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Generalized Fractional Calculus of the $H$-Function

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Abstract

The paper is devoted to study the generalized fractional calculus of arbitrary complex order for the $H$-function defined by the Mellin-Barnes integral

$$H_{p,q}^{m,n}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds,$$

where the function $\mathcal{H}_{p,q}^{m,n}(s)$ is a certain ratio of products of Gamma functions with the argument $s$ and the contour $\mathcal{L}$ is specially chosen. The considered generalized fractional integration and differentiation operators contain the Gauss hypergeometric function as a kernel and generalize classical fractional integrals and derivatives of Riemann-Liouville, Erdélyi-Kober type, etc. It is proved that the generalized fractional integrals and derivatives of $H$-functions are also $H$-functions but of greater order. In particular, the obtained results define more precisely and generalize known results.

1. Introduction

This paper deals with the $H$-function $H_{p,q}^{m,n}(z)$. For integers $m, n, p, q$ such that $0 \leq m \leq q$, $0 \leq n \leq p$, for $a_i, b_j \in \mathbb{C}$ with $\mathbb{C}$ of the field of complex numbers and for $\alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty)$ ($i = 1, 2, \cdots, p; j = 1, 2, \cdots, q$) the $H$-function $H_{p,q}^{m,n}(z)$ is defined via a Mellin-Barnes type integral in the following way:

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n} \left[ \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] = H_{p,q}^{m,n} \left[ \begin{array}{c} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{array} \right]$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] s^{-s} ds,$$  \hspace{1cm} (1.1)

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where the contour $\mathcal{L}$ is specially chosen and

$$
\mathcal{H}_{p,q}^{m,n}(s) = \mathcal{H}_{p,q}^{m,n} \left[ \frac{(a_i, \alpha_i)_{1,p}}{(b_j, \beta_j)_{1,q}} \right] s = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)}, \quad (1.2)
$$

in which an empty product, if it occurs, is taken to be one. Such a function was introduced by S. Pincherle in 1888 and its theory has been developed by Mellin [10], Dixon and Ferrar [2] (see [3, §1.19] in this connection). An interest to the $H$-function arose again in 1961 when Fox [4] has investigated such a function as a symmetrical Fourier kernel. Therefore this function is sometimes called as Fox's $H$-function. The theory of this function may be found in [1], [9, Chapter 1], [17, Chapter 2] and [11, 8.8.3].

Classical Riemann-Liouville fractional calculus of real order [17, §2.2] (see (2.1) - (2.6) below) was investigated in [12] - [14], [18] and [11]. The right-sided fractional integrals and derivatives of the $H$-function (1.1) were studied in [12] - [14] and the results were presented in [18, §2.7], where the case of left-sided fractional differentiation of the $H$-function was also considered. The left-sided fractional integration of the $H$-function was given in [11, 2.25.2]. Such results for the generalized fractional calculus operators with the Gauss hypergeometric function as a kernel (see (2.7) - (2.10) below), being introduced by the first author [15], were obtained in [16].

However, some of the results obtained in [12] - [14] (cited in [18]) and [16] can be taken to be more precisely. Moreover, these results were given provided that the parameters $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0, \beta_j > 0$ ($i = 1, 2, \cdots, p; j = 1, 2, \cdots, q$) of the $H$-function satisfy certain conditions. These conditions were based on asymptotic behavior of $H_{p,q}^{m,n}(z)$ at zero and infinity. In [5] we extended such the known asymptotic results for the $H$-function to more wide class of parameters.

In [7], [8] we have applied the obtained asymptotic estimates in [5] to find the Riemann-Liouville fractional integrals and derivatives of any complex order of the $H$-function. In particular, we could make more precisely the known results from [12] - [14], [18] and [11].

The present paper is devoted to obtain such type results for the generalized fractional integration and differentiation operators of any complex order with the Gauss hypergeometric function as a kernel. In particular, we give more precisely some of the results from [16] and generalize the results obtained in [7], [8]. The paper is organized as follow. In Section 2 we present classical and generalized fractional calculus operators and some facts from the theory of Gauss hypergeometric function. Sections 3 and 4 contain the result from the theory of the $H$-function. The existence of $H_{p,q}^{m,n}(z)$ and its asymptotic behavior at zero and infinity is considered in Section 3 and certain reduction and differentiation properties in Section 4. Sections 5 and 6 deal with generalized fractional differentiation of the $H$-function (1.1). Sections 7 and 8 are devoted to the generalized fractional differentiation of the $H$-function. Another type of fractional integro-differentiation of the $H$-function is given in Section 9.
2. Classical and Generalized Fractional Calculus Operators

For $\alpha \in \mathbb{C}, \Re(\alpha) > 0$, the Riemann-Liouville left- and right-sided fractional calculus operators are defined as follow [17, §2.3 and §2.4]:

\begin{align*}
(I_{0+}^{\alpha} f)(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t) dt}{(x-t)^{1-\alpha}} \quad (x > 0), \quad (2.1) \\
(I_{-}^\alpha f)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t) dt}{(t-x)^{1-\alpha}} \quad (x > 0), \quad (2.2)
\end{align*}

and

\begin{align*}
(D_{0+}^{\alpha} f)(x) &= \left( \frac{d}{dx} \right)^{[\Re(\alpha)]+1} (I_{0+}^{1-\alpha+\Re(\alpha)} f)(x) \\
&= \left( \frac{d}{dx} \right)^{[\Re(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+|\Re(\alpha)|)} \int_0^x \frac{f(t) dt}{(x-t)^{1-\alpha+|\Re(\alpha)|}} \quad (x > 0), \quad (2.3)
\end{align*}

\begin{align*}
(D_{-}^{\alpha} f)(x) &= \left( -\frac{d}{dx} \right)^{[\Re(\alpha)]+1} (I_{-}^{1-\alpha+\Re(\alpha)} f)(x) \\
&= \left( -\frac{d}{dx} \right)^{[\Re(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+|\Re(\alpha)|)} \int_x^\infty \frac{f(t) dt}{(t-x)^{1-\alpha+|\Re(\alpha)|}} \quad (x > 0), \quad (2.4)
\end{align*}

respectively, where the symbol $[\kappa]$ means the integral part of a real number $\kappa$, i.e. the largest integer not exceeding $\kappa$. In particular, for real $\alpha > 0$, the operators $D_{0+}^{\alpha}$ and $D_{-}^{\alpha}$ take more simple forms

\begin{align*}
(D_{0+}^{\alpha} f)(x) &= \left( \frac{d}{dx} \right)^{[\alpha]+1} (I_{0+}^{1-\alpha+\Re(\alpha)} f)(x) \\
&= \left( \frac{d}{dx} \right)^{[\alpha]+1} \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f(t) dt}{(x-t)^{\alpha}} \quad (x > 0), \quad (2.5)
\end{align*}

and

\begin{align*}
(D_{-}^{\alpha} f)(x) &= \left( -\frac{d}{dx} \right)^{[\alpha]+1} (I_{-}^{1-\alpha} f)(x) \\
&= \left( -\frac{d}{dx} \right)^{[\alpha]+1} \frac{1}{\Gamma(1-\alpha)} \int_x^\infty \frac{f(t) dt}{(t-x)^{\alpha}} \quad (x > 0), \quad (2.6)
\end{align*}

respectively, where $\{\kappa\}$ stands for the fractional part of $\kappa$, i.e. $\{\kappa\} = \kappa - [\kappa]$.

