数理解析研究所講究録
1061 巻 1998 年 13-26

On Conditions of Global Existence for Systems of Quasilinear Wave Equations in Three Space Dimensions

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1 Introduction.

We consider the Cauchy problem for a system of quasilinear wave equations

$$\square_i u^i = F^i(\partial u, \partial^2 u) \quad \text{in } [0,\infty)\times \mathbf{R}^n, \tag{1.1}$$

$$u^i(0,\cdot) = \varepsilon f^i,\ \partial_t u^i(0,\cdot) = \varepsilon g^i \quad \text{in } \mathbf{R}^n, i = 1,\dots,m, \tag{1.2}$$

where $\square_i = \partial_0^2 - c_i^2 \sum_{j=1}^n \partial_j^2, c_i > 0$, $\partial_\alpha = \partial/\partial x^\alpha, t = x^0$, $x = (x^1,\dots,x^n)$. We denote by ∂ the space-time derivatives, i.e.

$$\partial u = (\partial_\alpha u^i)_{\alpha,i}, \partial^2 u = (\partial_\alpha \partial_\beta u^i)_{\alpha,\beta,i},$$

where α,β range over $0,1,2,3$ and i over $1,2,3$. F^i are C^∞ functions near $(\partial u,\ \partial^2 u) = 0$ which are linear with respect to $\partial^2 u$ and satisfy $|F^i(\partial u, \partial^2 u)| \le C(|\partial u|^2 + |\partial^2 u|^2)$. f^i and g^i belong to $C_0^\infty(\mathbf{R}^n)$ and ε is a positive small parameter.

We give a condition of global existence for the initial value problem (1.1)-(1.2) in three space dimensions. Small solutions exist globally when $F^i(\partial u, \partial^2 u)$ do not have quadratic parts, but in general we cannot expect global solutions when they have quadratic parts even if ε is small. However, Klainerman [5] introduced the null condition for the quadratic parts of single wave equations (or systems of wave equations with the same propagation speeds) and proved a global existence theorem on that condition. We extend the Klainerman's null condition to the case where the propagation speeds are different. And we give a global existence theorem in this situation.

Let $n = 3$. We assume that each $F^i(\partial u, \partial^2 u)$ takes the form

$$F^i(\partial u, \partial^2 u) = \sum_{j=1}^{m} \sum_{\alpha,\beta=0}^{3} C^{ij}_{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u^j + D^i(\partial u), \tag{1.3}$$

$$C^{ij}_{\alpha\beta}(\partial u) = C^{ij}_{\beta\alpha}(\partial u), \tag{1.4}$$

$$C^{ij}_{\alpha\beta}(0) = 0, D^i(0) = \partial D^i/\partial(\partial_\alpha u^j) = 0. \tag{1.5}$$

We also assume that the propagation speeds c_i are different from each other. Then we call the following condition the null condition:

$$\begin{aligned} &\sum_{\alpha,\beta,\gamma=0}^{3} C^{iii}_{\alpha\beta\gamma} X^i_\alpha X^i_\beta X^i_\gamma = 0, \sum_{\alpha,\beta=0}^{3} D^{iii}_{\alpha\beta} X^i_\alpha X^i_\beta = 0 \quad (i = 1, \dots, m) \\ &\text{for all real vector } X^i = (X^i_0, X^i_1, X^i_2, X^i_3) \text{ satisfying} \\ &(X^i_0)^2 - c_i^2 \sum_{j=1}^{3} (X^i_j)^2 = 0. \end{aligned} \tag{1.6}$$

Here we have set

$$C^{ijk}_{\alpha\beta\gamma} = \frac{\partial C^{ij}_{\alpha\beta}}{\partial(\partial_\gamma u^k)}(0), D^{ijk}_{\alpha\beta} = \frac{\partial^2 D^i}{\partial(\partial_\alpha u^j)\partial(\partial_\beta u^k)}(0).$$

When $n = 2$, it was shown in [1] that the null condition for the systems (1.1) with different propagation speeds could be derived by applying John-Shatah observation provided $D^i(\partial u) = 0$. Similar argument is applicable to the case $n = 3$.

Theorem 1 *Let n=3 and c_i to be different from each other. Assume that the nonlinear terms $F^i(\partial u, \partial^2 u)$ given by (1.3)-(1.5) satisfy the null condition (1.6) and*

$$C^{ij}_{\alpha\beta}(\partial u) = C^{ji}_{\alpha\beta}(\partial u). \tag{1.7}$$

Then there exits a positive constant ε_0 such that the initial value problem (1.1)-(1.2) has a unique C^∞-solution in $[0, \infty) \times \mathbf{R}^3$ for ε with $0 \le \varepsilon < \varepsilon_0$.

