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Periodic Stability for Nonlinear Systems Generated by Time-Dependent Subdifferentials

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Abstract. A nonlinear time periodic system, which is governed by time-dependent subdifferentials, is considered in a (real) Hilbert space. Recent results on global attractors for our system are presented. Also, these abstract results are applied to a phase-field model with constraint of the Penrose-Fife type.

1. Introduction

Let us consider a nonlinear evolution system

\[(P)_s \quad u'(t) + \partial \varphi^t(u(t)) + g(t, u(t)) \ni f(t), \quad t > s (\geq 0), \tag{1.1} \]

which is governed by the subdifferential \(\partial \varphi^t\) of a time-dependent proper, l.s.c. convex function \(\varphi^t\) on a (real) Hilbert space \(H\), where \(u' = \frac{du}{dt}\), \(g(t, \cdot)\) is a perturbation and \(f\) is a forcing term. In this paper, assuming that \(\varphi^t(\cdot), g(t, \cdot)\) and \(f(t)\) are periodic in time \(t\) with a common period \(T_0(>0)\), we investigate the asymptotic behaviour of the dynamical process (evolution operator) \(U(t,s) : \overline{D(\varphi^s)} \to \overline{D(\varphi^t)}, \ 0 \leq s \leq t < +\infty\), associated to system \((P)_s\); in fact, we present that \((P)_s\) has at least one time-periodic solution with period \(T_0\) and for each \(\tau \in \mathbb{R}_+ := [0, +\infty)\) the discrete dynamical process \(\{T^n_\tau\}_{n=1}^\infty\) on \(\overline{D(\varphi^\tau)}\), generated by \(T_\tau := U(T_0 + \tau, \tau)\), possesses a global attractor \(\mathcal{A}_\tau\) which is periodic in \(\tau\) with period \(T_0\).

We recall some works (cf. [2]) treating similar topics for a class of semilinear evolution equations.

As an application of our abstract results we treat the large time behaviour of a phase-field model with constraint of the Penrose-Fife type, which is a
system of nonlinear PDEs as follows:

$$[\theta + \lambda(t, x, w)]_t - \Delta \left( -\frac{1}{\theta} + \mu \theta \right) = q(t, x) \quad \text{in } Q_s := (s, +\infty) \times \Omega, \ s \geq 0, \ (1.2)$$

$$w_t - \kappa \Delta w + \beta(w) + \sigma(w) + \frac{\lambda_w(t, x, w)}{\theta} \ni 0 \quad \text{in } Q_s, \quad (1.3)$$

with boundary conditions

$$\frac{\partial}{\partial n} \left( -\frac{1}{\theta} + \mu \theta \right) + n_0 \left( -\frac{1}{\theta} + \mu \theta \right) = h(t, x), \quad \text{on } \Sigma_s := (s, +\infty) \times \Gamma, \quad (1.4)$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma_s. \quad (1.5)$$

Here $\Omega$ is a bounded domain in $\mathbb{R}^N, 1 \leq N \leq 3$, with smooth boundary $\Gamma := \partial \Omega$; $\beta(\cdot)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$; $\lambda$ is a smooth function on $\mathbb{R}_+ \times \Omega \times \mathbb{R}$, convex in $w \in \mathbb{R}$ and periodic in $t$ with period $T_0$; $\sigma(\cdot)$ is a smooth function on $\mathbb{R}$; $n_0, \kappa$ and $\mu$ are positive constants and $q, h$ are given data.

The phase-field models with constraint were earlier studied in [5, 10, 13]. In [3, 4], the existence and uniqueness result for the Cauchy problem of the system (1.2)-(1.5) was obtained for good initial data $\theta_0$ and $w_0$ in the case of $\lambda(t, x, w) = \lambda(x, w)$ without convexity assumption with respect to $w$.

**Notation.** Throughout this paper, let $H$ be a (real) Hilbert space with norm $| \cdot |_H$ and inner product $(\cdot, \cdot)_H$. For a proper l.s.c. convex function $\varphi$ on $H$ we denote by $D(\varphi)$ and $\partial \varphi$ the effective domain and subdifferential of $\varphi$, respectively; the domain and range of $\partial \varphi$ are denoted by $D(\partial \varphi)$ and $R(\partial \varphi)$, respectively. We refer for fundamental properties of subdifferentials to [1].