For $\alpha, \beta, \eta \in \mathbb{C}$ and $x > 0$ the generalized fractional calculus operators are defined by [15]

\begin{equation}
(I_{0+}^{\alpha, \beta, \eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \binom{2}{1}(\alpha+\beta, -\eta; \alpha; 1 - \frac{t}{x}) f(t) dt \quad (2.7)
\end{equation}
\[(I_{0+}^{\alpha+\beta,\eta} f)(x) = \left(\frac{d}{dx}\right)^n (I_{0+}^{\alpha+\beta,\eta} f)(x) (\text{Re}(\alpha) \leq 0; n = \lfloor \text{Re}(\alpha) \rfloor + 1); \tag{2.8}\]
\[(I_{\cdot}^{\alpha,\beta,\eta} f)(x) = \left(\frac{d}{dx}\right)^n (I_{\cdot}^{\alpha,\beta,\eta} f)(x) (\text{Re}(\alpha) \leq 0; n = \lfloor \text{Re}(\alpha) \rfloor + 1); \tag{2.10}\]
and
\[(D_{0+}^{\alpha,\beta,\eta} f)(x) \equiv (I_{0+}^{\alpha+\beta,\eta} f)(x) \]
\[= \left(\frac{d}{dx}\right)^n (I_{0+}^{\alpha,\beta,\eta} f)(x) (\text{Re}(\alpha) > 0; n = \lfloor \text{Re}(\alpha) \rfloor + 1); \tag{2.11}\]
\[(D_{\cdot}^{\alpha,\beta,\eta} f)(x) \equiv (I_{\cdot}^{\alpha+\beta,\eta} f)(x) \]
\[= \left(\frac{d}{dx}\right)^n (I_{\cdot}^{\alpha,\beta,\eta} f)(x) (\text{Re}(\alpha) > 0; n = \lfloor \text{Re}(\alpha) \rfloor + 1)). \tag{2.12}\]

Here \(\text{Re}(\alpha) > 0\):

\[(I_{0+}^{\alpha,\beta,\eta} f)(x) = \left(\frac{d}{dx}\right)^n (I_{0+}^{\alpha,\beta,\eta} f)(x) (\text{Re}(\alpha) > 0; n = \lfloor \text{Re}(\alpha) \rfloor + 1); \tag{2.7}\]
\[(I_{\cdot}^{\alpha,\beta,\eta} f)(x) = \left(\frac{d}{dx}\right)^n (I_{\cdot}^{\alpha+\beta,\eta} f)(x) (\text{Re}(\alpha) \leq 0; n = \lfloor \text{Re}(\alpha) \rfloor + 1); \tag{2.9}\]
\[(D_{0+}^{\alpha,\beta,\eta} f)(x) = \left(\frac{d}{dx}\right)^n (D_{0+}^{\alpha,\beta,\eta} f)(x) (\text{Re}(\alpha) > 0; n = \lfloor \text{Re}(\alpha) \rfloor + 1); \tag{2.11}\]
\[(D_{\cdot}^{\alpha,\beta,\eta} f)(x) = \left(\frac{d}{dx}\right)^n (D_{\cdot}^{\alpha+\beta,\eta} f)(x) (\text{Re}(\alpha) \leq 0; n = \lfloor \text{Re}(\alpha) \rfloor + 1)). \tag{2.12}\]

The series in (2.13) is convergent for \(|z| < 1\) and for \(|z| = 1\) with \(\text{Re}(c - a - b) > 0\), and can be analytically continued into \(\{ z \in \mathbb{C} : |\text{arg}(1 - z)| < \pi \} \) (see [3, Chapter II]).

Since

\[2F_1(0, b; c; z) = 1 \quad (\text{Re}(\alpha) > 0); \tag{2.15}\]

for \(\beta = -\alpha\), the generalized fractional calculus operators (2.7), (2.9), (2.11) and (2.12) coincide with the Riemann-Liouville operators (2.1) - (2.4) for \(\text{Re}(\alpha) > 0\):

\[(I_{0+}^{\alpha,\beta,\eta} f)(x) = (I_{0+}^{\alpha} f)(x), \quad (I_{0}^{\alpha,\beta,\eta} f)(x) = (I^\alpha f)(x), \tag{2.16}\]
\[(D_{0+}^{\alpha,\beta,\eta} f)(x) = (D_{0+}^\alpha f)(x), \quad (D_{0}^{\alpha,\beta,\eta} f)(x) = (D^\alpha f)(x). \tag{2.17}\]
According to the relation [3, 2.8(4)]

\[ {}_2F_1(a, b; c; z) = (1 - z)^{-b}, \]  

(2.18)

when \( \beta = 0 \) the operators (2.7) and (2.9) coincide with the Erdélyi-Kober fractional integrals [17, §18.1]:

\[
\left( I_{0+}^{\alpha, \beta, \eta} f \right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x - t)^{-\beta-\eta} t^{-\alpha} f(t) dt \equiv \left( I_{\eta, \alpha}^{+} f \right)(x) \quad (x, \eta \in \mathbb{C}, \Re \alpha > 0),
\]

(2.19)

\[
\left( I_{-}^{-\alpha, \beta, \eta} f \right)(x) = \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty (t - x)^{-\beta-\eta} t^{-\alpha} f(t) dt \equiv \left( K_{\eta, \alpha}^{-} f \right)(x) \quad (\alpha, \eta \in \mathbb{C}, \Re \alpha > 0).
\]

(2.20)

Therefore the operators (2.7), (2.9) and (2.11), (2.12) are called "generalized" fractional integrals and derivatives, respectively. Moreover, the operators (2.11) and (2.12) are inverse to (2.7) and (2.9):

\[
D_{0+}^{\alpha, \beta, \eta} = \left( I_{0+}^{\alpha, \beta, \eta} \right)^{-1}, \quad D_{-}^{-\alpha, \beta, \eta} = \left( I_{-}^{-\alpha, \beta, \eta} \right)^{-1}.
\]

(2.21)

Fractional calculus operators (2.1), (2.3), (2.5), (2.7), (2.8), (2.11) and (2.2), (2.4), (2.6), (2.9), (2.10), (2.12) are called left-sided and right-sided, respectively [17, §2].