When $n = 2$, Hoshiga and Kubo have proved in [2] a corresponding global existence theorem. They obtained necessary estimates with the use of only the angular derivatives and the scaling operator to avoid the difficulty coming from the difference of speeds. We improve their method to prove our theorem.

2 Notation.

Set

$$\begin{aligned} &\partial_\alpha = \partial/\partial x^\alpha,\ x^0 = t, \\ &\partial = (\partial_0, \partial_1, \partial_2, \partial_3), \\ &\nabla = (\partial_1, \partial_2, \partial_3), \\ &r = |x| \quad \text{for } x \in \mathbf{R}^3. \end{aligned}$$

We denote by $\Gamma = (\Gamma_0, \ldots, \Gamma_7)$ the collection of differential operators ∂, Ω, S where

$$\Omega = x \wedge \nabla, \tag{2.1}$$

$$\Gamma_7 = \quad S = t\partial_t + r\partial_r, \tag{2.2}$$

$$\partial_r = \frac{x}{r} \cdot \nabla. \tag{2.3}$$

Then we find that the bracket $[\Gamma_\alpha, \Gamma_\beta]$ of any Γ_α and Γ_β is written by another Γ_γ. Moreover, we have

$$[\Gamma_\alpha, \Box_i] = 0 \text{ for } 0 \leq \alpha \leq 6 \text{ and } [\Gamma_7, \Box_i] = -2\Box_i. \tag{2.4}$$

We also note that

$$\nabla = \frac{x}{r}\partial_r - \frac{x}{r^2} \wedge \Omega. \tag{2.5}$$

For $a = (a_1, \ldots, a_k)$ $(a_i \in \{0, \ldots, 7\}, 1 \leq i \leq k)$ we define

$$\Gamma^a = \Gamma_{a_1} \cdots \Gamma_{a_k} \text{ and } |a| = k. \tag{2.6}$$

Let $u = {}^t(u^1, \ldots, u^m)$ be a vector and set

$$w_i(t, r) = (1 + r)(1 + |c_i t - r|) \quad (i = 1, \ldots, m). \tag{2.7}$$

Then we define

$$[\partial u]_{k,t} = \sum_{|a|\le k}\sum_{i=1}^{m}\sum_{\alpha=0}^{3} \sup_{0\le s\le t}\sup_{x\in\mathbf{R}^3} \left| w_i(s,|x|)\Gamma^a\partial_\alpha u^i(s,x)\right|, \tag{2.8}$$

$$\|\partial u(t)\|_k = \sum_{|a|\le k}\sum_{i=1}^{m}\sum_{\alpha=0}^{3} \|\Gamma^a\partial_\alpha u^i(s,\cdot)\|_{L^2(\mathbf{R}^3)}\,. \tag{2.9}$$

$$\|\partial u\|_{k,t} = \sum_{|a|\le k}\sum_{i=1}^{m}\sum_{\alpha=0}^{3} \sup_{0\le s\le t} \|\Gamma^a\partial_\alpha u^i(s,\cdot)\|_{L^2(\mathbf{R}^3)}\,. \tag{2.10}$$

3 Weighted L^∞-estimates.

Let $v = v(t,x)$ be the smooth solution of the Cauchy problem

$$\partial_t^2 v - c_0^2\Delta v = F \quad \text{in} \quad [0,T)\times\mathbf{R}^3, \tag{3.1}$$

$$v(0,\cdot) = \partial_t v(0,\cdot) = 0 \quad \text{in} \quad \mathbf{R}^3, \tag{3.2}$$

where $F \in C^\infty([0,T)\times\mathbf{R}^3)$ and $F(t,\cdot)\in C_0^\infty(\mathbf{R}^3)$ for each t. We first present the decay estimates for v that we will use later on.

Proposition 3.1 *Let v be the solution of (3.1)-(3.2). For $1\le\mu$, $0<\nu$ and $0\le c$, we set*

$$z_{\mu,\nu}(s,\lambda) = (1+|cs-\lambda|)^\mu(1+s+\lambda)^\nu,$$

$$\Phi_\theta(t) = \begin{cases} \log(2+t) & (\theta=0) \\ 1 & (\theta>0), \end{cases} \tag{3.3}$$

$$M_{\mu,\nu;k}(F) = \sum_{|a|\le k}\sup_{0\le s\le t}\sup_{y\in\mathbf{R}^3} |y| z_{\mu,\nu}(s,|y|)|\Gamma^a F(s,y)|. \tag{3.4}$$