When a given function is periodic in time with period $T_0$, we say simply that the function is $T_0$-periodic.

For a point $z$ in $H$ and non-empty subsets $X$ and $Y$ of $H$, we define

$$\text{dist}_H(z, Y) := \inf_{y \in Y} |z - y|_H, \quad \text{dist}_H(X, Y) := \sup_{x \in X} \inf_{y \in Y} |x - y|_H.$$  

**2. Abstract results (existence of a $T_0$-periodic solution)**

Evolution equation $(P)_s$ is formulated for any family $\{\varphi^t\}$ in the class $\Phi_p(\{a_r\}, \{b_r\}; T_0)$ specified below, where $\{a_r\} := \{a_r; r \geq 0\}$ and $\{b_r\} :=$
\{b_r; r \geq 0\} are families of real functions in $W_{loc}^{1,2}(R_+)$ and $W_{loc}^{1,1}(R_+)$, respectively, such that
\[
\sup_{t \geq 0} |a'_r|_{L^2(t,t+1)} + \sup_{t \geq 0} |b'_r|_{L^1(t,t+1)} < +\infty \quad \text{for every } r \geq 0.
\]

**Definition 2.1.** \{\varphi^t\} $\in \Phi_p(\{a_r\}, \{b_r\}; \tau_0)$ if and only if \varphi^t is a proper l.s.c. convex function on $H$ such that
\[
\varphi^{t+\tau_0}(\cdot) = \varphi^t(\cdot) \quad \text{on } H, \quad \forall t \in R_+,
\]
\{z \in H; |z|_H \leq k, \varphi^t(z) \leq k\} is compact in $H$ for every $t \geq 0$ and $k \geq 0$, and the following property (*) is fulfilled:

\begin{itemize}
  \item[(*)] For each $r \in R_+, s, t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that
  \[
  |\tilde{z} - z|_H \leq |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{\frac{1}{2}})
  \]
  and
  \[
  \varphi^t(\tilde{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).
  \]
\end{itemize}

Next, we introduce the class $\mathcal{G}_p(\{\varphi^t\}; \tau_0)$ associated with \{\varphi^t\} $\in \Phi_p(\{a_r\}, \{b_r\}; \tau_0)$.

**Definition 2.2.** \{g(t, \cdot)\} $\in \mathcal{G}_p(\{\varphi^t\}; \tau_0)$ if and only if $g(t, \cdot)$ is an operator from $H$ into $H$ which fulfills the following conditions (g1)-(g6):

\begin{itemize}
  \item[(g1)] $D(\varphi^t) \subset D(g(t, \cdot)) \subset H$ for all $t \in R_+$ and $g(\cdot, v(\cdot))$ is (strongly) measurable on $J$ for any interval $J \subset R_+$ and $v \in L_{loc}^2(J; H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$.
  \item[(g2)] There are positive constants $C_0, C_1$ and $C_2$ such that
  \[
  |g(t, z)|_H^2 \leq C_0 \varphi^t(z) + C_1|z|_H^2 + C_2, \quad \forall t \in R_+, \quad \forall z \in D(\varphi^t).
  \]
  \item[(g3)] (Demi-closedness) If \{t_n\} $\subset R_+$, \{z_n\} $\subset H$, $t_n \to t$, $z_n \to z$ in $H$ (as $n \to +\infty$) and $\{\varphi^n(z_n)\}$ is bounded, then $g(t_n, z_n) \to g(t, z)$ weakly in $H$.
\end{itemize}
For each $\epsilon > 0$, there exists a positive constant $C_{\epsilon} > 0$ such that
\[ |(g(t, z_{1}) - g(t, z_{2}), z_{1} - z_{2})_{H}| \leq \epsilon(z_{1}^{*} - z_{2}^{*}, z_{1} - z_{2})_{H} + C_{\epsilon}|z_{1} - z_{2}|^{2}_{H}, \]
\[ \forall t \in R_{+}, \forall z_{i} \in D(\varphi^{t}), \forall z_{i}^{*} \in \partial\varphi^{t}(z_{i}), i = 1, 2. \]

(Coerciveness) For each bounded set $B$ in $H$ there are positive constants $C_{0}(B)$ and $C_{1}(B)$ such that
\[ \varphi^{t}(z) + (g(t, z), z - b)_{H} \geq C_{0}(B)|z|_{H}^{2} - C_{1}(B), \]
\[ \forall t \in R_{+}, \forall z \in D(\varphi^{t}), \forall b \in B. \]

(T$_{0}$-periodicity) $g(t + T_{0}, \cdot) = g(t, \cdot)$ on $H$, $\forall t \in R_{+}$.

