finite dynamical system, one would need to know the explicit form of the IM. However, even when existence of an IM can be established, the theory does not provide us with an explicit form of IMs. In this paper, from the point of finite differences we construct such an approximate IM that reflects the true dynamics of the original PDE.

Each of the PDEs can be viewed as an evolution equation in a Hilbert space. To be more specific, let $X$ and $Y$ be Hilbert spaces with norms $\| \cdot \|$ and $| \cdot |$, respectively, such that $X$ is continuously embedded in $Y$. Let $\{S(t); t \geq 0\}$ be a $C_0$-semigroup on $Y$ and $F \in \text{Lip}(X, Y) \cap C^1(X, Y)$, the set of Lipschitz and continuously differentiable mappings from $X$ into $Y$. The evolution equations take the form

\begin{align}
(1.1) & \quad du(t)/dt = Au(t) + Fu(t), \quad t \geq 0 \\
(1.2) & \quad u(0) = x_0
\end{align}

where $x_0 \in X$ and $A$ is the infinitesimal generator of $\{S(t); t \geq 0\}$ satisfying $|S(t)y| \leq Me^{\omega t}|y|$ for $t \geq 0$ and $y \in Y$.

We assume the following conditions:

(S1) $S(t)Y \subset X$ for $t > 0$ and $S(t)x \in C([0, \infty); X)$ for $x \in X$.
(S2) $Y = Y_1 \oplus Y_2$ and $P_iS(t) = S(t)P_i$ for $i = 1, 2$ and $t \geq 0$, where $Y_i$ is a closed linear subspace and $P_i$ is a projection from $X$ onto $Y_i$.
(S3) $\{S(t)P_1; t \geq 0\}$ forms a uniformly continuous semigroup on $Y_1$.
(S4) There exist constants $\alpha, \beta > 0, \gamma \in [0, 1), \eta < -\max\{\alpha, \beta\}$ and $M_1, M_2, M_3, M_4, M_5 \geq 0$ such that

\begin{align}
(1.3) & \quad \|y\| \leq M_1|y|, \quad y \in Y_1, \\
(1.4) & \quad |e^{-\eta t}S(t)P_1| \leq M_2e^{\alpha t}|y|, \quad t \leq 0, y \in Y, \\
(1.5) & \quad \|e^{-\eta t}S(t)P_2x\| \leq M_3e^{-\beta t}\|x\|, \quad t \geq 0, x \in X, \\
(1.6) & \quad \|e^{-\eta t}S(t)P_2y\| \leq (M_4t^{-\gamma} + M_5)e^{-\beta t}|y|, \quad t > 0, y \in Y.
\end{align}
The above assumptions ensure the unique mild solution $u(t; x_0) \in C([0, \infty); X)$ of (1.1) and (1.2) for each $u_0 \in X$ (e.g., see [14]). It is known ([1],[5]) that we obtain the existence of IMs for (1.1) under the above conditions.

**Theorem 1.1.** Let (S1)-(S4) be satisfied. In addition we assume

\begin{equation}
K(\alpha, \beta)\text{Lip}(F) < 1 \quad \text{and} \quad \frac{M_2 M_3 K(\alpha, \beta)\text{Lip}(F)}{1 - K(\alpha, \beta)\text{Lip}(F)} < 1,
\end{equation}

where

\begin{equation}
K(\alpha, \beta) = M \{M_1 M_2 \alpha^{-1} + M_4 \Gamma(1 - \gamma)\beta^{\gamma - 1} + M_5\beta^{-1}\}
\end{equation}

$\text{Lip}(F)$ the Lipschitz constant of $F : X \rightarrow Y$, $\Gamma$ the gamma function. Then there exists $h \in C^1(Y_1, P_2 X)$ whose graph $\mathcal{M} = \{y + h(y) : y \in Y_1\}$ is an IM for (1.1), that is,

(a) If $x_0 \in \mathcal{M}$, then $u(t; x_0)$, the mild solution of (1.1) and (1.2), belongs to $\mathcal{M}$ for all $t > 0$.

