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Periodic Stability for Nonlinear Systems Generated by Time-Dependent Subdifferentials

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Abstract. A nonlinear time periodic system, which is governed by time-dependent subdifferentials, is considered in a (real) Hilbert space. Recent results on global attractors for our system are presented. Also, these abstract results are applied to a phase-field model with constraint of the Penrose-Fife type.

1. Introduction

Let us consider a nonlinear evolution system

$$(\mathrm{P})_s \qquad u'(t) + \partial\varphi^t(u(t)) + g(t, u(t)) \ni f(t), \quad t > s(\geq 0), \tag{1.1}$$

which is governed by the subdifferential $\partial\varphi^t$ of a time-dependent proper, l.s.c. convex function φ^t on a (real) Hilbert space H, where $u' = \frac{du}{dt}$, $g(t,\cdot)$ is a perturbation and f is a forcing term. In this paper, assuming that $\varphi^t(\cdot)$, $g(t,\cdot)$ and $f(t)$ are periodic in time t with a common period $T_0(>0)$, we investigate the asymptotic behaviour of the dynamical process (evolution operator) $U(t,s) : \overline{D(\varphi^s)} \to \overline{D(\varphi^t)}$, $0 \leq s \leq t < +\infty$, associated to system $(P)_s$; in fact, we present that $(\mathrm{P})_s$ has at least one time-periodic solution with period T_0 and for each $\tau \in R_+ := [0, +\infty)$ the discrete dynamical process $\{T_\tau^n\}_{n=1}^\infty$ on $\overline{D(\varphi^\tau)}$, generated by $T_\tau := U(T_0 + \tau, \tau)$, possesses a global attractor $\mathcal{A}_\tau$ which is periodic in τ with period T_0.

We recall some works (cf. [2]) treating similar topics for a class of semilinear evolution equations.

As an application of our abstract results we treat the large time behaviour of a phase-field model with constraint of the Penrose-Fife type, which is a system of nonlinear PDEs as follows:

$$[\theta+\lambda(t,x,w)]_t-\Delta\left(-\frac{1}{\theta}+\mu\theta\right)=q(t,x) \quad \text{in } Q_s:=(s,+\infty)\times\Omega,\ s\geq 0, \quad (1.2)$$

$$w_t-\kappa\Delta w+\beta(w)+\sigma(w)+\frac{\lambda_w(t,x,w)}{\theta}\ni 0 \quad \text{in } Q_s, \quad (1.3)$$

with boundary conditions

$$\frac{\partial}{\partial n}\left(-\frac{1}{\theta}+\mu\theta\right)+n_0\left(-\frac{1}{\theta}+\mu\theta\right)=h(t,x), \quad \text{on } \Sigma_s:=(s,+\infty)\times\Gamma, \quad (1.4)$$

$$\frac{\partial w}{\partial n}=0 \quad \text{on } \Sigma_s. \quad (1.5)$$

Here Ω is a bounded domain in $R^N, 1\leq N\leq 3$, with smooth boundary $\Gamma:=\partial\Omega$; $\beta(\cdot)$ is a maximal monotone graph in $R\times R$; λ is a smooth function on $R_+\times\Omega\times R$, convex in $w\in R$ and periodic in t with period T_0; $\sigma(\cdot)$ is a smooth function on R; n_0, κ and μ are positive constants and q, h are given data.

The phase-field models with constraint were earlier studied in [5, 10, 13]. In [3, 4], the existence and uniqueness result for the Cauchy problem of the system (1.2)-(1.5) was obtained for good initial data θ_0 and w_0 in the case of $\lambda(t,x,w)=\lambda(x,w)$ without convexity assumption with respect to w.

Notation. Throughout this paper, let H be a (real) Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot,\cdot)_H$. For a proper l.s.c. convex function φ on H we denote by $D(\varphi)$ and $\partial\varphi$ the effective domain and subdifferential of φ, respectively; the domain and range of $\partial\varphi$ are denoted by $D(\partial\varphi)$ and $R(\partial\varphi)$, respectively. We refer for fundamental properties of subdifferentials to [1].