The notion of a solution of $(P)_{s}$ is given in the next definition.

**Definition 2.3.** (1) A function $u : [s, T] \rightarrow H$, $0 \leq s < T < +\infty$, is a solution of $(P)_{s}$ on $[s, T]$, if $u \in C([s, T]; H) \cap W^{1,2}_{loc}((s, T]; H)$, $\varphi^{(\cdot)}(u(\cdot)) \in L^{1}(s, T)$, $g(\cdot, u(\cdot)) \in L^{2}(s, T, H)$ and
\[ f(t) - u'(t) - g(t, u(t)) \in \partial\varphi^{t}(u(t)) \quad \text{for a.e. } t \in [s, T]. \]

A function $u$ is called a solution of $(P)_{s}$ on $[s, +\infty)$, if it is a solution of $(P)_{s}$ on $[s, T]$ for every finite $T > s$. Also, $u : [s, T]$ or $[s, +\infty) \rightarrow H$ is called a solution of the Cauchy problem for $(P)_{s}$ with initial value $u_{0} \in H$, if it is a solution of $(P)_{s}$ and $u(s) = u_{0}$.

(2) $u$ is called a $T_{0}$-periodic solution of $(P)_{s}$ on $[s, +\infty)$, $s \geq 0$, if $u$ is a solution of $(P)_{s}$ which satisfies $T_{0}$-periodicity condition:
\[ u(t) = u(t + T_{0}) \quad \text{for any } t \in [s, +\infty). \]

**Theorem 2.1.** (cf. [14; Theorem 2.1.]) Assume that $\{\varphi^{t}\} \in \Phi_{p}(\{a_{r}\}, \{b_{r}\}; T_{0})$, $\{g(t, \cdot)\} \in G_{p}(\{\varphi^{t}\}; T_{0})$ and $f \in L^{2}_{loc}(R_{+}; H)$. Then, the Cauchy problem for $(P)_{s}$, $s \geq 0$, has one and only one solution $u$ on $J_{s} : = [s, +\infty)$ such that $(-s)^{\frac{1}{2}}u'(\cdot) \in L_{loc}^{2}(J_{s}; H)$, $(-s)\varphi^{(\cdot)}(u(\cdot)) \in L_{loc}^{\infty}(J_{s})$ and $\varphi^{(\cdot)}(u(\cdot))$ is absolutely continuous on any compact subinterval of $(s, +\infty)$, provided that $u_{0} \in D(\varphi^{s})$. In particular, if $u_{0} \in D(\varphi^{s})$, then the solution $u$ satisfies that $u' \in L_{loc}^{2}(J_{s}; H)$ and $\varphi^{(\cdot)}(u(\cdot))$ is absolutely continuous on any compact interval in $J_{s}$.

Based on this existence result, we can define the solution operator (dynamical process) associated to $(P)_{s}$.
**Definition 2.4.** For every $0 \leq s \leq t < +\infty$ we denote by $U(t, s)$ the mapping from $\overline{D(\varphi^s)}$ into $\overline{D(\varphi^t)}$ which assigns to each $u_0 \in \overline{D(\varphi^s)}$ the element $u(t) \in \overline{D(\varphi^t)}$, where $u$ is the unique solution of $(P)_s$ with initial condition $u(s) = u_0$.

It is easy to check the following properties of $\{U(t, s)\} := \{U(t, s); 0 \leq s \leq t < +\infty\}$:

(U1) $U(s, s) = I$ on $\overline{D(\varphi^s)}$ for any $s \in R_+$;

(U2) $U(t_2, s) = U(t_2, t_1) \circ U(t_1, s)$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$;

(U3) $U(t + T_0, s + T_0) = U(t, s)$ for any $0 \leq s \leq t < +\infty$, that is, $U$ is $T_0$-periodic.