(b) For each $x_0 \in X$ there exists a unique element $x_0^* \in \mathcal{M}$ such that

$$
\sup_{t \geq 0} e^{-\eta t} \|u(t; x_0) - u(t; x_0^*)\| < \infty.
$$

Since the solution on $\mathcal{M}$ must be of the form $u(t) = p(t) + h(p(t))$ with $p(t) = P_1 u(t)$, the restriction of (1.1) to $\mathcal{M}$ yields

\begin{equation}
\frac{dp}{dt} = Ap + P_1 F(p + h(p)), \quad p \in Y_1,
\end{equation}

whose long-time behavior is equivalent to that of (1.1) because by virtue of (b) the IM $\mathcal{M}$ attracts every orbit at an exponential rate. (1.9) is called an inertial form for (1.1).

### 2. Approximations of IMs

We approximate (1.1) by the following finite difference scheme of the form

\begin{equation}
x_{\ell}^n = C(\lambda_\ell)x_{\ell}^{n-1} + \lambda_\ell K_{\ell} F_{\ell}(x_{\ell}^{n-1}), \quad n, \ell \in \mathbb{N}
\end{equation}

in a space $Y_\ell$ approximating $Y$ in some sense, where $\lambda_\ell \downarrow 0$ as $\ell \rightarrow \infty$, $C(\lambda_\ell)$ and $K_{\ell}$ are given operators in $B(Y_\ell, Y_\ell)$ and $F_{\ell}$ is a given nonlinear operator in $Y_\ell$ stated below. We denote by $B(W, Z)$ the space of bounded linear operators from a Banach space $W$ into a Banach space $Z$. The norm in $B(W, Z)$ will be denoted by $\|\cdot\|_{W, Z}$. We make the following assumptions.

(C1) Let $X$ and $Y$ are reflexive Banach spaces such that $X$ is densely and continuously embedded in $Y$ and that $Y = Y_1 \oplus Y_2$, the direct sum of a finite dimensional subspace $Y_1$ and a closed subspace $Y_2$.

(C2) For each $\ell \in \mathbb{N}$ let $X_\ell$ and $Y_\ell$ be Banach spaces with norms $\|\cdot\|_\ell$ and $|\cdot|_\ell$, respectively, such that $X_\ell$ is continuously embedded in $Y_\ell$. Moreover, there exist $V_\ell \in B(Y, Y_\ell) \cap B(X, X_\ell)$ and $W_\ell \in B(Y_\ell, Y) \cap B(X_\ell, X)$ such that $\lim_{\ell \rightarrow \infty} |V_\ell y|_\ell = 0$ and $\lim_{\ell \rightarrow \infty} |W_\ell u|_\ell = 0$ for all $y \in Y$ and $u \in X$. We define

$$
V = \lim_{\ell \rightarrow \infty} V_\ell \quad \text{and} \quad W = \lim_{\ell \rightarrow \infty} W_\ell,
$$

and

$$
\lim_{\ell \rightarrow \infty} \|V_\ell y - V y\|_{Y_\ell, Y} = 0 \quad \text{and} \quad \lim_{\ell \rightarrow \infty} \|W_\ell u - W u\|_{X_\ell, X} = 0
$$

for all $y \in Y$ and $u \in X$.
$|y|, \lim_{\ell \to \infty} \|V_{\ell}x\|_{\ell} = \|x\|, \lim_{\ell \to \infty} |W_{\ell}V_{\ell}y - y| = 0$ and $V_{\ell}W_{\ell}y = y$ for $x \in X, y \in Y$ and that both $\|W_{\ell}\|_{Y_{\ell}, Y}$ and $\|W_{\ell}\|_{X_{\ell}, X}$ are bounded in $\ell$.

(C3) There exist closed subspaces $Y_{\ell 1}$ and $Y_{\ell 2}$ such that $Y_{\ell} = Y_{\ell 1} \oplus Y_{\ell 2}, V_{\ell}P_{\ell} = P_{\ell 1}V_{\ell}$ and $W_{\ell}P_{\ell 1} = P_{\ell 1}W_{\ell}$ for $i = 1, 2$, where $P_{\ell}$ (resp. $P_{\ell 1}$) denotes a projection from $Y$ onto $Y_{\ell}$ (resp. $Y_{\ell 1}$ onto $Y_{\ell 1}$).