When a given function is periodic in time with period T_0, we say simply that the function is T_0-periodic.

For a point z in H and non-empty subsets X and Y of H, we define

$$\mathrm{dist}_H(z,Y):=\inf_{y\in Y}|z-y|_H, \quad \mathrm{dist}_H(X,Y):=\sup_{x\in X}\inf_{y\in Y}|x-y|_H.$$

2. Abstract results (existence of a T_0-periodic solution)

Evolution equation $(\mathrm{P})_s$ is formulated for any family $\{\varphi^t\}$ in the class $\Phi_p(\{a_r\},\{b_r\};T_0)$ specified below, where $\{a_r\}:=\{a_r;r\geq 0\}$ and $\{b_r\}:=$

$\{b_r; r \geq 0\}$ are families of real functions in $W^{1,2}_{loc}(R_+)$ and $W^{1,1}_{loc}(R_+)$, respectively, such that

$$\sup_{t\geq 0} |a'_r|_{L^2(t,t+1)} + \sup_{t\geq 0} |b'_r|_{L^1(t,t+1)} < +\infty \quad \text{for every } r \geq 0.$$

Definition 2.1. $\{\varphi^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$ if and only if φ^t is a proper l.s.c. convex function on H such that

$$\varphi^{t+T_0}(\cdot) = \varphi^t(\cdot) \quad \text{on } H, \quad \forall t \in R_+,$$

$$\{z \in H; |z|_H \leq k, \ \varphi^t(z) \leq k\} \text{ is compact in } H \text{ for every } t \geq 0 \text{ and } k \geq 0,$$

and the following property $(*)$ is fulfilled:

$(*)$ For each $r \in R_+$, s, $t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \leq |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{\frac{1}{2}})$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).$$

Next, we introduce the class $\mathcal{G}_p(\{\varphi^t\}; T_0)$ associated with $\{\varphi^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$.

Definition 2.2. $\{g(t,\cdot)\} \in \mathcal{G}_p(\{\varphi^t\}; T_0)$ if and only if $g(t,\cdot)$ is an operator from H into H which fulfills the following conditions (g1)-(g6):

(g1) $D(\varphi^t) \subset D(g(t,\cdot)) \subset H$ for all $t \in R_+$ and $g(\cdot, v(\cdot))$ is (strongly) measurable on J for any interval $J \subset R_+$ and $v \in L^2_{loc}(J; H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$.

(g2) There are positive constants C_0, C_1 and C_2 such that

$$|g(t,z)|^2_H \leq C_0\varphi^t(z) + C_1|z|^2_H + C_2, \quad \forall t \in R_+, \quad \forall z \in D(\varphi^t).$$

(g3) (Demi-closedness) If $\{t_n\} \subset R_+$, $\{z_n\} \subset H$, $t_n \to t$, $z_n \to z$ in H (as $n \to +\infty$) and $\{\varphi^{t_n}(z_n)\}$ is bounded, then $g(t_n, z_n) \to g(t,z)$ weakly in H.

(g4) For each $\varepsilon > 0$, there exists a positive constant $C_\varepsilon > 0$ such that

$$|(g(t,z_1) - g(t,z_2), z_1 - z_2)_H| \leq \varepsilon(z_1^* - z_2^*, z_1 - z_2)_H + C_\varepsilon |z_1 - z_2|_H^2,$$

$$\forall t \in R_+,\ \forall z_i \in D(\varphi^t),\ \forall z_i^* \in \partial\varphi^t(z_i),\ i = 1,2.$$

(g5) (Coerciveness) For each bounded set B in H there are positive constants $C_0(B)$ and $C_1(B)$ such that

$$\varphi^t(z) + (g(t,z), z - b)_H \geq C_0(B)|z|_H^2 - C_1(B),$$

$$\forall t \in R_+,\ \forall z \in D(\varphi^t), \forall b \in B.$$

(g6) (T_0-periodicity) $g(t + T_0, \cdot) = g(t, \cdot)$ on H, $\forall t \in R_+$.

