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Generalized Fractional Calculus of the $H$-Function

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Abstract

The paper is devoted to study the generalized fractional calculus of arbitrary complex order for the $H$-function defined by the Mellin-Barnes integral

$$H_{p,q}^{m,n}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}^{m,n}_{p,q}(s)z^{-s}ds,$$

where the function $\mathcal{H}^{m,n}_{p,q}(s)$ is a certain ratio of products of Gamma functions with the argument $s$ and the contour $\mathcal{L}$ is specially chosen. The considered generalized fractional integration and differentiation operators contain the Gauss hypergeometric function as a kernel and generalize classical fractional integrals and derivatives of Riemann-Liouville, Erdélyi-Kober type, etc. It is proved that the generalized fractional integrals and derivatives of $H$-functions are also $H$-functions but of greater order. In particular, the obtained results define more precisely and generalize known results.

1. Introduction

This paper deals with the $H$-function $H_{p,q}^{m,n}(z)$. For integers $m, n, p, q$ such that $0 \leq m \leq q$, $0 \leq n \leq p$, for $a_i, b_j \in \mathbb{C}$ with $\mathbb{C}$ the field of complex numbers and for $\alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty)$ ($i = 1, 2, \cdots, p; j = 1, 2, \cdots, q$) the $H$-function $H_{p,q}^{m,n}(z)$ is defined via a Mellin-Barnes type integral in the following way:

$$H_{p,q}^{m,n}(z) \equiv H_{p,q}^{m,n} \left[ \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right] \equiv H_{p,q}^{m,n} \left[ \begin{array}{c} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{array} \right]$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}^{m,n}_{p,q} \left[ \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] s^{-s} ds, \quad \text{(1.1)}$$

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where the contour $\mathcal{L}$ is specially chosen and

$$
\mathcal{H}^{m,n}_{p,q}(s) \equiv \mathcal{H}^{m,n}_{p,q} \left[ \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)},
$$

(1.2)
in which an empty product, if it occurs, is taken to be one. Such a function was introduced by S. Pincherle in 1888 and its theory has been developed by Mellin [10], Dixon and Ferrar [2] (see [3, §1.19] in this connection). An interest to the $H$-function arose again in 1961 when Fox [4] has investigated such a function as a symmetrical Fourier kernel. Therefore this function is sometimes called as Fox's $H$-function. The theory of this function may be found in [1], [9, Chapter 1], [17, Chapter 2] and [11, 8.8.3].

Classical Riemann-Liouville fractional calculus of real order [17, §2.2] (see (2.1) - (2.6) below) was investigated in [12] - [14], [18] and [11]. The right-sided fractional integrals and derivatives of the $H$-function (1.1) were studied in [12] - [14] and the results were presented in [18, §2.7], where the case of left-sided fractional differentiation of the $H$-function was also considered. The left-sided fractional integration of the $H$-function was given in [11, 2.25.2]. Such results for the generalized fractional calculus operators with the Gauss hypergeometric function as a kernel (see (2.7) - (2.10) below), being introduced by the first author [15], were obtained in [16].

However, some of the results obtained in [12] - [14] (cited in [18]) and [16] can be taken to be more precisely. Moreover, these results were given provided that the parameters $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0, \beta_j > 0 \ (i = 1, 2, \ldots; p; j = 1, 2, \ldots, q)$ of the $H$-function satisfy certain conditions. These conditions were based on asymptotic behavior of $\Pi_{p,q}^{m,n}(z)$ at zero and infinity. In [5] we extended such the known asymptotic results for the $H$-function to more wide class of parameters.

In [7], [8] we have applied the obtained asymptotic estimates in [5] to find the Riemann-Liouville fractional integrals and derivatives of any complex order of the $H$-function. In particular, we could make more precisely the known results from [12] - [14], [18] and [11].

The present paper is devoted to obtain such type results for the generalized fractional integration and differentiation operators of any complex order with the Gauss hypergeometric function as a kernel. In particular, we give more precisely some of the results from [16] and generalize the results obtained in [7], [8]. The paper is organized as follow. In Section 2 we present classical and generalized fractional calculus operators and some facts from the theory of Gauss hypergeometric function. Sections 3 and 4 contain the result from the theory of the $H$-function. The existence of $\Pi_{p,q}^{m,n}(z)$ and its asymptotic behavior at zero and infinity is considered in Section 3 and certain reduction and differentiation properties in Section 4. Sections 5 and 6 deal with generalized fractional differentiation of the $H$-function (1.1). Sections 7 and 8 are devoted to the generalized fractional differentiation of the $H$-function. Another type of fractional integro-differentiation of the $H$-function is given in Section 9.
2. Classical and Generalized Fractional Calculus Operators