We give some other properties of \( {}_2F_1(a, b; c; z) \) [3, 2.8(46), 2.9(2), 2.10(14)] which will be used in the following calculations:

\[
{}_2F_1(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \quad (c \neq 0, -1, -2, \ldots; \Re(c - a - b) > 0);
\]

(2.22)

\[
{}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z);
\]

(2.23)

\[
{}_2F_1(a, b; a + b; z) = \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(k!)^2} \left[ 2\psi(1 + k) - \psi(a + k) + \psi(b + k) - \log(1 - z)(1 - z)^k \right] \quad (|\arg(z)| < \pi; a, b \neq 0, -1, -2, \ldots),
\]

(2.24)

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the Psi function [3, 1.7].

Formulas (2.22) - (2.24) mean the following asymptotic behavior of \( {}_2F_1(a, b; c; z) \) at the point \( z = 1 \):

**Lemma 1.** For \( a, b, c \in \mathbb{C} \) with \( \Re(c) > 0 \) and \( z \in \mathbb{C} \), there hold the following asymptotic relations near \( z = 1 \):

\[
{}_2F_1(a, b; c; z) = O(1) \quad (z \to 1-)
\]

(2.25)

for \( \Re(c - a - b) > 0 \);

\[
{}_2F_1(a, b; c; z) = O((1 - z)^{c-a-b}) \quad (z \to 1-)
\]

(2.26)

for \( \Re(c - a - b) < 0 \); and

\[
{}_2F_1(a, b; c; z) = O(\log(1 - z)) \quad (z \to 1-)
\]

(2.27)
for \(c - a - b = 0\), \(a, b \neq 0, -1, -2, \cdots\) and \(|\arg(z)| < \pi\).

3. Existence and Asymptotic Behavior of the \(H\)-Function

We shall consider the \(H\)-function (1.1) provided that the poles

\[
b_{jl} = -\frac{b_{j} - l}{\beta_{j}} \quad (j = 1, \cdots, m; l \in \mathbb{N}_{0})
\]

(3.1)

of the Gamma functions \(\Gamma(b_{j} + \beta_{j}s)\) and that

\[
a_{ik} = \frac{1 - a_{i} + k}{\alpha_{i}} \quad (i = 1, \cdots, n; k \in \mathbb{N}_{0})
\]

(3.2)

of \(\Gamma(1 - a_{i} - \alpha_{i}s)\) do not coincide:

\[
\alpha_{i}(b_{j} + l) \neq \beta_{j}(a_{i} - k - 1) \quad (i = 1, \cdots, n; j = 1, \cdots, m; k, l \in \mathbb{N}_{0}),
\]

(3.3)

where \(\mathbb{N}_{0} = \mathbb{N} \cup \{0\}\). \(\mathcal{L}\) in (1.1) is the infinite contour splitting poles \(b_{jl}\) in (3.1) to the left and poles \(a_{ik}\) in (3.2) to the right of \(\mathcal{L}\) and has one of the following forms:

(i) \(\mathcal{L} = \mathcal{L}_{-\infty}\) is a left loop situated in a horizontal strip starting at the point \(-\infty + i\varphi_{1}\) and terminating at the point \(-\infty + i\varphi_{2}\) with \(-\infty < \varphi_{1} < \varphi_{2} < +\infty\);

(ii) \(\mathcal{L} = \mathcal{L}_{+\infty}\) is a right loop situated in a horizontal strip starting at the point \(+\infty + i\varphi_{1}\) and terminating at the point \(+\infty + i\varphi_{2}\) with \(-\infty < \varphi_{1} < \varphi_{2} < +\infty\).

(iii) \(\mathcal{L} = \mathcal{L}_{\gamma+i\infty}\) is a contour starting at the point \(\gamma - i\infty\) and terminating at the point \(\gamma + i\infty\) with \(\gamma \in \mathbb{R} = (-\infty, +\infty)\).

The properties of the \(H\)-function \(II_{p,q}^{m,n}(z)\) depend on the numbers \(a^{*}, \Delta, \delta\) and \(\mu\) which are expressed via \(p, q, a_{i}, a_{i} (i = 1, 2, \cdots, p)\) and \(b_{j}, \beta_{j} (j = 1, 2, \cdots, q)\) by the following relations:

\[
a^{*} = \sum_{i=1}^{n} \alpha_{i} - \sum_{i=n+1}^{p} \alpha_{i} + \sum_{j=1}^{m} \beta_{j} - \sum_{j=m+1}^{q} \beta_{j},
\]

(3.4)

\[
\Delta = \sum_{j=1}^{q} \beta_{j} - \sum_{i=1}^{p} \alpha_{i},
\]

(3.5)

\[
\delta = \prod_{i=1}^{p} \alpha_{i}^{-\alpha_{i}} \prod_{j=1}^{q} \beta_{j}^{\beta_{j}},
\]

(3.6)

\[
\mu = \sum_{j=1}^{q} b_{j} - \sum_{i=1}^{p} a_{i} + \frac{p - q}{2}.
\]

(3.7)

Here an empty sum in (3.4), (3.5), (3.7) and an empty product in (3.6), if they occur, are taken to be zero and one, respectively.

The existence of the \(H\)-function is given by the following result [6].
**Theorem A.** Let \( a^* \), \( \Delta \), \( \delta \) and \( \mu \) be given by (3.4) - (3.7). Then the \( II \)-function \( H_{p,q}^{m,n}(z) \) defined by (1.1) and (1.2) makes sense in the following cases:

\[
\begin{align*}
\mathcal{L} = \mathcal{L}_{-\infty}, & \quad \Delta > 0, \quad z \neq 0; \\
\mathcal{L} = \mathcal{L}_{-\infty}, & \quad \Delta = 0, \quad 0 < |z| < \delta; \\
\mathcal{L} = \mathcal{L}_{-\infty}, & \quad \Delta = 0, \quad \text{Re}(\mu) < -1, \quad |z| = \delta; \\
\mathcal{L} = \mathcal{L}_{+\infty}, & \quad \Delta < 0, \quad z \neq 0; \\
\mathcal{L} = \mathcal{L}_{+\infty}, & \quad \Delta = 0, \quad |z| > \delta; \\
\mathcal{L} = \mathcal{L}_{+\infty}, & \quad \Delta = 0, \quad \text{Re}(\mu) < -1, \quad |z| = \delta; \\
\mathcal{L} = \mathcal{L}_{\infty}, & \quad a^* > 0, \quad |\arg z| < \frac{a^* \pi}{2}, \quad z \neq 0; \\
\mathcal{L} = \mathcal{L}_{\infty}, & \quad a^* = 0, \quad \Delta \gamma + \text{Re}(\mu) < -1, \quad \arg z = 0, \quad z \neq 0.
\end{align*}
\]

**Remark 1.** The results of Theorem A in the cases (3.10), (3.13) and (3.15) are more precisely than those in [11, §8.3.1].