The notion of a solution of $(\mathrm{P})_s$ is given in the next definition.

Definition 2.3. (1) A function $u : [s,T] \to H$, $0 \leq s < T < +\infty$, is a solution of $(\mathrm{P})_s$ on $[s,T]$, if $u \in C([s,T];H) \cap W_{loc}^{1,2}((s,T];H)$, $\varphi^{(\cdot)}(u(\cdot)) \in L^1(s,T)$, $g(\cdot, u(\cdot)) \in L^2(s,T;H)$ and

$$f(t) - u'(t) - g(t,u(t)) \in \partial\varphi^t(u(t)) \quad \text{for a.e. } t \in [s,T].$$

A function u is called a solution of $(\mathrm{P})_s$ on $[s,+\infty)$, if it is a solution of $(\mathrm{P})_s$ on $[s,T]$ for every finite $T > s$. Also, $u : [s,T]$ or $[s,+\infty) \to H$ is called a solution of the Cauchy problem for $(\mathrm{P})_s$ with initial value $u_0 \in H$, if it is a solution of $(\mathrm{P})_s$ and $u(s) = u_0$.
(2) u is called a T_0-periodic solution of $(\mathrm{P})_s$ on $[s,+\infty)$, $s \geq 0$, if u is a solution of $(\mathrm{P})_s$ which satisfies T_0-periodicity condition:

$$u(t) = u(t + T_0) \qquad \text{for any } t \in [s,+\infty).$$

Theorem 2.1. (cf. [14; Theorem 2.1.]) *Assume that* $\{\varphi^t\} \in \Phi_p(\{a_r\},\{b_r\};T_0)$, $\{g(t,\cdot)\} \in \mathcal{G}_p(\{\varphi^t\};T_0)$ *and* $f \in L_{loc}^2(R_+;H)$. *Then, the Cauchy problem for* $(P)_s$, $s \geq 0$, *has one and only one solution* u *on* $J_s := [s,+\infty)$ *such that* $(\cdot - s)^{\frac{1}{2}}u'(\cdot) \in L_{loc}^2(J_s;H)$, $(\cdot - s)\varphi^{(\cdot)}(u(\cdot)) \in L_{loc}^\infty(J_s)$ *and* $\varphi^{(\cdot)}(u(\cdot))$ *is absolutely continuous on any compact subinterval of* $(s,+\infty)$, *provided that* $u_0 \in \overline{D(\varphi^s)}$. *In particular, if* $u_0 \in D(\varphi^s)$, *then the solution* u *satisfies that* $u' \in L_{loc}^2(J_s;H)$ *and* $\varphi^{(\cdot)}(u(\cdot))$ *is absolutely continuous on any compact interval in* J_s.

Based on this existence result, we can define the solution operator (dynamical process) associated to $(\mathrm{P})_s$.

Definition 2.4. For every $0 \le s \le t < +\infty$ we denote by $U(t,s)$ the mapping from $\overline{D(\varphi^s)}$ into $\overline{D(\varphi^t)}$ which assigns to each $u_0 \in \overline{D(\varphi^s)}$ the element $u(t) \in \overline{D(\varphi^t)}$, where u is the unique solution of $(P)_s$ with initial condition $u(s) = u_0$.

It is easy to check the following properties of $\{U(t,s)\} := \{U(t,s); 0 \le s \le t < +\infty\}$:

(U1) $U(s,s) = I \quad$ on $\overline{D(\varphi^s)} \quad$ for any $s \in R_+$;

(U2) $U(t_2, s) = U(t_2, t_1) \circ U(t_1, s) \quad$ for any $0 \le s \le t_1 \le t_2 < +\infty$;

(U3) $U(t + T_0, s + T_0) = U(t,s) \quad$ for any $0 \le s \le t < +\infty$, that is, U is T_0-periodic.