(C4) The linear operators $C(\lambda_{\ell})$ and $K_{\ell}$ satisfy: (i) there exist $M \geq 0$ and $\omega \geq 0$ such that $|C(\lambda_{\ell})^{n}y|_{\ell} \leq Me^{\omega \lambda_{\ell}t}|y|_{\ell}$ and $|K_{\ell}y|_{\ell} \leq Me^{\omega \lambda_{\ell}t}|y|_{\ell}$ for $\ell, n \in \mathbb{N}, y \in Y_{\ell}$; (ii) $\lim_{\ell \to \infty} |(K_{\ell} - I)Y_{\ell}|_{\ell} = 0$ for $y \in Y$; (iii) for each $\ell, \ell' \in \mathbb{N}$ and $i = 1, 2$, $C(\lambda_{\ell})$ commutes with $P_{\ell 1}, P_{\ell 2}$ with $K_{\ell}, K_{\ell'}$ with $P_{\ell 1}, \tilde{C}(\lambda_{\ell})$ with $\tilde{K}_{\ell'}$, respectively, where $\tilde{C}(\lambda) = W_{\ell}C(\lambda)V_{\ell}$ and $\tilde{K}_{\ell} = W_{\ell}K_{\ell}V_{\ell}$.

(C5) $A$ is a densely defined linear operator in $Y$ such that $Y_{1} \subset D(A)$, the range of $I - \lambda_{0}A$ is dense in $Y$ for some $\lambda_{0} > 0$ and

$$\lim_{\ell \to \infty} |\lambda_{\ell}^{-1}(C(\lambda_{\ell}) - I)V_{\ell}y - V_{\ell}Ay|_{\ell} = 0 \quad \text{for } y \in D(A).$$

(C6) The inverse of $C(\lambda_{\ell})P_{\ell 1}$ exists in $B(Y_{\ell 1})$ and there exist constants $\alpha, \beta > 0, \gamma \in [0, 1), \eta < -\max\{\alpha, \beta\}$ and $M_{1}, \cdots, M_{5} \geq 0$ such that

\begin{align*}
(2.2) \quad & \|P_{\ell 1}y\|_{\ell} \leq M_{1}\|P_{\ell 1}y\|_{\ell} \\
(2.3) \quad & |(C(\lambda_{\ell})P_{\ell 1})^{-n}P_{\ell 1}y|_{\ell} \leq M_{2}e^{-(\alpha + \eta)n\lambda_{\ell}}\|y\|_{\ell} \\
(2.4) \quad & |C(\lambda_{\ell})^{n}P_{\ell 2}x|_{\ell} \leq M_{3}e^{(\eta - \beta)n\lambda_{\ell}}\|x\|_{\ell} \\
(2.5) \quad & |C(\lambda_{\ell})^{n}P_{\ell 2}K_{\ell}y|_{\ell} \leq \{M_{4}(n + 1)\lambda_{\ell}\}^{-\gamma} + M_{5}\|y\|_{\ell}
\end{align*}

for $n \geq 0, \ell \geq 1, x \in X_{\ell}, y \in Y_{\ell}$.

(C7) $F_{\ell} \in C^{1}(X_{\ell}, Y_{\ell})$ and there exists a constant $L_{F} \geq 0$ satisfying

$$|F_{\ell}(\xi_{1}) - F_{\ell}(\xi_{2})|_{\ell} \leq L_{F}\|\xi_{1} - \xi_{2}\|_{\ell} \quad \text{for } \ell \in \mathbb{N}, \xi_{1}, \xi_{2} \in X_{\ell}.$$

(C8) For each $x, z \in X$ and each positive sequence $\{\nu_{\ell}\}$ convergent to 0 we have

$$\lim_{\ell \to \infty} |F_{\ell}(V_{\ell}x) - V_{\ell}F(x)|_{\ell} = 0,$$

$$\lim_{\ell \to \infty} |DF_{\ell}(V_{\ell}x)V_{\ell}z - V_{\ell}DF(x)z|_{\ell} = 0 \quad \text{and} \quad \lim_{\ell \to \infty} (\sup_{\|\xi\|_{\ell} \leq \nu_{\ell}} |(DF_{\ell}(V_{\ell}x + \xi) - DF_{\ell}(V_{\ell}x))V_{\ell}z|_{\ell}) = 0.$$