In terms of $U(t, s)$, global estimates of solutions for $(P)_s$ are stated as follows:

**Theorem 2.2.** (cf. [14; Theorem 2.2]) (Global boundedness of the solution for $(P)_s$) In addition to all the assumptions of Theorem 2.1, suppose that

$$S_f := \sup_{t \geq 0} |f|_{L^2(t, t+1; H)} < +\infty.$$ 

Then, for any bounded set $B$ in $H$,

(i) There is a positive constant $R_1 := R_1(S_f, B)$ such that

$$|U(t, s)z|_H \leq R_1 \quad \text{for any } t \geq s \geq 0 \text{ and all } z \in \overline{D(\varphi^s)} \cap B.$$

(ii) There is a positive constant $R_2 := R_2(S_f, B)$ such that

$$\int_t^{t+1} |\varphi^s(U(\tau, s)z)| d\tau \leq R_2 \quad \text{for all } t \geq s \geq 0 \text{ and } z \in \overline{D(\varphi^s)} \cap B.$$ 

(iii) For each $\delta > 0$, there is a positive constant $R_3 := R_3(S_f, B, \delta)$ such that

$$|\varphi^s(U(t, s)z)| + \left| \frac{d}{dt} U(t, s)z \right|^2_{L^2(t, t+1; H)} \leq R_3,$$

for all $s \geq 0$, $t \geq s + \delta$ and $z \in \overline{D(\varphi^s)} \cap B$. 
With the help of global estimates mentioned in Theorem 2.2 as well as a convergence result [14; Lemma 4.1] we can prove:

**Theorem 2.3** Assume that the same assumptions are made as in Theorem 2.1 and \( f \in L_{\text{loc}}^{2}(R_{+};H) \) is \( T_{0} \)-periodic, i.e.

\[
f(t) = f(t + T_{0}) \quad \text{for any } t \in R_{+}.
\]

Then for each \( s \in R_{+} \), there exists a \( T_{0} \)-periodic solution \( u \) for \((P)_{s}\).

In the proof of Theorem 2.3, the crucial step is to show that the mapping

\[
T_{s} := U(T_{0} + s, s) : \overline{D(\varphi^{s})} \rightarrow \overline{D(\varphi^{s+T_{0}})} = \overline{D(\varphi^{s})}
\]

has a fixed point, which can be done by the Schauder's fixed point theorem. See [9] for a detailed proof.

3. Abstract results (global attractors)

In this section, we present some recent results on global attractors for the solution operator \( U(t, s) \) associated with \((P)_{s}\); all the assumptions of Theorem 2.1 are made as well.

For each \( \tau \geq 0 \) we define a mapping \( T_{\tau} \) by putting

\[
T_{\tau} := U(T_{0} + \tau, \tau) : \overline{D(\varphi^{\tau})} \rightarrow \overline{D(\varphi^{\tau})},
\]

and its \( k \)-th iteration by

\[
T_{\tau}^{k} := T_{\tau} \circ T_{\tau} \circ \cdots \circ T_{\tau}, \ k = 0, 1, 2, \ldots.
\]

Essentially using the theory of discrete dynamical systems (cf. [7, 15]), we have:

**Theorem 3.1.** Assume that \( \{\varphi^{t}\} \in \Phi_{p}({\{a_{r}\}, {b_{r}}};T_{0}), \{g(t, \cdot)\} \in \mathcal{G}_{p}({\{\varphi^{t}\};T_{0}}), \ f \in L_{\text{loc}}^{2}(R_{+};H) \) is \( T_{0} \)-periodic. Then, for each \( \tau \geq 0 \), there exists a subset \( A_{\tau} \)

of \( D(\varphi^{\tau}) \) such that

(i) \( A_{\tau} \) is non-empty, compact and connected in \( H \),

(ii) \( T_{\tau}^{k}A_{\tau} = A_{\tau} \) for all \( k = 0, 1, 2, \ldots \),

(iii) for each bounded set \( B \) in \( H \) and each number \( \epsilon > 0 \) there exists a positive integer \( N_{B,\epsilon} \) such that

\[
\text{dist}_{H}(T_{\tau}^{k}z, A_{\tau}) < \epsilon, \ \forall z \in \overline{D(\varphi^{\tau})} \cap B, \ \forall k \geq N_{B,\epsilon}.
\]
Moreover, for any $0 \leq s \leq \tau < +\infty$,

$$A_{\tau} = U(\tau, s)A_s$$

(3.1)

holds.