In terms of $U(t,s)$, global estimates of solutions for $(\mathrm{P})_s$ are stated as follows:

Theorem 2.2. (cf. [14; Theorem 2.2]) (Global boundedness of the solution for $(\mathrm{P})_s$) *In addition to all the assumptions of Theorem 2.1, suppose that*

$$S_f := \sup_{t \ge 0} |f|_{L^2(t,t+1;H)} < +\infty.$$

Then, for any bounded set B in H,

(i) *There is a positive constant $R_1 := R_1(S_f, B)$ such that*

$$|U(t,s)z|_H \le R_1 \quad \textit{for any } t \ge s \ge 0 \textit{ and all } z \in \overline{D(\varphi^s)} \cap B.$$

(ii) *There is a positive constant $R_2 := R_2(S_f, B)$ such that*

$$\int_t^{t+1} |\varphi^\tau(U(\tau,s)z)| d\tau \le R_2 \quad \textit{for all } t \ge s \ge 0 \textit{ and } z \in \overline{D(\varphi^s)} \cap B.$$

(iii) *For each $\delta > 0$, there is a positive constant $R_3 := R_3(S_f, B, \delta)$ such that*

$$|\varphi^t(U(t,s)z)| + \left| \frac{d}{dt} U(\cdot, s)z \right|^2_{L^2(t,t+1;H)} \le R_3,$$

for all $s \ge 0$, $t \ge s + \delta$ and $z \in \overline{D(\varphi^s)} \cap B$.

With the help of global estimates mentioned in Theorem 2.2 as well as a convergence result [14; Lemma 4.1] we can prove:

Theorem 2.3 *Assume that the same assumptions are made as in Theorem 2.1 and $f \in L^2_{loc}(R_+; H)$ is T_0-periodic, i.e.*

$$f(t) = f(t + T_0) \qquad \textit{for any } t \in R_+.$$

Then for each $s \in R_+$, there exists a T_0-periodic solution u for $(P)_s$.

In the proof of Theorem 2.3, the crucial step is to show that the mapping $T_s := U(T_0 + s, s) : \overline{D(\varphi^s)} \to \overline{D(\varphi^{s+T_0})} = \overline{D(\varphi^s)}$ has a fixed point, which can be done by the Schauder's fixed point theorem. See [9] for a detailed proof.

3. Abstract results (global attractors)

In this section, we present some recent results on global attractors for the solution operator $U(t, s)$ associated with $(\mathrm{P})_s$; all the assumptions of Theorem 2.1 are made as well.

For each $\tau \geq 0$ we define a mapping T_τ by putting

$$T_\tau := U(T_0 + \tau, \tau) : \overline{D(\varphi^\tau)} \to \overline{D(\varphi^\tau)},$$

and its k-th iteration by

$$T_\tau^k := T_\tau \circ T_\tau \circ \cdots \circ T_\tau, \ k = 0, 1, 2, \cdots.$$

Essentially using the theory of discrete dynamical systems (cf. [7, 15]), we have:

Theorem 3.1. *Assume that $\{\varphi^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, $\{g(t, \cdot)\} \in \mathcal{G}_p(\{\varphi^t\}; T_0)$, $f \in L^2_{loc}(R_+; H)$ is T_0-periodic. Then, for each $\tau \geq 0$, there exists a subset $\mathcal{A}_\tau$ of $D(\varphi^\tau)$ such that*

(i) *$\mathcal{A}_\tau$ is non-empty, compact and connected in H,*

(ii) *$T_\tau^k \mathcal{A}_\tau = \mathcal{A}_\tau$ for all $k = 0, 1, 2, \cdots$,*

(iii) *for each bounded set B in H and each number $\epsilon > 0$ there exists a positive integer $N_{B,\epsilon}$ such that*

$$dist_H(T_\tau^k z, \mathcal{A}_\tau) < \epsilon, \quad \forall z \in \overline{D(\varphi^\tau)} \cap B, \ \forall k \geq N_{B,\epsilon}.$$

Moreover, for any $0 \le s \le \tau < +\infty$,

$$\mathcal{A}_\tau = U(\tau, s)\mathcal{A}_s \tag{3.1}$$

holds.