The next statement being followed from the results in [5] characterizes the asymptotic behavior of the \( II \)-function at zero and infinity.

**Theorem B.** Let \( a^* \) and \( \Delta \) be given by (3.4) and (3.5) and let conditions in (3.3) be satisfied.

(i) If \( \Delta \geq 0 \) or \( \Delta < 0, a^* > 0 \), then the \( II \)-function has either of the asymptotic estimates at zero

\[
H_{p,q}^{m,n}(z) = O(z^{\varrho^*}) \quad (|z| \to 0)
\]

or

\[
H_{p,q}^{m,n}(z) = O(z^{\varrho^*} |\log(z)|^{N^*}) \quad (|z| \to 0),
\]

with the additional condition \(|\arg(z)| < a^* \pi/2\) when \( \Delta < 0, a^* > 0 \). Here

\[
\varrho^* = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right],
\]

and \( N^* \) is the order of one of the point \( b_j \) in (3.1) to which some other poles of \( \Gamma(b_j + \beta_j s) \) \((j = 1, \ldots, m)\) coincide.

(ii) If \( \Delta \leq 0 \) or \( \Delta > 0, a^* > 0 \), then the \( II \)-function has either of the asymptotic estimates at infinity

\[
H_{p,q}^{m,n}(z) = O(z^{\varrho}) \quad (|z| \to \infty)
\]
or

\[ H_{p,q}^{m,n}(z) = O \left( z^\theta |\log(z)|^N \right) \quad (|z| \to \infty), \]

(3.20)

with the additional condition \(|\arg(z)| < a^* \pi/2\) when \(\Delta > 0, a^* > 0\). Here

\[
\theta = \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right],
\]

(3.21)

and \(N\) is the order of one of the point \(\alpha_{ik}\) in (3.2) in which some other poles of \(\Gamma(1 - \alpha_i - \alpha_{is})\) \((i = 1, \cdots, n)\) coincide.

4. Reduction and Differentiation Properties of the \(H\)-Function

In this and next sections we suppose that the conditions for the existence of the \(H\)-function given in Theorem A are satisfied.

The following two Lemmas which characterize symmetric and reduction properties of the \(H\)-function follow from the definition of the \(H\)-function in (1.1) - (1.2).

**Lemma 2.** The \(H\)-function (1.1) is commutative in the set of pairs \((a_1, \alpha_1), \cdots, (a_n, \alpha_n); \quad (a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p); \quad (b_1, \beta_1), \cdots, (b_m, \beta_m)\) and in \((b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q)\).

**Lemma 3.** If one of \((a_i, \alpha_i)\) \((i = 1, \cdots, n)\) is equal to one of \((b_j, \beta_j)\) \((j = m + 1, \cdots, q)\) (or one of \((a_i, \alpha_i)\) \((i = n + 1, \cdots, p)\) is equal to one of \((b_j, \beta_j)\) \((j = 1, \cdots, m)\)), then the \(H\)-function reduces to the lower order one, that is, \(p, q\) and \(n\) (or \(m\)) decrease by unity. Two such results have the forms

\[
\frac{1}{p,q} \begin{bmatrix}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q-1,(a_1, \alpha_1)}
\end{bmatrix} = \frac{1}{p-1,q-1} \begin{bmatrix}
(a_i, \alpha_i)_{2,p} \\
(b_j, \beta_j)_{1,q-1}
\end{bmatrix}
\]

(4.1)

provided that \(n \geq 1\) and \(q > m\), and

\[
\frac{1}{p,q} \begin{bmatrix}
(a_i, \alpha_i)_{1,p-1,(b_1, \beta_1)} \\
(b_j, \beta_j)_{1,q}
\end{bmatrix} = \frac{1}{p-1,q-1} \begin{bmatrix}
(a_i, \alpha_i)_{1,p-1} \\
(b_j, \beta_j)_{2,q}
\end{bmatrix}
\]

(4.2)

provided that \(m \geq 1\) and \(p > n\).

The next differentiation formulae follow from the definition of the \(H\)-function given in (1.1) - (1.2) and from the functional equation for the Gamma function [3, §1.2(6)]

\[
\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}.
\]

(4.3)
Lemma 4. There hold the following differentiation formulae for $\omega, c \in \mathbb{C}, \sigma > 0$

\[
\left( \frac{d}{dz} \right)^k \left\{ z^\omega \Pi_{p,q}^m \begin{bmatrix} cz^\sigma \\ (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right\} = z^{\omega-k} \Pi_{p+1,q+1}^{m+1} \begin{bmatrix} cz^\sigma \\ (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (k-\omega, \sigma) \end{bmatrix}, \tag{4.4}
\]

\[
\left( \frac{d}{dz} \right)^k \left\{ z^\omega \Pi_{p,q}^m \begin{bmatrix} cz^\sigma \\ (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right\} = (-1)^k z^{\omega-k} \Pi_{p+1,q+1}^{m+1} \begin{bmatrix} cz^\sigma \\ (k-\omega, \sigma), (b_j, \beta_j)_{1,q} \\ (a_i, \alpha_i)_{1,p}, (-\omega, \sigma) \end{bmatrix}. \tag{4.5}
\]

5. Left-Sided Generalized Fractional Integration of the $H$-Function

In the following sections we treat the $H$-function (1.1) - (1.2) with $\mathcal{E} = \mathcal{E}_\infty$ and under the assumptions $a^* > 0$ or $a^* = 0, \Delta \gamma + \Re(\mu) < -1$ for $a^*, \Delta, \mu$ being given by (3.4), (3.5), (3.7).

Here we consider the left-sided generalized fractional integration $I^{\alpha, \beta, \eta}_{0+}$ defined by (2.7).