Then we have

$$|v(t,x)| \le C(1+t+|x|)^{-1}\Phi_{\mu-1}(t)\Phi_{\nu-1}(t)M_{\mu,\nu;0}(F) \tag{3.5}$$

for $1\le\mu, 1\le\nu$,

$$|\partial v(t,x)| \le C(1+|x|)^{-1}(1+|c_0t-|x||)^{-\nu}\Phi_{\mu-1}(t)M_{\mu,\nu;1}(F) \tag{3.6}$$

for $1\le\mu, 0<\nu, c\ne c_0$, and

$$|\partial v(t,x)| \le C(1+|x|)^{-1}\{(1+|c_0t-|x||)^{-\nu}\Phi_{\mu-1}(t) + (1+|c_0t-|x||)^{-\mu}\Phi_{\nu-1}(t)\}M_{\mu,\nu;1}(F) \tag{3.7}$$

for $1 \leq \mu, 1 \leq \nu, c = c_0$.

These estimates are proved by making use of the representation formula of F. John [3]. The proof of the proposition is elementary, but rather troublesome. For the proof, see [8].

Proposition 3.2 *Let* $u = (u^1, \ldots, u^m)$ *be the smooth solution of*

$$\Box_i u^i = F^i(\partial u, \partial^2 u) \quad \textit{in } [0,T) \times \mathbf{R}^3, \tag{3.8}$$

$$u^i(0,\cdot) = \varepsilon f^i,\ \partial_t u^i(0,\cdot) = \varepsilon g^i \quad \textit{in } \mathbf{R}^3, \tag{3.9}$$

$$i = 1, \ldots, m.$$

Assume that F^i *satisfy (1.6) and* c_i *are different from each other.*
If $\varepsilon < 1$ *and* $[\partial u]_{[(N+7)/2],t} < 1$, *then*

$$[\partial u]_{N,t} \leq C_N\{\varepsilon + [\partial u]_{[(N+7)/2]} \ \|\partial u\|^6_{N+10,t}\}. \tag{3.10}$$

Proof. Let u_0^i be the solutions of the homogeneous equations

$$\Box_i u^i = 0 \quad \text{in } [0,\infty) \times \mathbf{R}^3, \tag{3.11}$$

$$u^i(0,\cdot) = \varepsilon f^i,\ \partial_t u^i(0,\cdot) = \varepsilon g^i \quad \text{in } \mathbf{R}^3, \tag{3.12}$$

$$i = 1, \ldots, m.$$

From (2.4), each $\Gamma^a u_0^i$ satisfies (3.11). Hence

$$|\Gamma^a u_0^i(t,x)| \leq C_N \varepsilon (1+t+r)^{-1}, \tag{3.13}$$

$$|\partial \Gamma^a u_0^i(t,x)| \leq C_N \varepsilon (1+r)^{-1}(1+|c_i t - r|)^{-1}, \tag{3.14}$$

for $|a| \leq N$.

Set

$$u_1 = u - u_0. \tag{3.15}$$

Then, each $\Gamma^a u_1^i$ satisfies the equation of the form

$$\Box_i \Gamma^a u_1^i = \sum_{b \leq a} C_{ab} \Gamma^b F^i(\partial u, \partial^2 u) \quad \text{in } [0,T) \times \mathbf{R}^3, \tag{3.16}$$

$$\Gamma^a u_1^i(0,\cdot) = \partial_t \Gamma^a u_1^i(0,\cdot) = 0 \quad \text{in } \mathbf{R}^3. \tag{3.17}$$

We apply Proposition 3.1 to (3.16)-(3.17) by setting the weight $z_{\mu,\nu}(s,\lambda)$ to be

$$z(s,\lambda) = \min_{1\le i\le m+1} z^{(i)}_{1,1}(s,\lambda),$$
$$z^{(i)}_{\mu,\nu}(s,\lambda) = (1+|c_i s-\lambda|)^{\mu}(1+s+\lambda)^{\nu},\ c_{m+1}=0. \tag{3.18}$$

Then it follows from (3.5)-(3.7) that

$$|\Gamma^a u^i_1(t,x)| \le C_N(1+t+r)^{-1}\{\log(2+t)\}^2 M_N(F^i), \tag{3.19}$$
$$|\partial\Gamma^a u^i_1(t,x)| \le C_N(1+r)^{-1}(1+|c_i t-r|)^{-1}\log(2+t)M_{N+1}(F^i), \tag{3.20}$$

for $|a|\le N$, where

$$M_k(F^i) = \sum_{|a|\le k}\sup_{0\le s\le t}\sup_{y\in\mathbf{R}^3}|y|z(s,|y|)|\Gamma^a F^i(\partial u,\partial^2 u)(s,y)|.$$