For \( \alpha \in \mathbb{C}, \Re(\alpha) > 0 \), the Riemann-Liouville left- and right-sided fractional calculus operators are defined as follows [17, §2.3 and §2.4]:

\[
(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (x > 0),
\]

(2.1)

\[
(I_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x}^{\infty} \frac{f(t)dt}{(t-x)^{1-\alpha}} \quad (x > 0),
\]

(2.2)

and

\[
(D_{0+}^{\alpha} f)(x) = \left( \frac{d}{dx} \right)^{[\Re(\alpha)]+1} (I_{0+}^{1-\alpha+[\Re(\alpha)]} f)(x)
\]

\[
= \left( \frac{d}{dx} \right)^{[\Re(\alpha)]+1} \frac{1}{\Gamma(1-\alpha + |\Re(\alpha)|)} \int_{0}^{x} \frac{f(t)}{(x-t)^{\alpha-|\Re(\alpha)|}} dt \quad (x > 0),
\]

(2.3)

\[
(D_{-}^{\alpha} f)(x) = \left( -\frac{d}{dx} \right)^{[\Re(\alpha)]+1} (I_{-}^{1-\alpha+[\Re(\alpha)]} f)(x)
\]

\[
= \left( -\frac{d}{dx} \right)^{[\Re(\alpha)]+1} \frac{1}{\Gamma(1-\alpha + |\Re(\alpha)|)} \int_{x}^{\infty} \frac{f(t)}{(t-x)^{\alpha-|\Re(\alpha)|}} dt \quad (x > 0),
\]

(2.4)

respectively, where the symbol \([\kappa]\) means the integral part of a real number \( \kappa \), i.e. the largest integer not exceeding \( \kappa \). In particular, for real \( \alpha > 0 \), the operators \( D_{0+}^{\alpha} \) and \( D_{-}^{\alpha} \) take more simple forms

\[
(D_{0+}^{\alpha} f)(x) = \left( \frac{d}{dx} \right)^{[\alpha]+1} (I_{0+}^{1-\alpha+[\alpha]} f)(x)
\]

\[
= \left( \frac{d}{dx} \right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_{0}^{x} \frac{f(t)}{(x-t)^{\alpha-\{\alpha\}}} dt \quad (x > 0),
\]

(2.5)

and

\[
(D_{-}^{\alpha} f)(x) = \left( -\frac{d}{dx} \right)^{[\alpha]+1} (I_{-}^{1-\alpha+[\alpha]} f)(x)
\]

\[
= \left( -\frac{d}{dx} \right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_{x}^{\infty} \frac{f(t)}{(t-x)^{\alpha-\{\alpha\}}} dt \quad (x > 0),
\]

(2.6)

respectively, where \( \{\kappa\} \) stands for the fractional part of \( \kappa \), i.e. \( \{\kappa\} = \kappa - [\kappa] \).

For \( \alpha, \beta, \eta \in \mathbb{C} \) and \( x > 0 \) the generalized fractional calculus operators are defined by [15]

\[
(I_{0+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} {}_{2}F_{1} \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) f(t)dt
\]

(2.7)
$(I_{0+}^{\alpha,\beta,\eta}f)(x) = \left(\frac{d}{dx}\right)^n \left(I_{0+}^{\alpha+n,\beta-n,\eta-n} f\right)(x) \quad (\text{Re}(\alpha) > 0); \quad (\text{Re}(\alpha) \leq 0; n = [\text{Re}(\alpha)] + 1); \quad (2.8)$

$(I_{-}^{\alpha,\beta,\eta}f)(x) = (-\frac{d}{dx})^n \left(I_{-}^{\alpha+n,\beta-n,\eta-n} f\right)(x) \quad (\text{Re}(\alpha) \leq 0; n = [\text{Re}(\alpha)] + 1); \quad (2.10)$

and

$(D_{0+}^{\alpha,\beta,\eta}f)(x) \equiv (I_{0+}^{\alpha-n,\beta,\eta}f)(x), \quad (2.11)$

$(D_{-}^{\alpha,\beta,\eta}f)(x) \equiv (I_{-}^{\alpha-n,\beta,\eta}f)(x), \quad (2.12)$

Here $\frac{\Gamma(r)}{\Gamma(0.1)}$ is the Gauss hypergeometric function defined by the series

$$2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!} \quad (2.13)$$

with

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (k \in \mathbb{N}), \quad (2.14)$$

where $\Gamma(z)$ is the Gamma function [3, Chapter I] and $\mathbb{N}$ denotes the set of positive integers.

The series in (2.13) is convergent for $|z| < 1$ and for $|z| = 1$ with $\text{Re}(c - a - b) > 0$, and can be analytically continued into \{ $z \in \mathbb{C} : |\text{arg}(1-z)| < \pi$\} (see [3, Chapter II]).