To construct an IM for (2.1) we introduce the Banach space $c_{\eta}^{-}$ of sequences $\tilde{x} = \{x_{n}\}_{n \leq 0}$ in $X_{\ell}$ with the norm $\|\tilde{x}\|^{(n)}_{\ell} = \sup_{n \leq 0} e^{-\eta n\lambda_{\ell}}\|x_{n}\|_{\ell}$. Let $B_{\ell}$ be a bounded subset of $Y_{\ell 1}$. We denote by $BC(B_{\ell}, c_{\eta}^{-})$ the Banach space consisting of bounded and continuous functions $\psi : B_{\ell} \to c_{\eta}^{-}$ with the norm $\|\psi\|_{B_{\ell}} = \sup_{\xi \in B_{\ell}} \|\psi(\xi)\|_{\ell}^{(n)}$. We shall write $\psi \in BC(B_{\ell}, c_{\eta}^{-})$ as

$$\psi(\xi) = \{\psi(\xi, n)\}_{n \leq 0} \in c_{\eta}^{-} \quad \text{for } \xi \in B_{\ell}.$$
Then we define the mapping $H_{\ell}$ from $BC(B_{\ell}, C_{\eta})$ into itself by

\[(2.6) \quad (H_{\ell}\psi)(\xi, n) = R_{\ell}^{n}\xi - \lambda_{\ell} \sum_{i=-1}^{n} R_{\ell}^{n-i-1} p_{1} K_{\ell} F_{\ell}(\psi(\xi, i)) + \lambda_{\ell} \sum_{i=n+1}^{\infty} Q_{\ell}^{i-n-1} p_{2} K_{\ell} F_{\ell}(\psi(\xi, i))\]

for $\xi \in B_{\ell}$ and $n \leq 0$. Here $R_{\ell} = C(\lambda_{\ell}) p_{1}$ and $Q_{\ell} = C(\lambda_{\ell}) p_{2}$. Furthermore we define

\[(2.7) \quad h_{\ell k}(\xi) = ((H_{\ell})^{k}\psi_{0})(\xi, 0) - \xi\]

with

\[\psi_{0}(\xi, n) = \xi \quad \text{for} \quad n \leq 0.\]

Then we have ([10])

**Theorem 2.1.** Let (C1)-(C7) be satisfied. In addition we assume

\[(2.8) \quad K(\alpha, \beta)L_{F} < 1 \quad \text{and} \quad \frac{M_{2} M_{3}' K(\alpha, \beta)L_{F}}{1-K(\alpha, \beta)L_{F}} < 1\]

where

\[(2.9) \quad k(\alpha, \beta) = M\{M_{1} M_{2} \alpha^{-1} + M_{4}' \Gamma(1-\gamma) \beta^{-1} + M_{5}' \beta^{-1}\}\]

and

\[M_{i}' = M_{i} \max\{1, \lim_{\ell \to \infty} ||W_{\ell}||_{X_{\ell}}, x\}, \quad i = 3, 4, 5\]

Then, for every $\ell \in \mathbb{N}$ there exists $h_{\ell} \in C^{1}(Y_{\ell}, c_{\ell}^{-})$ whose graph $M_{\ell} = \{\xi + h_{\ell}(\xi); \xi \in Y_{\ell}\}$ is an IM for (2.1). Moreover, we have for each bounded set $B_{\ell} \subset Y_{\ell}$

\[(2.10) \quad \lim_{k \to \infty} \sup_{\xi \in B_{\ell}} ||h_{\ell k}(\xi) - h_{\ell}(\xi)||_{\ell} = 0\]

and

\[(2.11) \quad \lim_{k \to \infty} \sup_{\xi \in B_{\ell}} ||Dh_{\ell k}(\xi) - Dh_{\ell}(\xi)||_{B(Y_{\ell}, X_{\ell})} = 0.\]

From this theorem the inertial form for (2.1) is described by the system of equations

\[(2.12) \quad p_{\ell}^{n+1} = C(\lambda_{\ell}) p_{\ell}^{n} + \lambda_{\ell} K_{\ell} F_{\ell}(p_{\ell}^{n} + h_{\ell}(p_{\ell}^{n}))\]

$p_{\ell}^{n} \in Y_{\ell}, \quad n, \ell \in \mathbb{N}$

Furthermore, as an approximate inertial form for (2.1) we may employ the following system of equations with some $k$

\[(2.13) \quad p_{\ell}^{n+1} = C(\lambda_{\ell}) p_{\ell}^{n} + \lambda_{\ell} K_{\ell} F_{\ell}(p_{\ell}^{n} + h_{\ell k}(p_{\ell}^{n}))\]

$p_{\ell}^{n} \in Y_{\ell}, \quad n, \ell \in \mathbb{N}$

We emphasize that (2.13) can be solved for $p_{\ell}^{n}$ explicitly.