In order to estimate $M_k(F^i)$, we use the Sobolev inequality (Lemma 4.2 in [6]):

$$|y||f(y)| \le C\{\sum_{|a|\le 2}\|\Omega^a f\|_{L^2(\mathbf{R}^3)} + \sum_{|a|\le 1}\|\partial_r\Omega^a f\|_{L^2(\mathbf{R}^3)}\}. \tag{3.21}$$

If $|b|+|c|\le k$ and $0\le s\le t$, we have

$$|y|z(s,|y|)|\partial\Gamma^b u^j(s,y)||\partial\Gamma^c u^l(s,y)| \le C_k[\partial u]_{[k/2],t}\,\|\partial u\|_{k+2,t},$$

and hence

$$M_k(F^i) \le C_k[\partial u]_{[(k+1)/2],t}\,\|\partial u\|_{k+3,t}\,. \tag{3.22}$$

Therefore, from (3.19), (3.20) and (3.22) we obtain the estimates of $\Gamma^a u^i_1$ and $\partial\Gamma^a u^i_1$. Combining these estimates with (3.13) and (3.14), it follows

$$\begin{aligned}|\Gamma^a u^i(t,x)| &\le C_N(1+t+r)^{-1}\{\log(2+t)\}^2\cdot\\ &\quad\cdot(\varepsilon+[\partial u]_{[(N+1)/2],t}\,\|\partial u\|_{N+4,t}),\end{aligned} \tag{3.23}$$
$$\begin{aligned}|\partial\Gamma^a u^i(t,x)| &\le C_N(1+r)^{-1}(1+|c_i t-r|)^{-1}\log(2+t)\cdot\\ &\quad\cdot(\varepsilon+[\partial u]_{[(N+2)/2],t}\,\|\partial u\|_{N+5,t})\end{aligned} \tag{3.24}$$

for $|a|\le N$.

Next, we estimate the nonlinear terms by making use of (3.23), (3.24). We separate F^i into three parts:

$$F^i(\partial u, \partial^2 u) = N^i(\partial u, \partial^2 u) + R^i(\partial u, \partial^2 u) + G^i(\partial u, \partial^2 u), \tag{3.25}$$

where

$$N^i(\partial u, \partial^2 u) = \sum_{0\le\alpha,\beta,\gamma\le 3} C^{iii}_{\alpha\beta\gamma}\partial_\gamma u^i \partial_\alpha\partial_\beta u^i + \sum_{0\le\alpha,\beta\le 3} D^{iii}_{\alpha\beta}\partial_\alpha u^i \partial_\beta u^i, \tag{3.26}$$

$$R^i(\partial u, \partial^2 u) = \sum_{(j,k)\neq(i,i)} \left(\sum_{0\le\alpha,\beta,\gamma\le 3} C^{ijk}_{\alpha\beta\gamma}\partial_\gamma u^k \partial_\alpha\partial_\beta u^j + \sum_{0\le\alpha,\beta\le 3} D^{ijk}_{\alpha\beta}\partial_\alpha u^j \partial_\beta u^k \right), \tag{3.27}$$

and $G^i(\partial u, \partial^2 u)$ are higher order terms. Moreover, by the null condition (1.6), $N^i(\partial u, \partial^2 u)$ can be expressed in the form

$$\begin{aligned} N^i(\partial u, \partial^2 u) = & \sum_{0\le\alpha\le 3} T^i_\alpha Q^i(u^i, \partial_\alpha u^i) + \sum_{0\le\alpha,\beta,\gamma\le 3} T^i_{\alpha\beta\gamma} Q_{\alpha\beta}(u^i, \partial_\gamma u^i) \\ & + \sum_{0\le\alpha\le 3} \bar{T}^i_\alpha \partial_\alpha u^i \Box_i u^i + T^i Q^i(u^i, u^i), \end{aligned} \tag{3.28}$$

where

$$Q^i(u, v) = \partial_t u \partial_t v - c_i^2 \nabla u \cdot \nabla v, \tag{3.29}$$

$$Q_{\alpha\beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v. \tag{3.30}$$

These forms gain good decay near $c_i t - r = 0$. Indeed, the following estimates hold for $|c_i t - r| < c_i t/2$:

$$\begin{aligned} |Q^i(u,v)| \le\; & C|c_i t - r|(1+t+r)^{-1}|\partial u||\partial v| \\ & + C(1+t+r)^{-1}(|\Gamma u||\partial v| + |\partial u||\Gamma v|), \end{aligned} \tag{3.31}$$

$$|Q_{\alpha\beta}(u,v)| \le C(1+t+r)^{-1}(|\partial u||\Gamma v| + |\Gamma u||\partial v|), \tag{3.32}$$

$$\begin{aligned} |\Box_i u| \le\; & C|c_i t - r|(1+t+r)^{-1}|\partial^2 u| \\ & + C(1+t+r)^{-1}(|\partial u| + |\partial \Gamma u|). \end{aligned} \tag{3.33}$$

