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STARLIKE AND UNIVALANT SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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We consider the differential equation

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0,$$

where $a(z)$ and $b(z)$ are analytic in the unit disc Δ. In this note, we show that the above differential equation has a solution $w(z)$ univalent and starlike in Δ under some conditions. It is related to results of S.S. Miller and M.S. Robertson.

1. Introduction

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be an analytic function defined in the unit disc $\Delta = \{z : |z| < 1\}$. We denote the class of such functions by A. If in addition $f(z)$ is univalent, then we say $f(z) \in S$. Suppose $f'(z) \neq 0$ in Δ, then we define

$$S(f, z) = \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}(z)\right)^2$$

to be the <u>Schwarzian derivative</u> of $f(z)$.

Our starting point is the following result of S.S. Miller.

<u>Theorem A</u> (Miller [4]) Let $p(z)$ be analytic in the unit disc Δ with $|zp(z)| < 1$. Let $v(z)$, $z \in \Delta$, be the unique solution of

(1.1) $$v''(z) + p(z)v(z) = 0$$

with $v(0) = 0$ and $v'(0) = 1$. Then

(1.2) $$\left| \frac{zv'(z)}{v(z)} - 1 \right| < 1,$$

and $v(z)$ is a starlike conformal map of the unit disc.

Theorem A is related to the next results of M.S. Robertson and Z. Nehari.

<u>Theorem B</u> (Robertson [8]) Let $zp(z)$ be analytic in Δ and

$$(1.3) \qquad \mathrm{Re}\{z^2 p(z)\} \leqq \frac{\pi^2}{4}|z|^2 \quad (z \in \Delta)$$

Then the unique solution $w = w(z)$, $w(0)=0$, $w'(0)=1$ of

$$(1.4) \qquad w''(z) + p(z)w(z) = 0$$

is univalent and starlike in Δ. The constant $\pi^2/4$ is best possible one.

<u>Theorem C</u> (Nehari [6]) If $f(z) \in A$ and it satisfies

$$(1.5) \qquad |S(f,z)| \leqq \frac{\pi^2}{2} \quad (z \in \Delta),$$

then $f(z)$ is univalent. The result is sharp.

<u>Remark 1</u> The constant $\pi^2/2$ is best possible as shown by the example $\frac{e^{i\pi z}-1}{i\pi}$. We note that by putting $p(z) = \frac{1}{2}S(f,z)$ in (1.3) of Theorem B, then (1.5) of Theorem C implies (1.3). Therefore, Nehari's theorem has a stronger hypothesis. Thus Robertson proved that the unique solution of the equation (1.4) is starlike whereas Nehari proved the quotient of the linearly independent solution of (1.4) is univalent.

We also have

<u>Theorem D</u> (Gabriel [2]) Suppose $f(z) \in A$ and that

(1.6) $\quad |S(f, z)| \leqq 2C_0 \approx 2.73 \quad (z \in \Delta)$,

where C_0 is the smallest positive root of the equation $2\sqrt{x} - \tan\sqrt{x} = 0$, then $f(z)$ maps Δ onto a starlike domain.

Recall that $f(z) \in S$ is starlike with respect to the origin if and only if $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ for all $z \in \Delta$. We denote the class of starlike functions by S^*.

2. A class of bounded functions

Let B_J denote the class of bounded functions $w(z) = w_1 z + w_2 z^2 + \cdots$ analytic in the unit disc Δ for which $|w(z)| < J$. If $g(z) \in B_J$, then by using the Schwarz lemma we can show that the function $w(z)$ defined by $w(z) = z^{-\frac{1}{2}}\int_0^z g(t)t^{-\frac{1}{2}}dt$ is also in B_J. Writing this result in terms of derivatives we have

(2.1) $\left|\frac{1}{2}w(z)+zw'(z)\right|<J \ (z\in\Delta) \Rightarrow |w(z)|<J \ (z\in\Delta)$.

If we let $h(u,v)=\frac{1}{2}u+v$ we can write (2.1) as

(2.2) $|h(w(z),zw'(z))|<J \Rightarrow |w(z)|<J$.