Remark 3.1. (1) For any $\tau \ge 0$, choose $m_\tau \in Z_+$ and $\sigma_\tau \in [0, T_0)$ so that $\tau = \sigma_\tau + m_\tau T_0$. Then, Theorem 3.1 (ii) implies that $\mathcal{A}_\tau = \mathcal{A}_{\sigma_\tau}$, hence the set-valued mapping $\tau \longrightarrow \mathcal{A}_\tau$ is T_0-periodic.

(2) In [11, 12], periodic system $(\mathrm{P})_s$ with $g \equiv 0$ was studied, and it was shown that some solutions do not approach to any periodic solutions as $t \longrightarrow +\infty$; in other words the asymptotic behaviour (as $t \longrightarrow +\infty$) along a single solution is not periodic in time. However, as was seen in (1), the global attractor $\mathcal{A}_\tau$ is T_0-periodic.

(3) Relation (3.1) of Theorem 3.1 implies that $U(\tau, s)$ is a topological mapping from $\mathcal{A}_s$ onto $\mathcal{A}_\tau$.

4. Application to a phase-field model with constraint

In this section, let us consider the periodic problem $(\mathrm{PFC})_s$ of a phase-field model with constraint for the Penrose-Fife type:

$$(\mathrm{PFC})_s \begin{cases} [\theta + \lambda(t,x,w)]_t - \Delta\left(-\dfrac{1}{\theta} + \mu\theta\right) = q(t,x) \quad \text{in } Q_s, \\ w_t - \kappa\Delta w + \beta(w) + \sigma(w) + \dfrac{\lambda_w(t,x,w)}{\theta} \ni 0 \quad \text{in } Q_s, \\ \dfrac{\partial}{\partial n}\left(-\dfrac{1}{\theta} + \mu\theta\right) + n_0\left(-\dfrac{1}{\theta} + \mu\theta\right) = h(t,x) \quad \text{on } \Sigma_s, \\ \dfrac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma_s, \end{cases}$$

under the same notation as section 1.

We assume precisely that

- λ is a smooth function on $R_+ \times R^N \times R$ such that $\lambda(t,x,w)$ is convex with respect to $w \in R$ for each $(t,x) \in R_+ \times R^N$ and is T_0-periodic for each $(x,w) \in \Omega \times R$;

- λ and its partial derivatives $\lambda_w := \dfrac{\partial\lambda}{\partial w}$, $\lambda_t := \dfrac{\partial\lambda}{\partial t}$ are bounded on $R_+ \times \overline{\Omega} \times [-1,1]$, namely,

$$L_\lambda := \sup\{|\lambda(t,x,w)| + |\lambda_w(t,x,w)| + |\lambda_t(t,x,w)|\ ;$$

$$x \in \overline{\Omega},\ t \geq 0,\ |w| \leq 1\Big\} < +\infty;$$

- β is a maximal monotone graph in $R \times R$ such that $\overline{D(\beta)} = [-1, 1]$; we fix a proper l.s.c. convex and non-negative function $\hat{\beta}$ on R whose subdifferential $\partial\hat{\beta}$ coincides with β in R;
- σ is a smooth function on R;
- n_0, μ and κ are positive constants;
- $f \in L^2_{loc}(R_+; L^2(\Omega))$ and $h \in L^2_{loc}(R_+; L^2(\Gamma))$ are T_0-periodic in time.

We need some notation in order to reformulate $(\mathrm{PFC})_s$ as an evolution equation in terms of subdifferential.

Let V be the Sobolev space $H^1(\Omega)$ with norm

$$|v|_V := \left\{ \int_\Omega |\nabla v|^2 dx + n_0 \int_\Gamma |v|^2 d\Gamma \right\}^{\frac{1}{2}}, \qquad \forall v \in V,$$

V^* be the dual space of V and F be the duality mapping from V onto V^*, namely,

$$\langle Fv, z\rangle := \int_\Omega \nabla v \cdot \nabla z dx + n_0 \int_\Gamma vz d\Gamma, \quad \forall v, \forall z \in V,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V^* and V.