Let us give a sketch of (3.31)-(3.33). Following [2], we define

$$S_i^\pm = \partial_t \pm c_i \partial_r. \tag{3.34}$$

Noting

$$S_i^+u = t^{-1}(c_it - r)\partial_r u + t^{-1}Su,$$
$$|r^{-1}\Omega u| \le C|\nabla u|,$$

(3.31) follows from the identity

$$\begin{aligned} Q^i(u,v) &= 2^{-1}(S_i^+uS_i^-v + S_i^-uS_i^+v) \\ &\quad -c_i^2r^{-2}(\hat{x}\wedge\Omega u)\cdot(\hat{x}\wedge\Omega v), \hat{x} = x/r, \end{aligned}$$

which is derived from (2.5). If we rewrite $Q_{\alpha\beta}(u,v)$ by using (2.5) and

$$\partial_t u = -rt^{-1}\partial_r u + t^{-1}Su,$$

we can prove (3.32). Finally, (3.33) is the consequence of

$$\Box_i u = S_i^+S_i^-u - c_i^2(2r^{-1}\partial_r u + r^{-2}\Omega u\cdot\Omega u).$$

Hence, it follows from (3.28), (3.31)-(3.33) that

$$\begin{aligned} |\Gamma^a N^i(\partial u, \partial^2 u)| &\le C|c_it - r|(1+t+r)^{-1}\sum_{|b|+|c|\le|a|+1}|\partial\Gamma^b u^i||\partial\Gamma^c u^i| \\ &\quad +C(1+t+r)^{-1}\sum_{|b|+|c|\le|a|+2}|\Gamma^b u^i||\partial\Gamma^c u^i| \end{aligned} \tag{3.35}$$

for $|c_it - r| < c_it/2$. Therefore, if $|c_it - r| < c_it/2$, the estimates (3.23), (3.24) and (3.35) yield

$$\begin{aligned} |\Gamma^a N^i(\partial u, \partial^2 u)| &\le C_N(1+t+r)^{-3}(1+|c_it-r|)^{-1}\{\log(2+t)\}^3\cdot \\ &\quad \cdot(\varepsilon + [\partial u]_{[(N+4)/2],t}\,\|\partial u\|^2_{N+7,t}) \end{aligned} \tag{3.36}$$

for $|a| \le N$. In case $|c_it - r| \ge c_it/2$, from (3.24) we obtain

$$\begin{aligned} |\Gamma^a N^i(\partial u, \partial^2 u)| &\le C_N\sum_{|b|+|c|\le N+1}|\partial\Gamma^b u^i||\partial\Gamma^c u^i| \\ &\le C_N(1+r)^{-2}(1+t+r)^{-2}\{\log(2+t)\}^2\cdot \\ &\quad \cdot(\varepsilon + [\partial u]_{[(N+4)/2],t}\,\|\partial u\|^2_{N+7,t}). \end{aligned} \tag{3.37}$$

Similarly, by (3.24)

$$\begin{aligned}|\Gamma^a R^i(\partial u,\partial^2 u)| &\le C_N \sum_{(j,k)\neq(i,i)} \sum_{|b|+|c|\le N+1} |\partial\Gamma^b u^j||\partial\Gamma^c u^k| \\ &\le C_N\{(1+r)^{-2}(1+t+r)^{-2} \\ &\quad + \sum_{j\neq i}(1+t+r)^{-2}(1+|c_j t-r|)^{-2}\}\{\log(2+t)\}^2\cdot \\ &\quad \cdot(\varepsilon+[\partial u]_{[(N+4)/2],t}\ \|\partial u\|^2_{N+7,t}), \end{aligned} \tag{3.38}$$

$$\begin{aligned}|\Gamma^a G^i(\partial u,\partial^2 u)| &\le C_N \sum_{1\le j,k,l\le m} \sum_{|b|+|c|+|d|\le N+1} |\partial\Gamma^b u^j||\partial\Gamma^c u^k||\partial\Gamma^d u^l| \\ &\le C_N\{(1+r)^{-3}(1+t+r)^{-3} \\ &\quad + \sum_{1\le j\le m}(1+t+r)^{-3}(1+|c_j t-r|)^{-3}\}\{\log(2+t)\}^3\cdot \\ &\quad \cdot(\varepsilon+[\partial u]_{[(N+4)/2],t}\ \|\partial u\|^3_{N+7,t}), \end{aligned} \tag{3.39}$$

for $|a|\le N$.