**Remark 3.1.** (1) For any $\tau \geq 0$, choose $m_{\tau} \in \mathbb{Z}_+$ and $\sigma_{\tau} \in [0, T_0)$ so that $\tau = \sigma_{\tau} + m_{\tau}T_0$. Then, Theorem 3.1 (ii) implies that $A_{\sigma_{\tau}}$, hence the set-valued mapping $\tau \rightarrow A_{\tau}$ is $T_0$-periodic.

(2) In [11, 12], periodic system $(P)_s$ with $g \equiv 0$ was studied, and it was shown that some solutions do not approach to any periodic solutions as $t \rightarrow +\infty$; in other words the asymptotic behaviour (as $t \rightarrow +\infty$) along a single solution is not periodic in time. However, as was seen in (1), the global attractor $A_{\tau}$ is $T_0$-periodic.

(3) Relation (3.1) of Theorem 3.1 implies that $U(\tau, s)$ is a topological mapping from $\mathcal{A}_s$ onto $\mathcal{A}_{\tau}$.

4. Application to a phase-field model with constraint

In this section, let us consider the periodic problem $(PFC)_s$ of a phase-field model with constraint for the Penrose-Fife type:

$$[\theta + \lambda(t, x, w)]_t - \Delta \left( -\frac{1}{\theta} + \mu \theta \right) = q(t, x) \quad \text{in } Q_s,$$

$$w_t - \kappa \Delta w + \beta(w) + \sigma(w) + \frac{\lambda_w(t, x, w)}{\theta} \ni 0 \quad \text{in } Q_s,$$

$$\frac{\partial}{\partial n} \left( -\frac{1}{\theta} + \mu \theta \right) + n_0 \left( -\frac{1}{\theta} + \mu \theta \right) = h(t, x) \quad \text{on } \Sigma_s,$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma_s,$$

under the same notation as section 1.

We assume precisely that

- $\lambda$ is a smooth function on $\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}$ such that $\lambda(t, x, w)$ is convex with respect to $w \in \mathbb{R}$ for each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$ and is $T_0$-periodic for each $(x, w) \in \Omega \times \mathbb{R}$;

- $\lambda$ and its partial derivatives $\lambda_w := \frac{\partial \lambda}{\partial w}$, $\lambda_t := \frac{\partial \lambda}{\partial t}$ are bounded on $\mathbb{R}_+ \times \overline{\Omega} \times [-1, 1]$, namely,

$$L_{\lambda} := \sup \{|\lambda(t, x, w)| + |\lambda_w(t, x, w)| + |\lambda_t(t, x, w)| \} ;$$
$x \in \overline{\Omega}, \ t \geq 0, \ |w| \leq 1 \} < +\infty$;

- \(\beta\) is a maximal monotone graph in \(R \times R\) such that \(D(\beta) = [-1, 1]\); we fix a proper l.s.c. convex and non-negative function \(\beta\) on \(R\) whose subdifferential \(\partial \beta\) coincides with \(\beta\) in \(R\);

- \(\sigma\) is a smooth function on \(R\);

- \(n_0, \mu\) and \(\kappa\) are positive constants;

- \(f \in L_{loc}^{2}(R_+; L^{2}(\Omega))\) and \(h \in L_{loc}^{2}(R_+; L^{2}(\Gamma))\) are \(T_0\)-periodic in time.

We need some notation in order to reformulate (PFC)$_s$ as an evolution equation in terms of subdifferential.

Let \(V\) be the Sobolev space \(H^1(\Omega)\) with norm

\[
|v|_V := \left\{ \int_{\Omega} |\nabla v|^2 dx + n_0 \int_{\Gamma} |v|^2 d\Gamma \right\}^{\frac{1}{2}}, \quad \forall v \in V,
\]

\(V^*\) be the dual space of \(V\) and \(F\) be the duality mapping from \(V\) onto \(V^*\), namely,

\[
\langle Fv, z \rangle := \int_{\Omega} \nabla v \cdot \nabla z dx + n_0 \int_{\Gamma} vz d\Gamma, \quad \forall v, \forall z \in V,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the duality pairing between \(V^*\) and \(V\).