**Theorem 1.** Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\Re(\alpha) > 0, \Re(\beta) \neq \Re(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

\[
\sigma \min_{1 \leq i \leq m} \left[ \frac{\Re(b_j)}{\beta_j} \right] + \Re(\omega) + \min[0, \Re(\eta-\beta)] + 1 > 0, \tag{5.1}
\]

\[
\sigma \gamma < \Re(\omega) + \min[0, \Re(\eta-\beta)] + 1. \tag{5.2}
\]

Then the generalized fractional integral $I^{\alpha, \beta, \eta}_{0+}$ of the $H$-function (1.1) exists and the following relation holds:

\[
\left( I^{\alpha, \beta, \eta}_{0+} z^\omega \Pi_{p,q}^m \begin{bmatrix} t^{\sigma} \\ (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right) (x) = x^{\omega-\beta} \Pi_{p+2,q+2}^{m+2} \begin{bmatrix} t^{\sigma} \\ (-\omega, \sigma), (-\omega + \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \end{bmatrix}. \tag{5.3}
\]

**Proof.** By (2.7) we have

\[
\left( I^{\alpha, \beta, \eta}_{0+} z^\omega \Pi_{p,q}^m \begin{bmatrix} t^{\sigma} \\ (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right) (x) = x^{-\alpha-\beta} \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\omega \begin{bmatrix} F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x}) \Pi_{p,q}^m \begin{bmatrix} t^{\sigma} \\ (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right] dt. \tag{5.4}
\]
According to (2.25), (2.26), (3.16) and (3.17), the integrand in (5.4) for any \( x > 0 \) has the asymptotic estimate at zero

\[
(x - t)^{\alpha - 1} t^{\omega} \binom{2F_1}{(\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x})} H_{p,q}^{m,n} \left[ t^{\sigma} \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] = O \left( t^{\omega + \sigma^* + \min(0, \Re(\eta - \beta))] \right) \quad (t \to +0)
\]
or

\[
(x - t)^{\alpha - 1} t^{\omega} \binom{2F_1}{(\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x})} H_{p,q}^{m,n} \left[ t^{\sigma} \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] = O \left( t^{\omega + \sigma^* + \min(0, \Re(\eta - \beta))} \log(t)^{N^*} \right) \quad (t \to +0).
\]

Here \( \sigma^* \) is given by (3.18) and \( N^* \) is indicated in Theorem B(i). Therefore the condition (5.1) ensures the existence of the integral (5.4).

Applying (1.2), making the change of variable \( t = x\tau \), changing the order of integration and taking into account the formula [11, §2.2.1.11]

\[
\int_0^x t^{\alpha - 1}(x - t)^{\beta - 1} \binom{2F_1}{(\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x})} dt = \frac{\Gamma(c)\Gamma(\alpha)\Gamma(\alpha + c - a - b)}{\Gamma(\alpha + c - a)\Gamma(\alpha + c - b)} x^{\alpha + \beta - 1} (5.5)
\]

\[
(a, b, c, \alpha \in \mathbb{C}, \Re(\alpha) > 0, \Re(c) > 0, \Re(\alpha + c - a - b) > 0),
\]

we obtain

\[
\left( I_{0+}^{\alpha, \beta, \omega, \eta} t^{\omega} H_{p,q}^{m,n} \left[ t^{\sigma} \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x)
\]

\[
= \frac{x^{-\alpha - \beta}}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} t^{\omega} \binom{2F_1}{(\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x})} H_{p,q}^{m,n} \left[ t^{\sigma} \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] dt
\]

\[
= \frac{x^{-\alpha - \beta}}{2\pi i} \int_\mathcal{L} \Gamma(\alpha) \left( a_i, \alpha_i \right)_{1,p} \left( b_j, \beta_j \right)_{1,q} \left( s \right) \Gamma(1 + \omega - s\sigma)\Gamma(1 + \omega - \beta + \eta - s\sigma) x^{-\sigma s} ds, (5.6)
\]

We note that since \( \mathcal{L} = \mathcal{L}_{\gamma \infty} \), \( \Re(s) = \gamma \) and therefore the condition (5.2) ensures the existence of the Mellin-Barnes integral above. Hence in view of (1.2)

\[
\left( I_{0+}^{\alpha, \beta, \omega, \eta} t^{\omega} H_{p,q}^{m,n} \left[ t^{\sigma} \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) (x)
\]

\[
= x^{\omega - \beta} H_{p+2,q+2}^{m,n+2} \left[ x^{\sigma} \left| \begin{array}{c} (-\omega, \sigma), (-\omega + \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \end{array} \right. \right], (5.7)
\]

and in accordance with (1.1) we obtain (5.3) which completes the proof of Theorem 1.
Corollary 1.1. Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \cdots, p; j = 1, \cdots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy
\[
\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + 1 > 0,
\]
(5.8)
\[
\sigma \gamma < \text{Re}(\omega) + 1.
\]
(5.9)
Then the Riemann-Liouville fractional integral $I^\alpha_{0+}$ of the $H$-function (1.1) exists and the following relation holds:
\[
\left( I^\alpha_{0+} e^\omega H^{m,n}_{p,q} \begin{Bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{Bmatrix} (x) \right) = e^{\omega + \alpha} I^{m,n+1}_{p+1,q+1} \begin{Bmatrix} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \alpha, \sigma) \end{Bmatrix} .
\]
(5.10)

Corollary 1.2. Let $\alpha, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \cdots, p; j = 1, \cdots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy
\[
\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\eta)] + 1 > 0,
\]
(5.11)
\[
\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\eta)] + 1.
\]
(5.12)
Then the Erdélyi-Kober fractional integral $I^\alpha_{\eta+}$ of the $H$-function (1.1) exists and the following relation holds:
\[
\left( I^\alpha_{\eta+} e^\omega H^{m,n}_{p,q} \begin{Bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{Bmatrix} (x) \right) = e^{\omega + \alpha} I^{m,n+1}_{p+1,q+1} \begin{Bmatrix} (-\omega - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \alpha - \eta, \sigma) \end{Bmatrix} .
\]
(5.13)

Remark 2. In the case $\alpha^* > 0, \Delta \geq 0$ the relation (5.3) was indicated in [16, (4.2)], but in the assumptions of the result the condition (5.2) of Theorem 1 should be added.

Remark 3. Corollary 1.1 coincides with Theorem 1 in [7]. For real $\alpha > 0$ and $\alpha^* > 0$ the relation (5.10) was indicated in [11, 2.25.2.2], but the conditions of its validity have to be also corrected according to (5.8) and (5.9).