Therefore, it follows from (3.36)-(3.39) that

$$\begin{aligned}&|\Gamma^a F^i(\partial u,\partial^2 u)| \\ &\le C_N\{(1+t+r)^{-1} z^{(i)}_{1+\gamma,1+\kappa}(t,r)^{-1} \\ &\quad +(1+t+r)^{-1}\sum_{j\neq i} z^{(j)}_{2,1-\rho}(t,r)^{-1} \\ &\quad +(1+r)^{-1} z^{(m+1)}_{1+\gamma,1+\kappa}(t,r)^{-1}\}(\varepsilon+[\partial u]_{[(N+4)/2],t}\ \|\partial u\|^3_{N+7,t})\end{aligned}$$

for $0<\gamma<1$, $0<\kappa<1-\gamma$ and $0<\rho<1$. Hence, by Proposition 3.1 and (3.14),

$$\begin{aligned}|\partial\Gamma^a u^i(t,x)| &\le C_N(1+r)^{-1}(1+|c_i t-r|)^{-1+\rho}\cdot \\ &\quad \cdot(\varepsilon+[\partial u]_{[(N+5)/2],t}\ \|\partial u\|^3_{N+8,t}). \end{aligned} \tag{3.40}$$

Using (3.40), we estimate $\Gamma^a R^i(\partial u,\partial^2 u)$ again. Then

$$\begin{aligned}|\Gamma^a R^i(\partial u,\partial^2 u)| &\le C_N\{(1+r)^{-2}(1+t+r)^{-2+2\rho} \\ &\quad + \sum_{j\neq i}(1+t+r)^{-2}(1+|c_j t-r|)^{-2+2\rho}\}\cdot \\ &\quad \cdot(\varepsilon+[\partial u]_{[(N+6)/2],t}\ \|\partial u\|^6_{N+9,t}), \end{aligned} \tag{3.41}$$

for $|a| \leq N$. We take $\rho < 1/2$ and replace the estimate (3.38) with (3.41). Then we have

$$
\begin{aligned}
&|\Gamma^a F^i(\partial u, \partial^2 u)| \\
&\leq C_N\{(1+t+r)^{-1} z^{(i)}_{1+\gamma,1+\kappa}(t,r)^{-1} \\
&\quad +(1+t+r)^{-1}\sum_{j\neq i} z^{(j)}_{1+\gamma,1}(t,r)^{-1} \\
&\quad +(1+r)^{-1} z^{(m+1)}_{1+\gamma,1+\kappa}(t,r)^{-1}\}(\varepsilon + [\partial u]_{[(N+6)/2],t}\ \|\partial u\|^6_{N+9,t})
\end{aligned} \tag{3.42}
$$

for some $0 < \gamma,\ \kappa < 1$. Then, applying Proposition 3.1 again, we have

$$
\begin{aligned}
|\partial \Gamma^a u^i(t,x)| \ &\leq\ C_N(1+r)^{-1}(1+|c_i t - r|)^{-1}\cdot \\
&\quad \cdot(\varepsilon + [\partial u]_{[(N+7)/2],t}\ \|\partial u\|^6_{N+10,t}).
\end{aligned} \tag{3.43}
$$

Consequently we have finished the proof.

4 Energy estimates.

Proposition 4.1 *Let u be the solution of (3.8)-(3.9). Assume that F^i satisfy (1.6) and c_i are different from each other. Assume moreover that $C^{ij}_{\alpha\beta}(\partial u)$ satisfy (1.7). Then,*

$$\|\partial u\|_{N,t} \leq C_N \varepsilon, \tag{4.1}$$

provided $[\partial u]_{[(N+11)/2],t}$ is small.

Since we know the local existence of the smooth solution to the Cauchy problem (1.1)-(1.2), we can prove Theorem 1 by Proposition 3.2, Proposition 4.1 and the usual continuation argument.