Given \(q \in L^2(\Omega)\) and \(h \in L^2(\Gamma)\), an element \(q^* \in V^*\) is uniquely determined by

\[
\langle q^*, z \rangle := \int_{\Omega} qz dx + \int_{\Gamma} hz d\Gamma, \quad \forall z \in V,
\]

and it is easy to check that \(Fv = q^*\) is formally equivalent to

\[
-\Delta v = q \text{ in } \Omega, \quad \frac{\partial v}{\partial n} + n_0 v = h \text{ on } \Gamma; \quad (4.1)
\]

in fact, (4.1) is satisfied in the variational sense that

\[
\int_{\Omega} \nabla v \cdot \nabla z dx + n_0 \int_{\Gamma} vz d\Gamma = \int_{\Omega} qz dx + \int_{\Gamma} hz d\Gamma \ (= \langle q^*, z \rangle), \quad \forall z \in V.
\]

By notation \(\Delta_N\) we denote the Laplacian, with homogeneous Neumann boundary condition, in \(L^2(\Omega)\), more precisely,

\[
D(\Delta_N) = \left\{ z \in H^2(\Omega) \left| \frac{\partial z}{\partial n} = 0 \text{ in } H^{\frac{1}{2}}(\Gamma) \right. \right\}
\]
and

\[ \Delta_N z = \Delta z \text{ a.e. in } \Omega \text{ for any } z \in D(\Delta_N). \]

It is well known that \(-\Delta_N\) is singlevalued and maximal monotone in \(L^2(\Omega)\).

As was seen in the recent paper [6], we can reformulate \((\text{PFC})_s\) as an evolution equation with a new variable \(e := \theta + \lambda(\cdot, \cdot, w)\), in the following form:

\[
\frac{d}{dt} \begin{pmatrix} e(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} F(\alpha(e(t) - \lambda(t, \cdot, w(t))) + \mu e(t)) \\ -\kappa \Delta_N w(t) + \xi(t) - \alpha(e(t) - \lambda(t, \cdot, w(t)))\lambda_w(t, \cdot, w(t)) \end{pmatrix} + \begin{pmatrix} -\mu F \lambda(t, \cdot, w(t)) \\ \sigma(w(t)) \end{pmatrix} = \begin{pmatrix} q^*(t) \\ 0 \end{pmatrix},
\]

in the product space

\[
V^* \quad \times \quad L^2(\Omega)
\]

where \(H\) is a Hilbert space with inner product \((\cdot, \cdot)_H\) given by

\[
(U_1, U_2)_H := \langle e_1, F^{-1}e_2 \rangle + \int_\Omega w_1 w_2 dx,
\]

for all \(U_i := \begin{pmatrix} e_i \\ w_i \end{pmatrix} \in H \ (i = 1, 2)\), \(q^*(t)\) is the element of \(V^*\) determined by

\[
\langle q^*(t), z \rangle = \int_\Omega q(t)z dx + \int_\Gamma h(t)zd\Gamma, \quad \forall z \in V,
\]

and \(\alpha(r) := -\frac{1}{r}\) for \(r > 0\).

Let us define \(\varphi^t\) on \(H\) by putting

\[
\varphi^t(u) := \begin{cases} 
\int_\Omega \left\{-\log(e - \lambda(t, \cdot, w)) + \frac{\mu}{2}|e|^2\right\} dx + \frac{\kappa}{2} \int_\Omega |\nabla w|^2 dx + \int_\Omega \hat{\beta}(w) dx 
& \text{if } u := \begin{pmatrix} e \\ w \end{pmatrix} \in L^2(\Omega) \times H^1(\Omega), \\
+\infty & \text{otherwise,}
\end{cases}
\]

with \(\log(e - \lambda(t, \cdot, w)) \in L^1(\Omega), \hat{\beta}(w) \in L^1(\Omega)\),
According to the result of [6, 14], we have the following lemmas.