6. Right-Sided Generalized Fractional Integration of the $H$-Function

In this section we consider the right-sided generalized fractional integration $I^{\alpha, \beta, \eta}$ defined by (2.9).
**Theorem 2.** Let \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \Re(\alpha) > 0, \Re(\beta) \neq \Re(\eta) \). Let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \ldots, p; j = 1, \ldots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\Re(a_i) - 1}{\alpha_i} \right] + \Re(\omega) < \min[\Re(\beta), \Re(\eta)],
\]

\[
\sigma \gamma > \Re(\omega) - \min[\Re(\beta), \Re(\eta)].
\]

Then the generalized fractional integral \( I_{\alpha, \beta, \eta}^{\omega} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( I_{\alpha, \beta, \eta}^{\omega} H_{p,q}^{m,n} \left\{ \gamma \begin{vmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{vmatrix} \right\} \right) (x) = x^{\omega-\beta} I_{\alpha, \beta, \eta}^{m,n+2} \left\{ \gamma \begin{vmatrix} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\ (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{vmatrix} \right\}. \tag{6.3}
\]

**Proof.** By (2.9) we have

\[
\left( I_{\alpha, \beta, \eta}^{\omega} H_{p,q}^{m,n} \left\{ \gamma \begin{vmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{vmatrix} \right\} \right) (x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} t^{\omega-\alpha-\beta} F_1 (\alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t}) H_{p,q}^{m,n} \left\{ \gamma \begin{vmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{vmatrix} \right\} dt. \tag{6.4}
\]

Due to (2.25), (2.26), (3.19) and (3.20), the integrand in (6.4) for any \( x > 0 \) has the asymptotic at infinity

\[
(t-x)^{\alpha-1} t^{\omega-\alpha-\beta} F_1 (\alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t}) H_{p,q}^{m,n} \left\{ \gamma \begin{vmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{vmatrix} \right\}
\]

\[
= O \left( t^{\omega - \min[\Re(\beta), \Re(\eta)] - 1 + \sigma \varrho} \right) \quad (t \to +\infty)
\]

or

\[
= O \left( t^{\omega - \min[\Re(\beta), \Re(\eta)] - 1 + \sigma \varrho \log(t)} \right) \quad (t \to +\infty).
\]

Here \( \varrho \) is given by (3.21) and \( N \) is indicated in Theorem B(ii). Therefore the condition (6.1) ensures the existence of the integral (6.4). Applying (1.2), making the change \( t = 1/\tau \) and using (5.5), we obtain

\[
\left( I_{\alpha, \beta, \eta}^{\omega} H_{p,q}^{m,n} \left\{ \gamma \begin{vmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{vmatrix} \right\} \right) \left( \frac{1}{x} \right) = \frac{1}{\Gamma(\alpha)} \int_{1/x}^{\infty} \left( t - \frac{1}{x} \right)^{\alpha-1} t^{\omega-\alpha-\beta} F_1 (\alpha + \beta, -\eta; \alpha; 1 - \frac{1}{tx}) H_{p,q}^{m,n} \left\{ \gamma \begin{vmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{vmatrix} \right\} dt
\]
\[= \frac{x^{1-\alpha}}{2\pi i \Gamma(\alpha)} \int \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c|c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] x^{\sigma \sigma} ds \]

\[= \frac{x^{-\omega+\beta}}{2\pi i} \int \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c|c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \frac{\Gamma(-\omega+\beta+\sigma s)\Gamma(-\omega+\eta+\sigma s)}{\Gamma(-\omega+\alpha+\beta+\eta+\sigma s)} x^{\sigma \sigma} ds. \quad (6.5)\]

Since \( \mathcal{L} = \mathcal{L}_{\gamma \infty} \), Re(s) = \gamma and therefore the condition (6.2) guarantees the existence of the Mellin-Barnes integral above. Replacing in (6.5) \( x \) by \( 1/x \), we obtain (6.3).

**Corollary 2.1.** Let \( \alpha \in \mathbb{C} \) with Re(\( \alpha \)) > 0, and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) (\( i = 1, \ldots, p; j = 1, \ldots, q \)) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) + \text{Re}(\alpha) < 0, \quad (6.6)\]

\[\sigma \gamma > \text{Re}(\omega) + \text{Re}(\alpha). \quad (6.7)\]

Then the Riemann-Liouville fractional integral \( I^\alpha \) of the II-function (1.1) exists and the following relation holds:

\[\left( I^\alpha L \mathbb{H}^{m,n}_{p,q} \left[ \begin{array}{c|c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] (x) \right) = x^{\omega+\alpha} H^{m+1,n}_{p+1,q+1} \left[ \begin{array}{c|c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma) \\ (-\omega - \alpha, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right]. \quad (6.8)\]

**Corollary 2.2.** Let \( \alpha, \eta \in \mathbb{C} \) with Re(\( \alpha \)) > 0, and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) (\( i = 1, \ldots, p; j = 1, \ldots, q \)) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \text{Re}(\eta), \quad (6.9)\]

\[\sigma \gamma > \text{Re}(\omega) - \text{Re}(\eta). \quad (6.10)\]

Then the Erdélyi-Kober fractional integral \( I_{\eta,\alpha} \) of the H-function (1.1) exists and the following relation holds:

\[\left( I_{\eta,\alpha} E \mathbb{H}^{m,n}_{p,q} \left[ \begin{array}{c|c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] (x) \right) = x^{\omega+\alpha} H^{m+1,n}_{p+1,q+1} \left[ \begin{array}{c|c} (a_i, \alpha_i)_{1,p}, (-\omega + \eta + \alpha, \sigma) \\ (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right]. \quad (6.11)\]

**Remark 4.** In the case \( a^* > 0, \Delta \geq 0 \) the relation of the form (6.3) was indicated in [16, (4.3)]. But it includes a mistake and should be replaced by (6.3) with the conditions (6.1) and (6.2).
Remark 5. Corollary 2.1 coincides with Theorem 2 in [7]. For real \( \alpha > 0 \) and \( \alpha^* > 0 \) the relation (6.8) was indicated in [18, (2.5)], but the conditions of its validity have to be also corrected in accordance with (6.6) and (6.7).

7. Left-Sided Generalized Fractional Differentiation of the \( H \)-Function

Now we treat the left-sided generalized fractional derivative \( D_{0+}^{\alpha, \beta, \eta} \) given by (2.11).