Proof. If $v = (v^1, \ldots, v^m)$ satisfies

$$\sum_{0\leq\alpha,\beta\leq 3}\ \sum_{1\leq j\leq m} a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta v^j = b^i,\ i = 1,\ldots,m \tag{4.2}$$

with

$$a^{ij}_{\alpha\beta} = a^{ij}_{\beta\alpha} = a^{ji}_{\alpha\beta}, \tag{4.3}$$

then we have the energy identity

$$\sum_{0\le\alpha,\beta\le 3}\sum_{1\le i,j\le m}\{\partial_\alpha(a^{ij}_{\alpha\beta}\partial_0 v^i\partial_\beta v^j)-\partial_\alpha a^{ij}_{\alpha\beta}\partial_0 v^i\partial^j_\beta-2^{-1}\partial_0(a^{ij}_{\alpha\beta}\partial^i_\alpha\partial^j_\beta)+2^{-1}\partial_0 a^{ij}_{\alpha\beta}\partial_\alpha v^i\partial_\beta v^j\}$$
$$=\sum_{1\le i\le m} b^i\partial_0 v^i, \tag{4.4}$$

by multiplying both side of (4.2) by $\partial_0 v^i$. Integrating (4.4) on $[0,t]\times\mathbf{R}^3$, we obtain

$$2^{-1}\int_{\mathbf{R}^3}\langle\partial v(t),\partial v(t)\rangle dx-2^{-1}\int_{\mathbf{R}^3}\langle\partial v(0),\partial v(0)\rangle dx$$
$$=\int_0^t ds\int_{\mathbf{R}^3}\{\sum_{0\le\alpha,\beta\le 3}\sum_{1\le i,j\le m}(\partial_\alpha a^{ij}_{\alpha\beta}\partial_0 v^i\partial_\beta v^j-2^{-1}\partial_0 a^{ij}_{\alpha\beta}\partial_\alpha v^i\partial_\beta v^j)+\sum_{1\le i\le m} b^i\partial_0 v^i\}dx, \tag{4.5}$$

where

$$\langle\partial v,\partial w\rangle=\sum_{1\le i,j\le m}(a^{ij}_{00}\partial_0 v^i\partial_0 w^j-\sum_{1\le k,l\le 3}a^{ij}_{kl}\partial_k v^i\partial_l w^j). \tag{4.6}$$

We first set

$$a^{ij}_{\alpha\beta} = \eta^i_{\alpha\beta}\delta^{ij}-C^{ij}_{\alpha\beta}(\partial u), \tag{4.7}$$
$$b^i = \sum_{1\le j\le m}\sum_{0\le\alpha,\beta\le 3}\{a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^a u^j-\Gamma^a(a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta u^j)\}$$
$$+\Gamma^a\{D^i(\partial u)+G^i(\partial u,\partial^2 u)\}, \tag{4.8}$$

where

$$(\eta^i_{\alpha\beta})_{0\le\alpha,\beta\le 3}=\begin{pmatrix}1&0&0&0\\0&-c_i^2&0&0\\0&0&-c_i^2&0\\0&0&0&-c_i^2\end{pmatrix}. \tag{4.9}$$

Then each $a^{ij}_{\alpha\beta}$ satisfies (4.3), and $\Gamma^a u$ is a solution of (4.2). Therefore, it follows from (4.5) that

$$2^{-1}\int_{\mathbf{R}^3}\langle\partial\Gamma^a u(t),\partial\Gamma^a u(t)\rangle dx-2^{-1}\int_{\mathbf{R}^3}\langle\partial\Gamma^a u(0),\partial\Gamma^a u(0)\rangle dx$$
$$=\int_0^t ds\int_{\mathbf{R}^3}\{\sum_{0\le\alpha,\beta\le 3}\sum_{1\le i,j\le m}(\partial_\alpha a^{ij}_{\alpha\beta}\partial_0\Gamma^a u^i\partial_\beta\Gamma^a u^j-2^{-1}\partial_0 a^{ij}_{\alpha\beta}\partial_\alpha\Gamma^a u^i\partial_\beta\Gamma^a u^j)$$
$$+\sum_{1\le i\le m} b^i\partial_0\Gamma^a u^i\}dx.$$

Noting

$$\sum_{1\le j\le m}\sum_{0\le\alpha,\beta\le 3}\{a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^a u^j-\Gamma^a(a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta u^j)\}$$
$$=[\Box_i,\Gamma^a]u^i-\sum_{1\le j,k\le m}\sum_{0\le\alpha,\beta,\gamma\le 3}\{C^{ij}_{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^j-\Gamma^a(C^{ij}_{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^j\},$$

we obtain

$$\begin{aligned}
&2^{-1}\int_{\mathbf{R}^3}\langle\partial\Gamma^a u(t),\partial\Gamma^a u(t)\rangle dx-2^{-1}\int_{\mathbf{R}^3}\langle\partial\Gamma^a u(0),\partial\Gamma^a u(0)\rangle dx\\
&\le C_N\sum_{\substack{|b|+|c|\le N+1\\ |b|,|c|\ne 0}}\sum_{1\le i,j,k\le m}\int_0^t ds\int_{\mathbf{R}^3}|\Gamma^a\partial u^i||\Gamma^b\partial u^j||\Gamma^c\partial u^k|dx\\
&\le C_N[\partial u]_{[(N+1)/2],t}\int_0^t(1+s)^{-1}\,\|\partial u(s)\|_N^2\,ds
\end{aligned}$$