Given $q \in L^2(\Omega)$ and $h \in L^2(\Gamma)$, an element $q^* \in V^*$ is uniquely determined by

$$\langle q^*, z\rangle := \int_\Omega qz dx + \int_\Gamma hz d\Gamma, \quad \forall z \in V,$$

and it is easy to check that $Fv = q^*$ is formally equivalent to

$$-\Delta v = q \text{ in } \Omega, \quad \frac{\partial v}{\partial n} + n_0 v = h \text{ on } \Gamma; \tag{4.1}$$

in fact, (4.1) is satisfied in the variational sense that

$$\int_\Omega \nabla v \cdot \nabla z dx + n_0 \int_\Gamma vz d\Gamma = \int_\Omega qz dx + \int_\Gamma hz d\Gamma \ (= \langle q^*, z\rangle), \quad \forall z \in V.$$

By notation Δ_N we denote the Laplacian, with homogeneous Neumann boundary condition, in $L^2(\Omega)$, more precisely,

$$D(\Delta_N) = \left\{ z \in H^2(\Omega) \;\middle|\; \frac{\partial z}{\partial n} = 0 \text{ in } H^{\frac{1}{2}}(\Gamma) \right\}$$

and

$$\Delta_N z = \Delta z \text{ a.e. in } \Omega \text{ for any } z \in D(\Delta_N).$$

It is well known that $-\Delta_N$ is singlevalued and maximal monotone in $L^2(\Omega)$.

As was seen in the recent paper [6], we can reformulate $(\mathrm{PFC})_s$ as an evolution equation with a new variable $e := \theta + \lambda(\cdot,\cdot,w)$, in the following form:

$$\frac{d}{dt}\begin{pmatrix} e(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} F(\alpha(e(t) - \lambda(t,\cdot,w(t))) + \mu e(t)) \\ -\kappa\Delta_N w(t) + \xi(t) - \alpha(e(t) - \lambda(t,\cdot,w(t)))\lambda_w(t,\cdot,w(t)) \end{pmatrix}$$
$$+ \begin{pmatrix} -\mu F\lambda(t,\cdot,w(t)) \\ \sigma(w(t)) \end{pmatrix} = \begin{pmatrix} q^*(t) \\ 0 \end{pmatrix}, \tag{4.2}$$

in the product space

$$H := \begin{array}{c} V^* \\ \times \\ L^2(\Omega) \end{array},$$

where H is a Hilbert space with inner product $(\cdot,\cdot)_H$ given by

$$(U_1, U_2)_H := \left\langle e_1, F^{-1}e_2 \right\rangle + \int_\Omega w_1 w_2 dx,$$

for all $U_i := \begin{pmatrix} e_i \\ w_i \end{pmatrix} \in H$ $(i = 1, 2)$, $q^*(t)$ is the element of V^* determined by

$$\langle q^*(t), z\rangle = \int_\Omega q(t) z dx + \int_\Gamma h(t) z d\Gamma, \quad \forall z \in V,$$

and $\alpha(r) := -\dfrac{1}{r}$ for $r > 0$.

Let us define φ^t on H by putting

$$\varphi^t(u) := \begin{cases} \displaystyle\int_\Omega \left\{ -\log(e - \lambda(t,\cdot,w)) + \frac{\mu}{2}|e|^2 \right\} dx + \frac{\kappa}{2}\int_\Omega |\nabla w|^2 dx + \int_\Omega \hat{\beta}(w) dx \\ \quad \text{if } u := \begin{pmatrix} e \\ w \end{pmatrix} \in \begin{array}{c} L^2(\Omega) \\ \times \\ H^1(\Omega) \end{array} \\ \quad\quad \text{with } \log(e - \lambda(t,\cdot,w)) \in L^1(\Omega),\ \hat{\beta}(w) \in L^1(\Omega), \\ +\infty \quad \text{otherwise.} \end{cases}$$

According to the result of [6, 14], we have the following lemmas.