**Theorem 3.** Let \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \text{Re}(\alpha + \beta + \eta) \neq 0 \). Let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \( (i = 1, \ldots, p; j = 1, \ldots, q) \) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \min_{1 \leq i \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\alpha + \beta + \eta)] + 1 > 0, \tag{7.1}
\]

\[
\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\alpha + \beta + \eta)] + 1. \tag{7.2}
\]

Then the generalized fractional derivative \( D_{0+}^{\alpha, \beta, \eta} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( D_{0+}^{\alpha, \beta, \eta} \right)^{\sigma} \left[ \left( a_i, \alpha_i \right)_{1, p} \left( b_j, \beta_j \right)_{1, q} \right] (x) = x^{\omega + \beta} \left[ \left( -\omega, \sigma \right), \left( -\omega - \eta - \alpha - \beta, \sigma \right), \left( a_i, \alpha_i \right)_{1, p} \left( b_j, \beta_j \right)_{1, q} \right] \left( -\omega - \eta, \sigma \right). \tag{7.3}
\]

**Proof.** Let \( n = [\text{Re}(\alpha)] + 1 \). From (2.11) we have

\[
\left( D_{0+}^{\alpha, \beta, \eta} \right)^{\sigma} \left[ \left( a_i, \alpha_i \right)_{1, p} \left( b_j, \beta_j \right)_{1, q} \right] (x) = \left( \frac{d}{dx} \right)^n \left( I_{0+}^{\alpha - n, -\beta - n, \alpha + \eta - n} \left[ \left( a_i, \alpha_i \right)_{1, p} \left( b_j, \beta_j \right)_{1, q} \right] \right) (x), \tag{7.4}
\]

which exists according to Theorem 1 with \( \alpha, \beta \) and \( \eta \) being replaced by \( -\alpha + n, -\beta - n \) and \( \alpha + \eta - n \), respectively. Then we find

\[
\left( D_{0+}^{\alpha, \beta, \eta} \right)^{\sigma} \left[ \left( a_i, \alpha_i \right)_{1, p} \left( b_j, \beta_j \right)_{1, q} \right] (x) = \left( \frac{d}{dx} \right)^n x^{\omega + \beta + n + 1} \left[ \left( -\omega, \sigma \right), \left( -\omega - \alpha - \beta - \eta, \sigma \right), \left( a_i, \alpha_i \right)_{1, p} \left( b_j, \beta_j \right)_{1, q} \right] \left( -\omega - \eta, \sigma \right). \tag{7.5}
\]
Taking into account the differentiation formula (4.1) we have

\[
\left( D_{0+}^{\alpha, \beta, \eta} H_{p,q}^{m,n+1} \left[ \begin{array}{c}
\left( a_i, \alpha_i \right)_{1,p} \\
\left( b_j, \beta_j \right)_{1,q}
\end{array} \right] \right) (x) = x^{\omega+\beta} H_{p,q}^{m,n} + \int \frac{1}{x} \left( \omega + \beta \right) \left( x^{\sigma-1} \right) \left( \frac{\eta}{\beta}, \frac{\alpha}{\sigma} \right) d\tau,
\]

and Lemma 2 and the reduction relation (4.1) imply (7.3), which completes the proof of theorem.

**Corollary 3.1.** Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \ldots, p; j = 1, \ldots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy the conditions in (5.8) and (5.9). Then the Riemann-Liouville fractional derivative \( D_{0+}^{\alpha, \beta, \eta} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( D_{0+}^{\alpha, \beta, \eta} H_{p,q}^{m,n+1} \left[ \begin{array}{c}
\left( a_i, \alpha_i \right)_{1,p} \\
\left( b_j, \beta_j \right)_{1,q}
\end{array} \right] \right) (x) = x^{\omega-\alpha} H_{p,q}^{m,n+2} \left[ \begin{array}{c}
\left( a_i, \alpha_i \right)_{1,p} \\
\left( b_j, \beta_j \right)_{1,q}
\end{array} \right] (x).
\]

**Remark 6.** For real \( \alpha > 0 \) and \( \alpha' > 0 \) the relation (7.3) was given in [18, (2.7.13)], but the conditions of its validity have to be corrected in accordance with (7.1) and (7.2).

**Remark 7.** Corollary 3.1 coincides with Theorem 3 in [7].

**8. Right-Sided Generalized Fractional Differentiation of the \( H \)-Function**

Here we deal with the right-sided generalized fractional derivative \( D_{-}^{\alpha, \beta, \eta} \) given by (2.12).

**Theorem 4.** Let \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \text{Re}(\alpha + \beta + \eta) + [\text{Re}(\alpha)] + 1 \neq 0 \). Let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \ldots, p; j = 1, \ldots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) + \max\{\text{Re}(\beta) + [\text{Re}(\alpha)] + 1, -\text{Re}(\alpha + \eta)\} < 0,
\]

\[
\sigma \gamma > \text{Re}(\omega) + \max\{\text{Re}(\beta) + [\text{Re}(\alpha)] + 1, -\text{Re}(\alpha + \eta)\}.
\]

Then the generalized fractional derivative \( D_{-}^{\alpha, \beta, \eta} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( D_{-}^{\alpha, \beta, \eta} H_{p,q}^{m,n} \left[ \begin{array}{c}
\left( a_i, \alpha_i \right)_{1,p} \\
\left( b_j, \beta_j \right)_{1,q}
\end{array} \right] \right) (x) = (-1)^{\text{Re}(\alpha) + 1} x^{\omega+\beta} H_{p+2,q+2}^{m+2,n} \left[ \begin{array}{c}
\left( a_i, \alpha_i \right)_{1,p} \\
\left( b_j, \beta_j \right)_{1,q}
\end{array} \right] (x).
\]
Proof. Let \( n = [\text{Re}(\alpha)] + 1 \). Owing to (2.12) we have

\[
\left( D_{-}^{-\beta, \eta} \omega \mathcal{H}_{p,q}^{m,n} \left[ t^{\sigma} \begin{pmatrix} (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q} \end{pmatrix} \right] \right) (x)
\]

\[
= \left( -\frac{d}{dx} \right)^{n} \left( I_{-}^{-\alpha+n, -\beta-n, \alpha+\eta} \omega \mathcal{H}_{p,q}^{m,n} \left[ t^{\sigma} \begin{pmatrix} (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q} \end{pmatrix} \right] \right) (x),
\]

which exists according to Theorem 2 with \( \alpha, \beta \) and \( \eta \) being replaced by \(-\alpha + n, -\beta - n \) and \( \alpha + \eta \), respectively. Then applying the differentiation formula (4.5), similarly to (7.5), (7.6), we find in view of the reduction formula (1.2) that

\[
\left( D_{-}^{-\beta, \eta} \omega \mathcal{H}_{p,q}^{m,n} \left[ t^{\sigma} \begin{pmatrix} (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q} \end{pmatrix} \right] \right) (x)
\]

\[
= \left( -\frac{d}{dx} \right)^{n} \omega^{\alpha+n} \mathcal{H}_{p,q}^{m+2,n} \left[ t^{\sigma} \begin{pmatrix} (a_{i}, \alpha_{i})_{1,p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma) \\ (-\omega - \beta - n, \sigma), (-\omega + \alpha + \eta, \sigma), (b_{j}, \beta_{j})_{1,q} \end{pmatrix} \right]
\]

\[
= (-1)^{n} \omega^{\alpha+n} \mathcal{H}_{p,q}^{m+3,n} \left[ t^{\sigma} \begin{pmatrix} (a_{i}, \alpha_{i})_{1,p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma), (-\omega - \beta - n, \sigma) \\ (-\omega - \beta, \sigma), (-\omega - \beta - n, \sigma), (-\omega + \alpha + \eta, \sigma), (b_{j}, \beta_{j})_{1,q} \end{pmatrix} \right],
\]

which implies the formula (8.3).