for $|a|\le N$. Since

$$\sum_{|a|\le N}\int_{\mathbf{R}^3}\langle\partial\Gamma^a u(t),\partial\Gamma^a u(t)\rangle dx\ge C_N\,\|\partial u(t)\|_N^2$$

if $[\partial u]_{0,t}$ is small, it follows that

$$\|\partial u(t)\|_N^2\le C_N\{\varepsilon^2+[\partial u]_{[(N+1)/2],t}\int_0^t(1+s)^{-1}\,\|\partial u(s)\|_N^2\,ds\}.$$

Hence, by Gronwall's lemma,

$$\|\partial u(t)\|_N^2\le C_N\varepsilon^2(1+t)^{C_N[\partial u]_{[(N+1)/2],t}}. \tag{4.10}$$

Next, we set

$$a^{ij}_{\alpha\beta}=\eta^i_{\alpha\beta}\delta^{ij}, \tag{4.11}$$
$$b^i=[\Box_i,\Gamma^a]u^i+\Gamma^a F^i(\partial u,\partial^2 u). \tag{4.12}$$

Then by (4.5), we have

$$\|\partial u(t)\|_N^2\le C_N(\varepsilon^2+\sum_{|a|,|b|\le N}\sum_{1\le i\le m}\int_0^t ds\int_{\mathbf{R}^3}|\Gamma^b F^i(\partial u,\partial^2 u)||\partial_0\Gamma^a u^i|dx). \tag{4.13}$$

Using (3.42) and (3.43), we have

$$|\Gamma^b F^i(\partial u,\partial^2 u)||\partial_0\Gamma^a u^i|$$

$$
\begin{aligned}
&\le C_N\{(1+s+r)^{-1} z^{(i)}_{1+\gamma,1+\kappa}(s,r)^{-1}\\
&\quad +(1+s+r)^{-1}\sum_{j\neq i} z^{(j)}_{1+\gamma,1}(s,r)^{-1}\\
&\quad +(1+r)^{-1} z^{(m+1)}_{1+\gamma,1+\kappa}(s,r)^{-1}\}(1+r)^{-1}(1+|c_i s-r|)^{-1}\cdot\\
&\quad \cdot(\varepsilon^2+[\partial u]_{[(N+7)/2],s}\,\|\partial u\|^{12}_{N+10,s})\\
&\le C_N\{(1+s+r)^{-3-\kappa}(1+|c_i s-r|)^{-2-\gamma}\\
&\quad +(1+s+r)^{-4}\sum_{j\neq i}(1+|c_j s-r|)^{-1-\gamma}\\
&\quad +(1+r)^{-3-\gamma}(1+s+r)^{-2-\kappa}\}(\varepsilon^2+[\partial u]_{[(N+7)/2],s}\,\|\partial u\|^{12}_{N+10,s})\\
&\le C_N(1+s)^{-1-\kappa}\sum_{j=1}^{m+1}(1+|c_j s-r|)^{-1-\gamma}r^{-2}(\varepsilon^2+[\partial u]_{[(N+7)/2],s}\,\|\partial u\|^{12}_{N+10,s}). \quad (4.14)
\end{aligned}
$$

Moreover, by (4.10) and (4.14) it follows that

$$
\begin{aligned}
&|\Gamma^b F^i(\partial u,\partial^2 u)||\partial_0\Gamma^a u^i|\\
&\quad\le C_N(1+s)^{-1-\kappa}\sum_{j=1}^{m+1}(1+|c_j s-r|)^{-1-\gamma}r^{-2}(\varepsilon^2+[\partial u]_{[(N+7)/2],s}\varepsilon^{12}(1+s)^{C_N[\partial u]_{[(N+11)/2],s}})\\
&\quad\le C_N\varepsilon^2(1+[\partial u]_{[(N+7)/2],s})(1+s)^{-1-\kappa+C_N[\partial u]_{[(N+11)/2],s}}\sum_{j=1}^{m+1}(1+|c_j s-r|)^{-1-\gamma}r^{-2}. \quad (4.15)
\end{aligned}
$$

Therefore, if $[\partial u]_{[(N+11)/2],t}$ is sufficiently small, we obtain (4.1) from (4.13) and (4.15).

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