In this section, we will that (2.2) holds for functions $h(u,v)$ satisfying the following definition.

<u>Definition 1</u> Let H_J be the set of complex functions $h(u,v)$ satisfying :

(i) $h(u,v)$ is continuous in a domain $D\subset\mathbb{C}\times\mathbb{C}$,

(ii) $(0,0)\in D$ and $|h(0,0)|<J$,

(iii) $|h(Je^{i\theta},Ke^{i\theta})|>J$ when $(Je^{i\theta},Ke^{i\theta})\in D$, θ is real and $K\geqq J$.

<u>Example 1</u> It is easy to check that the following function $h(u,v)$ is in H_J :

$h(u,v)=\alpha u+v$ where α is complex with $\operatorname{Re}\alpha\geqq 0$, and $D=\mathbb{C}\times\mathbb{C}$.

<u>Definition 2</u> Let $h\in H_J$ with corresponding

domain D. We denote by $B_J(h)$ those functions $w(z)=w_1z+w_2z^2+\cdots$ which are analytic in Δ satisfying

(i) $(w(z), zw'(z)) \in D$,

(ii) $|h(w(z), zw'(z))| < J \quad (z \in \Delta)$.

The set $B_J(h)$ is not empty since for any $h \in H_J$ it is true that $w(z)=w_1z \in B_J(h)$ for $|w_1|$ sufficiently small depending on h.

We need the following lemma to prove our results.

Lemma (Miller and Mocanu [5]) Let $w(z)=w_1z+w_2z^2+\cdots$ be analytic in Δ with $w(z) \not\equiv 0$. If $z_0=r_0e^{i\theta_0}$, $0<r_0<1$, and $|w(z_0)| = \max_{|z|\leq r_0} |w(z)|$, then

(i) $$\frac{z_0w'(z_0)}{w(z_0)} = m$$

and

(ii) $$\mathrm{Re}\left[\frac{z_0w''(z_0)}{w'(z_0)}\right]+1 \geq m$$

where $m \geq 1$.

Theorem 1 For any $h \in H_J$, $B_J(h) \subset B_J$.

<proof> Let $w(z) \in B_J(h)$. Suppose that $z_0 = r_0 e^{i\varphi_0} \in \Delta$ $(0<r_0<1)$ such that

$$\max_{|z|\le r_0} |w(z)| = |w(z_0)| = J .$$

Then $w(z_0) = Je^{i\theta}$ and since by Lemma

$$\frac{z_0 w'(z_0)}{w(z_0)} = m \geqq 1 ,$$ we have

$$z_0 w'(z_0) = Ke^{i\theta} \quad (K = mJ \geqq J)$$ and thus

$$h(w(z_0), z_0 w'(z_0)) = h(Je^{i\theta}, Ke^{i\theta}) .$$

Since $h \in H_J$ this implies that

$|h(w(z_0), z_0 w'(z_0))| \geqq J$ which contradiction of $w(z) \in B_J(h)$. Hence $|w(z)| < J$ $(z \in \Delta)$, and thus $w(z) \in B_J$. Q.E.D.

Remark 2 In other words, above theorem shows that if $h \in H_J$, with corresponding domain D, and if $w(z) = w_1 z + w_2 z^2 + \cdots$ is analytic in Δ and $(w(z), zw'(z)) \in D$ then

$$|h(w(z), zw'(z))| < J \Rightarrow |w(z)| < J .$$

Furthermore, Theorem 1 can be used to show that certain first order differential equations have bounded solutions. The proof of the following theorem follows immediately from Theorem 1.

Theorem 2 Let $h \in H_J$ and $b(z)$ be a analytic function in Δ with $|b(z)| < J$. If the differential equation

$$h(w(z), zw'(z)) = b(z) \qquad (w(0)=0)$$

has a solution $w(z)$ analytic in Δ, then $|w(z)| < J$.

3. Main result

Our main result is the following theorem.