Lemma 4.1. *(1) For each $t \in R_+$, φ^t is proper l.s.c. convex on H and T_0-periodic, and* $D(\varphi^t) \subset \begin{array}{c} L^2(\Omega) \\ \times \\ H^1(\Omega) \end{array}$. *Moreover, there are positive constants ν_0, ν_1, independent of $t \in R_+$, such that*

$$\varphi^t(u) \geq \nu_0(|e|^2_{L^2(\Omega)} + |w|^2_{H^1(\Omega)}) - \nu_1, \quad \forall u := \begin{pmatrix} e \\ w \end{pmatrix} \in D(\varphi^t).$$

(2) $\{\varphi^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, where $a_r(t) = b_r(t) := R_0 t$ for all $r \geq 0$ and $t \in R_+$, with a (sufficiently large) constant $R_0 > 0$; in fact, we can choose as R_0 a constant of the form const.L_λ.

Lemma 4.2. *For each $t \in R_+$,*

$$D(\partial\varphi^t) = \left\{ \begin{pmatrix} e \\ w \end{pmatrix} \in \begin{array}{c} L^2(\Omega) \\ \times \\ H^2(\Omega) \end{array} ; \begin{array}{l} \alpha(e - \lambda(t, \cdot, w)) + \mu e \in V, \ \dfrac{\partial w}{\partial n} = 0 \ \textit{in } H^{\frac{1}{2}}(\Gamma), \\ \exists \xi \in L^2(\Omega) \ \textit{such that } \xi \in \beta(w) \ \textit{a.e. on } \Omega \end{array} \right\}$$

and if $\begin{pmatrix} e^* \\ w^* \end{pmatrix} \in \partial\varphi^t \begin{pmatrix} e \\ w \end{pmatrix}$, *then*

$$\begin{aligned} e^* &= F\left(\alpha(e - \lambda(t, \cdot, w)) + \mu e\right), \\ w^* &= -\kappa\Delta_N w + \xi - \alpha(e - \lambda(t, \cdot, w))\lambda_w(t, \cdot, w) \\ &\textit{for some } \xi \in L^2(\Omega) \textit{ such that } \xi \in \beta(w) \textit{ a.e. on } \Omega. \end{aligned} \tag{4.3}$$

Moreover, we have

$$(u_1^* - u_2^*, u_1 - u_2)_H \geq \mu|e_1 - e_2|^2_{L^2(\Omega)} + \kappa|\nabla(w_1 - w_2)|^2_{L^2(\Omega)} \tag{4.4}$$

$$\forall t \in R_+, \ \forall u_i := \begin{pmatrix} e_i \\ w_i \end{pmatrix} \in D(\partial\varphi^t), \ \forall u_I^* \in \partial\varphi^t(u_i), \ i = 1, 2.$$

Now, combining expressions (4.2) and (4.3), we see that our system $(\text{PFC})_s$ is reformulated as the evolution equation

$$u'(t) + \partial\varphi^t(u(t)) + g(t, u(t)) \ni f(t) \quad \text{in } H, \ t > s (\geq 0),$$

where

$$g(t,u) := \begin{pmatrix} -\mu F\lambda(t,\cdot,w) \\ \sigma(w) \end{pmatrix} \text{ for } u := \begin{pmatrix} e \\ w \end{pmatrix} \in \begin{matrix} L^2(\Omega) \\ \times \\ H^1(\Omega) \end{matrix}, \; f(t) := \begin{pmatrix} q^*(t) \\ 0 \end{pmatrix}. \tag{4.5}$$

It is not difficult to check with the help of (4.4) that the operator $g(t,\cdot)$ defined by (4.5) satisfies all the conditions (g1)-(g6) in Definition 2.2.

As direct consequences of Theorems 2.3 and 3.1, we see that the periodic system (4.1)-(4.4) has at least one T_0-periodic solution and the global attractor $\mathcal{A}_\tau$ for each $\tau \geq 0$. Namely, for any bounded subset $B \in X$ any solution $[\theta(nT_0+\tau)+\lambda(nT_0+\tau,\cdot,w(nT_0+\tau)), w(nT_0+\tau)]$ of (PFC)$_s$ starting from B converges uniformly in τ to the global attractor $\mathcal{A}_\tau$ of the periodic system (PFC)$_s$.

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