Corollary 4.1. Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_{i}, b_{j} \in \mathbb{C}, \alpha_{i}, \beta_{j} > 0 \) \((i=1, \cdots, p; j=1, \cdots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left( \frac{\text{Re}(\alpha_{i}) - 1}{\alpha_{i}} \right) + \text{Re}(\omega) - \{ \text{Re}(\alpha) \} + 1 < 0,
\]

\[
\sigma \gamma + \text{Re}(\omega) - \{ \text{Re}(\alpha) \} + 1 > 0.
\]

Then the Riemann-Liouville fractional derivative \( D_{-}^{\alpha} \) of the \( \mathcal{H} \)-function (1.1) exists and there holds the relation:

\[
\left( D_{-}^{\alpha} \omega \mathcal{H}_{p,q}^{m,n} \left[ t^{\sigma} \begin{pmatrix} (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q} \end{pmatrix} \right] \right) (x)
\]

\[
= (-1)^{\text{Re}(\alpha)+1} \omega^{\alpha+n+1} \mathcal{H}_{p,q}^{m+1,n+1} \left[ t^{\sigma} \begin{pmatrix} (a_{i}, \alpha_{i})_{1,p}, (-\omega, \sigma) \\ (-\omega + \alpha, \sigma), (b_{j}, \beta_{j})_{1,q} \end{pmatrix} \right].
\]

Remark 8. The relation of the form (8.7) with real \( \alpha > 0 \) and \( \alpha^{*} > 0 \) was proved in [13, formula (14a)] (see also [12], [14, (2.2)] and [18, (2.7.9)]). But such a formula contains mistakes and should be replaced by (8.7) with the condition (8.5) and (8.6).
 Remark 9. When $\alpha = k \in \mathbb{N}$, the relations (7.7) and (8.7) coincide with (4.4) and (4.5), respectively.

9. Generalized Fractional Integro-Differentiation of the $H$-Function

Here we investigate the generalized fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{-}^{\alpha,\beta,\eta}$ given by (2.8) and (2.10). The following statements are proved similarly to Theorems 3 and 4 by using the relations (2.8) and (2.10), Theorems 1 and 2, and the properties of the $H$-function in Sections 3 and 4.

Theorem 5. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) \leq 0, \text{Re}(\beta) \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha, \beta > 0 (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\begin{align*}
\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1 > 0, \\
\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1.
\end{align*}$$

Then the generalized fractional integro-differentiation $I_{0+}^{\alpha,\beta,\eta}$ of the $H$-function (1.1) exists and there holds the relation

$$\begin{align*}
&\left( I_{0+}^{\alpha,\beta,\eta} \omega I_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right. \right] \right) (x) \\
&= x^{\omega - \beta} I_{p+2,q+2}^{m,n+2} \left[ x^\sigma \left| \begin{array}{c}
(-\omega, \sigma), (-\omega - \eta + \beta, \sigma), (a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma)
\end{array} \right. \right].
\end{align*}$$

Theorem 6. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) \leq 0, \text{Re}(\beta) + [\text{Re}(\alpha)] - 1 \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha, \beta > 0 (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\begin{align*}
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \min[\text{Re}(\beta) - [\text{Re}(\alpha)] - 1, \text{Re}(\eta)], \\
\sigma \gamma > \text{Re}(\omega) - \min[\text{Re}(\beta) - [\text{Re}(\alpha)] - 1, \text{Re}(\eta)].
\end{align*}$$

Then the generalized fractional integro-differentiation $I_{-}^{\alpha,\beta,\eta}$ of the $H$-function (1.1) exists and there holds the relation

$$\begin{align*}
&\left( I_{-}^{\alpha,\beta,\eta} \omega I_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right. \right] \right) (x) \\
&= x^{\omega - \beta} I_{p+2,q+2}^{m,n+2} \left[ x^\sigma \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\
(-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q}
\end{array} \right. \right].
\end{align*}$$
Remark 10. The relation (9.3) with $a^* > 0, \Delta \geq 0$ was indicated in [16, (4.2)], but conditions of its validity have to be corrected in accordance with (9.1) and (9.2).

Remark 11. The relations (9.3) and (9.6) for the fractional integro-differentiation operators $I_0^{\alpha,\beta,\gamma}$ and $I^{\alpha,\beta,\gamma}$, defined in (2.8) and (2.10) for $\alpha \in \mathbb{C}, \Re(\alpha) \leq 0$ coincide with that (5.3) and (6.3) for the fractional integration operators $I_0^{\alpha,\beta,\gamma}$ and $I^{\alpha,\beta,\gamma}$, defined in (2.7) and (2.9) for $\alpha \in \mathbb{C}, \Re(\alpha) > 0$. Though the conditions for validity of (5.3) and (9.3) in Theorems 1 and 5 have the same form, that of (6.3) and (9.6) presented in Theorems 2 and 4 are slightly different.

In conclusion we note that, as it was mentioned in Remarks 2, 4 and 10, the relations (5.3), (6.3) and (9.3) for generalized calculus operator $I_0^{\alpha,\beta,\gamma}$ were already known in the case $a^* > 0, \Delta \geq 0$. Further, Remarks 3, 5, 6 and 8 indicate that the relations (5.10) and (6.8) for the Riemann-Liouville fractional integrals $I_0^\alpha$, $I^\alpha$ and (7.3) and (8.7) for the fractional derivative $D_0^\alpha$, in the case real $\alpha > 0$ and $a^* > 0$ were established. However, the II-function's asymptotic estimates (3.16), (3.17) at zero and (3.19), (3.20) at infinity allow us to prove such results under more general assumptions $a^* > 0$ and $a^* = 0, \Delta > 0, \Re(\mu) < -1$.

References