Theorem 3 Let $a(z)$ and $b(z)$ be analytic in Δ with $\left| z\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}a^2(z)\right) \right| < \frac{1}{2}$ and $|a(z)| < 1$. Let $w(z)$ $(z \in \Delta)$ be the solution of the following second order linear differential equation

$$(3.1) \qquad w''(z) + a(z)w'(z) + b(z)w(z) = 0$$

with $w(0)=0$, $w'(0)=1$. Then $w(z)$ is starike in Δ.

< proof > The transformation

(3.2) $$w(z) = \exp\left(-\frac{1}{2}\int_0^z a(\zeta)d\zeta\right) v(z)$$

to the normal form

(3.3) $$v''(z) + \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}a^2(z)\right) v(z) = 0$$

and $v(0)=0$, $v'(0)=1$. If we put

(3.4) $$u(z) = \frac{zv'(z)}{v(z)} - 1 \quad (z \in \Delta),$$

then $u(z)$ is analytic in Δ, $u(0)=0$ and (3.3) becomes

(3.5) $$u^2(z) + u(z) + zu'(z) = -z^2\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}a^2(z)\right),$$

or equivalently

(3.6) $$h(u(z), zu'(z)) = -z^2\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}a^2(z)\right),$$

where $h(u,v) = u^2 + u + v$. ($h(u,v)$ is obtained by (3.3) and (3.4)) It is easy to check $h(u,v) \in H_{\frac{1}{2}}$. i.e.,

(i) $h(u,v)$ is continuous in $D = \mathbb{C} \times \mathbb{C}$,

(ii) $(0,0) \in D$, $|h(0,0)| = 0 < \frac{1}{2}$

(iii) $\left|h\left(\frac{1}{2}e^{i\theta}, Ke^{i\theta}\right)\right| > \frac{1}{2} \quad \left(K \geq \frac{1}{2}\right)$.

From assumption, we have

$$\left| -z^2\left(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}a^2(z)\right)\right| < \frac{1}{2} \quad (z\in\Delta).$$

By using Theorem 2, we have

$$|u(z)| < \frac{1}{2} \quad (z\in\Delta).$$

Therefore, we obtain

$$\left| \frac{zv'(z)}{v(z)} - 1 \right| < \frac{1}{2} \quad (z\in\Delta).$$

This implies that

$$(3.7) \quad \frac{1}{2} < \mathrm{Re}\left\{\frac{zv'(z)}{v(z)}\right\} < \frac{3}{2} \quad (z\in\Delta).$$

From (3.2), we have

$$(3.8) \quad \exp\left(\frac{1}{2}\int_0^z a(\zeta)d\zeta\right) w(z) = v(z) \quad .$$

Logarithmically differentiating of (3.8) leads to

$$(3.9) \quad \frac{zw'(z)}{w(z)} = \frac{zv'(z)}{v(z)} - \frac{z}{2}a(z) \quad .$$

Combining (3.9) and $|a(z)| < 1$, we obtain

$$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\} \geqq \mathrm{Re}\left\{\frac{zv'(z)}{v(z)}\right\} - \frac{1}{2}|za(z)| > 0 \quad (z\in\Delta),$$

and thus $w(z)$ is starlike in Δ.

Q.E.D.

References

1. P. L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.
2. R. F. Gabriel, The Schwarzian derivative and convex functions, Proc. Amer. Math. Soc. 6 (1955), 58-66.
3. E. Hille, Ordinary Differential Equations in the Complex Plane, Wiley, New York, 1976.
4. S. S. Miller, A class of differential inequalities implying boundedness, Illinois J. Math. 20 (1976), 647-649.
5. S. S. Miller and P. T. Mocanu, Second order differential inequalities in the complex plane, Jour. Math. Anal. & Appl. 65 (1978), 289-305.
6. Z. Nehari, The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc. 55 (1949), 545-551.
7. Ch. Pommerenke, Univalent Functions, Vanderhoeck and Ruprecht, Gottingen, 1975.
8. M. S. Robertson, Schlicht solution of $W'' + pW = 0$, Trans. Amer. Math. Soc. 76 (1954), 254-274.

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