**Lemma 4.1.** (1) For each $t \in R_+$, $\varphi^t$ is proper l.s.c. convex on $H$ and $T_0$-periodic, and $D(\varphi^t) \subset \times$. Moreover, there are positive constants $\nu_0, \nu_1$, independent of $t \in R_+$, such that

$$\varphi^t(u) \geq \nu_0(|e|^2_{L^2(\Omega)} + |w|^2_{H^1(\Omega)}) - \nu_1, \quad \forall u := \begin{pmatrix} e \\ w \end{pmatrix} \in D(\varphi^t).$$

(2) $\{\varphi^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, where $a_r(t) = b_r(t) := R_0 t$ for all $r \geq 0$ and $t \in R_+$, with a (sufficiently large) constant $R_0 > 0$; in fact, we can choose as $R_0$ a constant of the form $\text{const.} L_\lambda$.

**Lemma 4.2.** For each $t \in R_+$,

$$D(\partial \varphi^t) = \left\{ \begin{pmatrix} e \\ w \end{pmatrix} \in \times \begin{pmatrix} L^2(\Omega) \\ H^2(\Omega) \end{pmatrix} ; \begin{array}{l}
\alpha(e - \lambda(t, \cdot, w)) + \mu e \in V, \\
\frac{\partial w}{\partial n} = 0 \text{ in } H^{\frac{1}{2}}(\Gamma), \\
\exists \xi \in L^2(\Omega) \text{ such that } \xi \in \beta(w) \text{ a.e. on } \Omega \end{array} \right\}$$

and if $\begin{pmatrix} e^* \\ w^* \end{pmatrix} \in \partial \varphi^t \begin{pmatrix} e \\ w \end{pmatrix}$, then

$$e^* = F(\alpha(e - \lambda(t, \cdot, w)) + \mu e),$$

$$w^* = -\kappa \Delta_N w + \xi - \alpha(e - \lambda(t, \cdot, w))\lambda_w(t, \cdot, w) \quad (4.3)$$

for some $\xi \in L^2(\Omega)$ such that $\xi \in \beta(w)$ a.e. on $\Omega$.

Moreover, we have

$$\begin{pmatrix} u_1^* - u_2^* \\ u_1 - u_2 \end{pmatrix}_H \geq \mu |e_1 - e_2|^2_{L^2(\Omega)} + \kappa |\nabla(w_1 - w_2)|^2_{L^2(\Omega)} \quad (4.4)$$

$$\forall t \in R_+, \forall u_i := \begin{pmatrix} e_i \\ w_i \end{pmatrix} \in D(\partial \varphi^t), \forall u_i^* \in \partial \varphi^t(u_i), \ i = 1, 2.$$

Now, combining expressions (4.2) and (4.3), we see that our system $(\text{PFC})_s$ is reformulated as the evolution equation

$$u'(t) + \partial \varphi^t(u(t)) + g(t, u(t)) \ni f(t) \quad \text{in } H, \ t > s(\geq 0),$$
where
\[
g(t, u) := \left( -\mu F\lambda(t, \cdot, w) \right) \quad \text{for} \quad u := \begin{pmatrix} e \\ w \end{pmatrix} \in L^2(\Omega) \times H^1(\Omega), \quad f(t) := \begin{pmatrix} q^*(t) \\ 0 \end{pmatrix}.
\]

(4.5)

It is not difficult to check with the help of (4.4) that the operator \(g(t, \cdot)\) defined by (4.5) satisfies all the conditions (g1)-(g6) in Definition 2.2.

As direct consequences of Theorems 2.3 and 3.1, we see that the periodic system (4.1)-(4.4) has at least one \(T_0\)-periodic solution and the global attractor \(\mathcal{A}_\tau\) for each \(\tau \geq 0\). Namely, for any bounded subset \(B \subseteq X\) any solution \([\theta(nT_0 + \tau) + \lambda(nT_0 + \tau, \cdot, w(nT_0 + \tau)), w(nT_0 + \tau)]\) of \((\text{PFC})_s\) starting from \(B\) converges uniformly in \(\tau\) to the global attractor \(\mathcal{A}_\tau\) of the periodic system \((\text{PFC})_s\).

References


4. P. Colli, Ph. Laurençot and J. Sprekels, Global solution to the Penrose-Fife phase field model with special heat flux laws, preprint.


