<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>スタリゼーションおよび単極解の2次方程式の応用 (複素関数論に於ける微分方程式)</td>
</tr>
<tr>
<td>作者</td>
<td>Saitoh, Hitoshi</td>
</tr>
<tr>
<td>引用</td>
<td>数理解析研究所講究録 1998, 1062: 108-119</td>
</tr>
<tr>
<td>発行日</td>
<td>1998-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62400">http://hdl.handle.net/2433/62400</a></td>
</tr>
<tr>
<td>テキストバージョン</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>連絡先</td>
<td>Department of Mathematics, Kyoto University</td>
</tr>
</tbody>
</table>
STARLIKE AND UNIVALVANT SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

群馬高専 渋藤 斉 (HITOSHI SAITOH)

We consider the differential equation

$$w''(z) + a(z) w'(z) + b(z) w(z) = 0,$$

where $a(z)$ and $b(z)$ are analytic in the unit disc $\Delta$. In this note, we show that the above differential equation has a solution $w(z)$ univalent and starlike in $\Delta$ under some conditions. It is related to results of S.S. Miller and M.S. Robertson.

1. Introduction

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be an analytic function defined in the unit disc $\Delta = \{z : |z| < 1\}$. We denote the class of such functions by $A$. If in addition $f(z)$ is univalent, then we say $f(z) \in S$. Suppose $f'(z) \neq 0$ in $\Delta$, then we define
\[ S(f, z) = \left( \frac{f''}{f'} \right)'(z) - \frac{1}{2} \left( \frac{f''}{f'}(z) \right)^2 \]

to be the \textit{Schwarzian derivative} of \( f(z) \).

Our starting point is the following result of S.S. Miller.

\textbf{Theorem A (Miller [4])} Let \( p(z) \) be analytic in the unit disc \( \Delta \) with \( |zp(z)| < 1 \). Let \( \nu(z), z \in \Delta \), be the unique solution of

\[ \nu''(z) + p(z) \nu(z) = 0 \]

with \( \nu(0) = 0 \) and \( \nu'(0) = 1 \). Then

\[ | \frac{z
u'(z)}{\nu(z)} - 1 | < 1, \]

and \( \nu(z) \) is a starlike conformal map of the unit disc.

Theorem A is related to the next results of M. S. Robertson and Z. Nehari.

\textbf{Theorem B (Robertson [8])} Let \( \varepsilon p(z) \) be analytic in \( \Delta \) and

\[ \frac{z}{2} \frac{\varepsilon'}{\varepsilon} + \varepsilon' < 1, \]
(1.3) \[ \text{Re}\{z^2 p(z)\} \leq \frac{\pi^2}{4} |z|^2 \quad (z \in \Delta) \]

Then the unique solution \( W = W(z), \ W(0) = 0, \ W'(0) = 1 \) of

(1.4) \[ W''(z) + p(z) W(z) = 0 \]

is univalent and starlike in \( \Delta \). The constant \( \frac{\pi^2}{4} \) is best possible one.

**Theorem C (Nehari [6])** If \( f(z) \in A \) and it satisfies

(1.5) \[ |S(f, z)| \leq \frac{\pi^2}{2} \quad (z \in \Delta), \]

then \( f(z) \) is univalent. The result is sharp.

**Remark 1** The constant \( \frac{\pi^2}{2} \) is best possible as shown by the example \( \frac{e^{ix} z^2 - 1}{i \pi} \). We note that by putting \( p(z) = \frac{1}{2} S(f, z) \) in (1.3) of Theorem B, then (1.5) of Theorem C implies (1.3). Therefore, Nehari's theorem has a stronger hypothesis. Thus Robertson proved that the unique solution of the equation (1.4) is starlike whereas Nehari proved the quotient of the linearly independent solution of (1.4) is univalent.
We also have

**Theorem D** (Gabriel1[2]) Suppose \( f(z) \in A \) and that

\[
(1.6) \quad |S(f, z)| \leq 2c_0 \approx 2.73 \quad (z \in \Delta),
\]

where \( c_0 \) is the smallest positive root of the equation

\[
2\sqrt{x} - \tan \sqrt{x} = 0,
\]

then \( f(z) \) maps \( \Delta \) onto a starlike domain.

Recall that \( f(z) \in S \) is starlike with respect to the origin if and only if \( \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \) for all \( z \in \Delta \). We denote the class of starlike functions by \( S^* \).