Now we have our main result which is proved in [12].
Theorem 2.2. Let \((C1)-(C8)\) and \((2.7)\) are satisfied. Then, conditions \((S1)-(S4)\) and \((1.7)\) hold true with the semigroup generated by the operator \(A\) in \((C5)\). Consequently, there exists \(h \in C^1(Y_1, P_2X)\) whose graph is an IM for \((1.1)\). Moreover we have for each bounded set \(B \subset Y_1\)

\[
\lim_{\ell \to \infty} \sup_{y \in B} \|h_\ell(V_\ell y) - V_\ell h(y)\|_\ell = 0
\]

and

\[
\lim_{\ell \to \infty} \sup_{y \in B} \|Dh_\ell(V_\ell y) - V_\ell Dh(y)\|_{B(\ell)} = 0.
\]

From this theorem we can employ \((2.13)\) as an explicit \(C^1\)-approximation of the inertial form \((1.9)\). The \(C^1\) closeness would be a necessary and important step toward establishing a relationship between the dynamics of the PDE and its approximation.

3. Kuramoto-Sivashinsky equations

We consider the renormalized Kuramoto-Sivashinsky equation with periodic boundary condition, with period \(L\)

\[
\begin{cases}
u_t + D^4u + D^2u + uDu = 0 & (x, t) \in \mathbb{R} \times \mathbb{R}^+,
\end{cases}
\]

\[
\begin{cases}
u(x, t) = u(x + L, t) & (x, t) \in \mathbb{R} \times \mathbb{R}^+,
\end{cases}
\]

\[
\begin{cases}
u(x, 0) = u_0(x) & x \in \mathbb{R}.
\end{cases}
\]

\(D\) denotes \(\partial/\partial x\) or \(d/dx\). Let \(H^m_{\text{per}}(0, L)\) denote the subspace of the Sobolev space \(H^m(0, L)\) consisting of functions which, along with all their derivatives up to order \(m - 1\), are periodic with period \(L\). A function \(u\) defined a.e. on \((0, L)\) is said to be odd whenever \(u(x) = -u(L - x)\) a.e. in \((0, L)\). Following Foias et al. [4] and Foias and Titi [6] we set

\[
Y = \{u \in L^2_{\text{per}}(0, L); u \text{ is odd}\}
\]

\[
< u, v > = \int_0^L u(x)v(x)dx \quad \text{for } u, v \in Y
\]

\[
|u| = \sqrt{< u, u >} \quad \text{for } u \in Y
\]

\[
X = \{u \in H^2_{\text{per}}(0, L); u \text{ is odd}\}
\]

\[
\|u\| = |D^2u| \quad \text{for } u \in X
\]

\[
Au = -D^4u \quad \text{for } u \in D(A) \equiv H^4_{\text{per}}(0, L) \cap Y
\]

and

\[
Ru = -D^2u - uDu \quad \text{for } u \in X.
\]
Then (3.1) is written as the following evolution equation in the Hilbert space $Y$

\[
\begin{align*}
\frac{du(t)}{dt} &= Au(t) + Ru(t), \quad t \geq 0 \\
u(0) &= u_0
\end{align*}
\]

(3.2)

It is known (see [4]) that for every $u_0 \in Y$ there exists a unique solution $u(t)$ of (3.2). Moreover, for every $r > 0$ there exists a time $T^*(r) > 0$ such that $\|u(t)\| \leq r_0$ for all $t \geq T^*(r)$ and $u_0 \in Y$ with $|u_0| \leq r$, where $r_0$ is a constant which is independent of $r$. Hence, the study of asymptotic behavior of solutions to (3.2) can be reduced to the study of the prepared equation

(3.3)

\[\frac{du}{dt} = Au + Fu, \quad t \geq 0\]

where

\[
\begin{align*}
F &= -D^2u - \rho(\|u\|)uDu, \\
\rho &\in \mathcal{C}_0^\infty(\mathbb{R}), 0 \leq \rho \leq 1, \\
\rho(s) &= 1 \text{ for } |s| \leq r_0, \quad \rho(s) = 0 \text{ for } |s| > 2r_0
\end{align*}
\]