Since

$$2F_1(0, b; c; z) = 1 \quad (2.15)$$

for $\beta = -\alpha$, the generalized fractional calculus operators (2.7), (2.9), (2.11) and (2.12) coincide with the Riemann-Liouville operators (2.1) - (2.4) for $\text{Re}(\alpha) > 0$:

$$(I_{0+}^{\alpha,\eta}f)(x) = \left(I_{0+}^{\alpha}f\right)(x), \quad (I_{-}^{\alpha,\eta}f)(x) = \left(I_{-}^{\alpha}f\right)(x), \quad (2.16)$$

$$(D_{0+}^{\alpha,\eta}f)(x) = \left(D_{0+}^{\alpha}f\right)(x), \quad (D_{-}^{\alpha,\eta}f)(x) = \left(D_{-}^{\alpha}f\right)(x). \quad (2.17)$$
According to the relation [3, 2.8(4)]
\[
2F_1(a, b; c; z) = (1 - z)^{-b},
\]
when \( \beta = 0 \) the operators (2.7) and (2.9) coincide with the Erdélyi-Kober fractional integrals [17, §18.1]:
\[
\left( I_{0+}^{0, \alpha, \eta} f \right)(x) = \Gamma(\alpha) \int_0^x (x - t)^{\alpha - 1} t^{\eta} f(t) \, dt \equiv \left( I_{\eta, \alpha}^+ f \right)(x) \quad (\alpha, \eta \in \mathbb{C}, \Re(\alpha) > 0),
\]
\[
\left( I_{-}^{0, \alpha, \eta} f \right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) \, dt \equiv \left( K_{\eta, \alpha}^- f \right)(x) \quad (\alpha, \eta \in \mathbb{C}, \Re(\alpha) > 0).
\]
Therefore the operators (2.7), (2.9) and (2.11), (2.12) are called "generalized" fractional integrals and derivatives, respectively. Moreover, the operators (2.11) and (2.12) are inverse to (2.7) and (2.9):
\[
D_{0+}^{\alpha, \beta, \eta} = \left( I_{0+}^{0, \alpha, \eta} \right)^{-1}, \quad D_{-}^{\alpha, \beta, \eta} = \left( I_{-}^{0, \alpha, \eta} \right)^{-1}.
\]
Fractional calculus operators (2.1), (2.3), (2.5), (2.7), (2.8), (2.11) and (2.2), (2.4), (2.6), (2.9), (2.10), (2.12) are called left-sided and right-sided, respectively [17, §2].

We give some other properties of \( 2F_1(a, b; c; z) \) [3, 2.8(46), 2.9(2), 2.10(14)] which will be used in the following calculations:
\[
2F_1(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \quad (c \neq 0, -1, -2, \ldots; \Re(c-a-b) > 0); \quad (2.22)
\]
\[
2F_1(a, b; c; z) = (1 - z)^{c-a-b} 2F_1(c - a, c - b; c; z); \quad (2.23)
\]
\[
2F_1(a, b; a + b; z) = \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(k!)^2} [2 \psi(1 + k) - \psi(a + k) + \psi(b + k) - \log(1 - z)] (1 - z)^k \quad (|\arg(z)| < \pi; a, b \neq 0, -1, -2, \ldots), \quad (2.24)
\]
where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the Psi function [3, 1.7].

Formulas (2.22) - (2.24) mean the following asymptotic behavior of \( 2F_1(a, b; c; z) \) at the point \( z = 1 \).

**Lemma 1.** For \( a, b, c \in \mathbb{C} \) with \( \Re(c) > 0 \) and \( z \in \mathbb{C} \), there hold the following asymptotic relations near \( z = 1 \):
\[
2F_1(a, b; c; z) = O(1) \quad (z \to 1-)
\]
for \( \Re(c-a-b) > 0 \);
\[
2F_1(a, b; c; z) = O \left( (1 - z)^{c-a-b} \right) \quad (z \to 1-)
\]
for \( \Re(c-a-b) < 0 \); and
\[
2F_1(a, b; c; z) = O \left( \log(1 - z) \right) \quad (z \to 1-)
\]
for $c - a - b = 0$, $a, b \neq 0, -1, -2, \cdots$ and $|\arg(z)| < \pi$.