2. A class of bounded functions

Let \( B_J \) denote the class of bounded functions \( w(z) = w_1 z + w_2 z^2 + \cdots \) analytic in the unit disc \( \Delta \) for which \( |w(z)| < J \). If \( g(z) \in B_J \), then by using the Schwarz lemma we can show that the function \( w(z) \) defined by

\[
w(z) = z^{-\frac{1}{2}} \int_0^z g(t)t^{-\frac{1}{2}} dt
\]
is also in \( B_J \). Writing this
result in terms of derivatives we have

\( 2.1 \) \[ \left| \frac{1}{2} w(z) + z w'(z) \right| < J \quad (z \in \Delta) \implies |w(z)| < J (z \in \Delta). \]

If we let \( h(u,v) = \frac{1}{2} u + v \) we can write (2.1) as

\( 2.2 \) \[ |h(w(z), zw'(z))| < J \implies |w(z)| < J. \]

In this section, we will that (2.2) holds for functions \( h(u,v) \) satisfying the following definition.

**Definition 1** Let \( H_J \) be the set of complex functions \( h(u,v) \) satisfying:

(i) \( h(u,v) \) is continuous in a domain \( D \subset \mathbb{C} \times \mathbb{C} \),

(ii) \( (0,0) \in D \) and \( |h(0,0)| < J \),

(iii) \( |h(Je^{i\theta}, Ke^{i\theta})| > J \) when \( (Je^{i\theta}, Ke^{i\theta}) \in D \), \( \theta \) is real and \( K \geq J \).

**Example 1** It is easy to check that the following function \( h(u,v) \) is in \( H_J \):

\[ h(u,v) = \alpha u + v \quad \text{where} \quad \alpha \text{ is complex with} \]

\( \text{Re} \alpha \geq 0 \), and \( D = \mathbb{C} \times \mathbb{C} \).

**Definition 2** Let \( h \in H_J \) with corresponding
domain $D$. We denote by $B_J(h)$ those functions $w(z) = w_iz + w_iz^2 + \cdots$ which are analytic in $\Delta$ satisfying

(i) $(w(z), z w'(z)) \in D$,

(ii) $|h(w(z), z w'(z))| < 1$ ($z \in \Delta$).

The set $B_J(h)$ is not empty since for any $h \in H_J$ it is true that $w(z) = w_i z \in B_J(h)$ for $|w_i| \leq 1$ sufficiently small depending on $h$.

We need the following lemma to prove our results.

**Lemma** (Miller and Mocanu [5]). Let $w(z) = w_iz + w_iz^2 + \cdots$ be analytic in $\Delta$ with $w(z) \not= 0$. If $z_0 = r_0 e^{i\theta_0}$, $0 < r_0 < 1$, and $|w(z_0)| = \max_{1z \leq r_0} |w(z)|$, then

(i) $\frac{z_0 w'(z_0)}{w(z_0)} = m$

and

(ii) $\Re\left[\frac{z_0 w''(z_0)}{w'(z_0)}\right] + 1 \geq m$

where $m \geq 1$. 
Theorem 1  For any $h \in H_J$, $B_J(h) \subset B_J$.

Proof > Let $w(z) \in B_J(h)$. Suppose that $z_0 = r_0 e^{i\varphi_0} \in \Delta \ (0 < r_0 < 1)$ such that

$$\max_{|z| \leq r_0} |w(z)| = |w(z_0)| = J.$$

Then $w(z_0) = Je^{i\theta}$ and since by Lemma

$$\frac{z_0w'(z_0)}{w(z_0)} = m \geq 1,$$

we have

$$z_0w'(z_0) = Ke^{i\theta} \ (K = mJ \geq J) \quad \text{and thus}$$

$$h(w(z_0), z_0w'(z_0)) = h(Je^{i\theta}, Ke^{i\theta}).$$

Since $h \in H_J$ this implies that

$$|h(w(z_0), z_0w'(z_0))| \geq J$$

which contradiction of $w(z) \in B_J(h)$. Hence $|w(z)| < J \ (z \in \Delta)$, and thus $w(z) \in B_J$. Q.E.D.

Remark 2  In other words, above theorem shows that if $h \in H_J$, with corresponding domain $D$, and if $w(z) = w_1z + w_2z^2 + \cdots$ is analytic in $\Delta$ and $(w(z), zw'(z)) \in D$ then

$$\left| h(w(z), zw'(z)) \right| < J \Rightarrow |w(z)| < J.$$

Furthermore, Theorem 1 can be used to show that
certain first order differential equations have bounded solutions. The proof of the following theorem follows immediately from Theorem 1.