The operator $-A$ is a positive selfadjoint operator in $Y$ and the functions

\[e_k(x) = \sin(2\pi kx/L)\]

are eigenfunctions of the operator $A$ with corresponding eigenvalues $\nu_k = (2\pi k/L)^4$ for $k = 1, 2, \ldots$. The conditions (S1)-(S4) and (1.7) in Section 1 are satisfied with $Y_1 = \text{span}\{e_1, e_2, \cdots, e_N\}$, $Y_2 = \text{span}\{e_{N+1}, e_{N+2}, \cdots\}$, $\alpha' = (\nu_{N+1} - \nu_N)/2$, $\eta' = -((\nu_{N+1} + \nu_N)/2, \gamma' = 1/2, M'_1 = M'_2 = M'_3 = 1, M'_4 = \sqrt{\nu_N}$ and $M'_5 = \sqrt{\nu_{N+1}}$ if $N$ is sufficiently large. Therefore, (3.3) has an inertial manifold. We shall approximate (3.1) by finite difference schemes. Following Foias and Titi [6], we introduce the set $S^\ell_{\text{odd, per}}$ consisting of $\ell$-dimensional vectors $\xi = (\xi_0, \cdots, \xi_{\ell-1})$ which satisfy

\[\xi_j = -\xi_{\ell-j} \quad \text{for} \quad j = 1, 2, \cdots, \ell - 1, \quad \xi_0 = 0\]

and are extended periodically to a double infinite sequence such that

\[\xi_{j+\ell} = \xi_j, \quad j = 0, \pm 1, \pm 2, \ldots\]

For $\ell \geq 1$ we set

\[Y_\ell = X_\ell = S^\ell_{\text{odd, per}}, \quad \langle \xi, \zeta \rangle_\ell = \frac{L}{\ell} \sum_{k=0}^{\ell-1} \xi_k \zeta_k,\]

\[|\xi|_\ell = \sqrt{\langle \xi, \xi \rangle_\ell} \quad \text{for} \quad \xi, \zeta \in Y_\ell; \quad \text{and} \quad ||\xi||_\ell = |\Delta \ell \xi|_\ell \quad \text{for} \quad \xi \in X_\ell,\]
Define $\theta_{\ell} : C([0, L]) \to \mathbb{R}^{\ell}$ by

$$\theta_{\ell}(u) = (u(x_{0}), u(x_{1}), \cdots, u(x_{\ell-1})),$$

where $x_{j} = jh$ for $j = 0, 1, \cdots, \ell - 1$, and $h = L/\ell$.

**Lemma 3.1.** Let $\ell_{0} = [(\ell - 1)/2]$, the integer part of $(\ell - 1)/2$. $Y_{\ell}$ is an $\ell_{0}$-dimensional Banach space with the norm $|\cdot|_{\ell}$. $\{\theta_{\ell}(e_{1}), \theta_{\ell}(e_{2}), \cdots, \theta_{\ell}(e_{\ell_{0}})\}$ forms an orthogonal basis for $Y_{\ell}$ with $|\theta_{\ell}(e_{j})|_{\ell} = \sqrt{L/2}$.

**Lemma 3.2.** $\theta_{\ell}(e_{k})$ are eigenvectors of $\Delta_{\ell} : Y_{\ell} \to Y_{\ell}$ with corresponding eigenvalue $-(2/h)^{2}\sin^{2}(\pi k/\ell)$ for $1 \leq k \leq \ell_{0}$.

In what follows we set

$$\mu_{k}^{\ell} = (2/h)^{4}\sin^{4}(\pi k/\ell), \quad k = 1, 2, \cdots, \ell_{0}.$$

Notice that $(2/\pi)^{4}\nu_{k} \leq \mu_{k}^{\ell} \leq \nu_{k}$ for $1 \leq k \leq \ell_{0}$.

Define linear operators $V_{\ell} : Y \to Y_{\ell}$ and $W_{\ell} : Y_{\ell} \to Y$ as follows.

$$V_{\ell}u = \theta_{\ell}(u_{\ell}) \quad \text{for} \quad u \in Y,$$

where $u_{\ell} = \sum_{i=1}^{\ell_{0}} \alpha_{i}e_{i}$ with $\alpha_{i} = 2L^{-1} < u, e_{i} >$. Next, thanks to Lemma 3.1, every $\xi \in Y_{\ell}$ can be written uniquely as

$$\xi = \alpha_{1}\theta_{\ell}(e_{1}) + \cdots + \alpha_{\ell_{0}}\theta_{\ell}(e_{\ell_{0}}).$$

We then set

$$W_{\ell}\xi = \alpha_{1}e_{1} + \cdots + \alpha_{\ell_{0}}e_{\ell_{0}}.$$

Finally, we set

$$Y_{\ell_{1}} = \text{span}\{\theta_{\ell}(e_{1}), \cdots, \theta_{\ell}(e_{N})\}, \quad \text{and}$$

$$Y_{\ell_{2}} = \text{span}\{\theta_{\ell}(e_{N+1}), \cdots, \theta_{\ell}(e_{\ell_{0}})\} \quad \text{for} \quad N < \ell_{0}.$$

It is easy to see that conditions (C1)-(C3) in Section 2 hold true in this case.

We here consider the following semi-implicit discrete scheme for (3.1):

$$(3.4) \quad \frac{\xi^{i+1} - \xi^{i}}{\lambda_{\ell}} + \Delta_{\ell}^{2}(1 - \theta)\xi^{i} + \theta\xi^{i+1} + F_{\ell}(\xi^{i}) = 0, \quad \xi^{i} \in Y_{\ell}$$
where $\lambda_\ell \to +0$ as $\ell \to \infty$, $2^{-1} < \theta \leq 1$, $F_\ell(\xi) = -\rho(||\xi||^2_\ell)(\Delta_\ell \xi + B^\ell(\xi, \xi))$ and $B^\ell : Y_\ell \times Y_\ell \to Y_\ell$ is defined as follows: For every $\xi, \hat{\xi} \in Y_\ell$ the $k$-th element $B^\ell_k(\xi, \hat{\xi})$ of $B^\ell(\xi, \hat{\xi})$ is given by

$$B^\ell_k(\xi, \hat{\xi}) = \frac{1}{6h} \{\xi_k(\hat{\xi}_k+1-\hat{\xi}_{k-1}) + \xi_{k+1}\hat{\xi}_{k+1} - \xi_{k-1}\hat{\xi}_{k-1}\}.$$

To apply the preceding results put

$$C(\lambda_\ell) = (I - (1 - \theta)\lambda_\ell \Delta_\ell^2)(I + \theta \lambda_\ell \Delta_\ell^2)^{-1}$$

and

$$K_\ell = (I + \theta \lambda_\ell \Delta_\ell^2)^{-1}.$$

Then (3.4) can be rewritten as (2.1). We have already shown in [2] that conditions (C4)-(C6) hold with $M = M_2 = M_3 = 1$, $\omega = 0$, $M_1 = \sqrt{\mu_N}$, $M_4 = 2$, $M_5 = \sqrt{2\mu_{N+1}}$, $\alpha = \beta = (\nu_{N+1} - \nu_N)/4$, $\eta = (\nu_{N+1} + \nu_N)/2$ and $\gamma = 1/2$.

Finally, to see (C7) and (C8) it suffices to note that

$$DF(u)v = -D^2v - 2\rho'(||u||^2) < D^2u, D^2v > uDu - \rho(||u||^2)(uDv + vDu) \quad \text{for} \quad u, v \in X$$

and

$$DF_\ell(\xi)\eta = -\Delta_\ell \eta - 2\rho'(||\xi||^2_\ell) < \Delta_\ell \xi, \Delta_\ell \eta > \ell \cdot B^\ell(\xi, \xi) - \rho(||\xi||^2_\ell)DB^\ell(\xi, \xi)\eta$$

for $\xi = (\xi_0, \cdots, \xi_{\ell-1})$, $\eta = (\eta_0, \cdots, \eta_{\ell-1}) \in Y_\ell$, where the $k$-th element of $DB^\ell(\xi, \xi)\eta$ is defined by

$$\{DB^\ell(\xi, \xi)\eta\}_k = (6h)^{-1}(\xi_{k+1} + \xi_k + \xi_{k-1})(\eta_{k+1} - \eta_{k-1}) + (6h)^{-1}(\xi_{k+1} - \xi_{k-1})(\eta_{k+1} + \eta_k + \eta_{k-1}).$$

As a result, one can apply Theorem 2.2 to the Kuramoto-Sivashinsky equation (3.1).

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