**Theorem 2.** Let $h \in H_{\overline{J}}$ and $b(z)$ be a analytic function in $\Delta$ with $|b(z)|<J$. If the differential equation

$$h(w(z), zw'(z)) = b(z) \quad (w(0)=0)$$

has a solution $w(z)$ analytic in $\Delta$, then $|w(z)|<J$.

3. Main result

Our main result is the following theorem.

**Theorem 3.** Let $a(z)$ and $b(z)$ be analytic in $\Delta$ with $|z(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}a^2(z))|<\frac{1}{2}$ and $|a(z)|<1$.

Let $w(z)$ ($z \in \Delta$) be the solution of the following second order linear differential equation

$$(3.1) \quad w''(z) + a(z)w'(z) + b(z)w(z) = 0$$

with $w(0)=0, w'(0)=1$. Then $w(z)$ is starlike in $\Delta$. 
<proof> The transformation

\[(3.2)\quad \omega (z) = \exp \left( -\frac{1}{2} \int_0^z a(\xi) d\xi \right) \nu(z)\]

to the normal form

\[(3.3)\quad \nu''(z) + \left(b(z) - \frac{1}{2} a'(z) - \frac{1}{4} a^2(z)\right) \nu(z) = 0\]

and \(\nu(0) = 0, \nu'(0) = 1\). If we put

\[(3.4)\quad U(z) = \frac{z \nu'(z)}{\nu(z)} - 1\quad (z \in \Delta),\]

then \(U(z)\) is analytic in \(\Delta\), \(U(0) = 0\) and (3.3) becomes

\[(3.5)\quad U^2(z) + U(z) + z U'(z) = -z^2 \left(b(z) - \frac{1}{2} a'(z) - \frac{1}{4} a^2(z)\right),\]

or equivalently

\[(3.6)\quad h(U(z), z U'(z)) = -z^2 \left(b(z) - \frac{1}{2} a'(z) - \frac{1}{4} a^2(z)\right),\]

where \(h(u, v) = u^2 + u + v\). \(h(u, v)\) is obtained by (3.3) and (3.4). It is easy to check \(h(u, v)\in H_{\frac{1}{2}}\). i.e.,

(i) \(h(u, v)\) is continuous in \(D = \mathbb{C} \times \mathbb{C}\).

(ii) \((0, 0) \in D\), \(|h(0, 0)| = 0 < \frac{1}{2}\).

(iii) \(|h(\frac{1}{2} e^{i\theta}, ke^{i\theta})| > \frac{1}{2}\) \((k \geq \frac{1}{2})\).

From assumption, we have
$$|-z^2 \left( b(z) - \frac{1}{2} a'(z) - \frac{1}{4} a^2(z) \right) | < \frac{1}{2} \quad (z \in \Delta).$$

By using Theorem 2, we have

$$|u(z)| < \frac{1}{2} \quad (z \in \Delta).$$

Therefore, we obtain

$$\left| \frac{z v'(z)}{v(z)} - 1 \right| < \frac{1}{2} \quad (z \in \Delta).$$

This implies that

$$(3.7) \quad \frac{1}{2} < \text{Re} \left\{ \frac{z v'(z)}{v(z)} \right\} < \frac{3}{2} \quad (z \in \Delta).$$

From (3.2), we have

$$(3.8) \quad \exp \left( \frac{1}{2} \int_0^z a(\xi) d\xi \right) w(z) = v(z).$$

Logarithmically differentiating of (3.8) leads to

$$(3.9) \quad \frac{z w'(z)}{w(z)} = \frac{z v'(z)}{v(z)} - \frac{z}{2} a(z).$$

Combining (3.9) and $|a(z)| < 1$, we obtain

$$\text{Re} \left\{ \frac{z w'(z)}{w(z)} \right\} \geq \text{Re} \left\{ \frac{z v'(z)}{v(z)} \right\} - \frac{1}{2} |z a(z)| > 0 \quad (z \in \Delta),$$

and thus $w(z)$ is starlike in $\Delta$.

Q.E.D.
References


DEPARTMENT OF MATHEMATICS, GUNMA NATIONAL COLLEGE OF TECHNOLOGY, 580 TORIBA, MAEBASHI, GUNMA 371-8530, JAPAN

E-mail address: saitoh@nat.gunma-ct.ac.jp