3. Existence and Asymptotic Behavior of the $H$-Function

We shall consider the $H$-function (1.1) provided that the poles

$$b_j = \frac{-b_j - l}{\beta_j} \quad (j = 1, \ldots, m; l \in \mathbb{N}_0)$$

(3.1)

of the Gamma functions $\Gamma(b_j + \beta_j s)$ and that

$$a_{ik} = \frac{1 - a_i + k}{\alpha_i} \quad (i = 1, \ldots, n; k \in \mathbb{N}_0)$$

(3.2)

of $\Gamma(1 - a_i - \alpha_i s)$ do not coincide:

$$\alpha_i(b_j + l) \neq \beta_j(a_i - k - 1) \quad (i = 1, \ldots, n; j = 1, \ldots, m; k, l \in \mathbb{N}_0),$$

(3.3)

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathfrak{L}$ in (1.1) is the infinite contour splitting poles $b_j$ in (3.1) to the left and poles $a_{ik}$ in (3.2) to the right of $\mathfrak{L}$ and has one of the following forms:

- (i) $\mathfrak{L} = \mathfrak{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty + i\varphi_1$ and terminating at the point $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$;
- (ii) $\mathfrak{L} = \mathfrak{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at the point $+\infty + i\varphi_1$ and terminating at the point $+\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$;
- (iii) $\mathfrak{L} = \mathfrak{L}_{y,\infty}$ is a contour starting at the point $\gamma - i\infty$ and terminating at the point $\gamma + i\infty$ with $\gamma \in \mathbb{R} = (-\infty, +\infty)$.

The properties of the $H$-function $\Pi_{p,q}^{m,n}(z)$ depend on the numbers $a^*$, $\Delta$, $\delta$ and $\mu$ which are expressed via $p, q, a_i, \alpha_i$ ($i = 1, 2, \cdots, p$) and $b_j, \beta_j$ ($j = 1, 2, \cdots, q$) by the following relations:

$$a^* = \sum_{i=1}^{p} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j,$$

(3.4)

$$\Delta = \sum_{j=1}^{p} \beta_j - \sum_{i=1}^{n+1} \alpha_i,$$

(3.5)

$$\delta = \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j},$$

(3.6)

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{n} a_i + \frac{p - q}{2}.$$  

(3.7)

Here an empty sum in (3.4), (3.5), (3.7) and an empty product in (3.6), if they occur, are taken to be zero and one, respectively.

The existence of the $H$-function is given by the following result [6].
Theorem A. Let $a^*$, $\Delta$, $\delta$ and $\mu$ be given by (3.4) - (3.7). Then the $H$-function $H_{p,q}^{m,n}(z)$ defined by (1.1) and (1.2) makes sense in the following cases:

\[ \mathcal{L} = \mathcal{L}_{-\infty}, \quad \Delta > 0, \quad z \neq 0; \quad (3.8) \]
\[ \mathcal{L} = \mathcal{L}_{-\infty}, \quad \Delta = 0, \quad 0 < |z| < \delta; \quad (3.9) \]
\[ \mathcal{L} = \mathcal{L}_{-\infty}, \quad \Delta = 0, \quad \text{Re}(\mu) < -1, \quad |z| = \delta; \quad (3.10) \]
\[ \mathcal{L} = \mathcal{L}_{+\infty}, \quad \Delta < 0, \quad z \neq 0; \quad (3.11) \]
\[ \mathcal{L} = \mathcal{L}_{+\infty}, \quad \Delta = 0, \quad |z| > \delta; \quad (3.12) \]
\[ \mathcal{L} = \mathcal{L}_{+\infty}, \quad \Delta = 0, \quad \text{Re}(\mu) < -1, \quad |z| = \delta; \quad (3.13) \]
\[ \mathcal{L} = \mathcal{L}_{+\infty}, \quad a^* > 0, \quad |\arg z| < \frac{a^* \pi}{2}, \quad z \neq 0; \quad (3.14) \]
\[ \mathcal{L} = \mathcal{L}_{+\infty}, \quad a^* = 0, \quad \Delta \gamma + \text{Re}(\mu) < -1, \quad \arg z = 0, \quad z \neq 0. \quad (3.15) \]

Remark 1. The results of Theorem A in the cases (3.10), (3.13) and (3.15) are more precisely than those in [11, §8.3.1].

The next statement being followed from the results in [5] characterizes the asymptotic behavior of the $H$-function at zero and infinity.

Theorem B. Let $a^*$ and $\Delta$ be given by (3.4) and (3.5) and let conditions in (3.3) be satisfied.

(i) If $\Delta \geq 0$ or $\Delta < 0, a^* > 0$, then the $H$-function has either of the asymptotic estimates at zero

\[ H_{p,q}^{m,n}(z) = O \left( z^{\varrho^*} \right) \quad (|z| \to 0) \quad (3.16) \]

or

\[ H_{p,q}^{m,n}(z) = O \left( z^{\varrho^*} |\log(z)|^{N^*} \right) \quad (|z| \to 0), \quad (3.17) \]

with the additional condition $|\arg(z)| < a^* \pi/2$ when $\Delta < 0, a^* > 0$. Here

\[ \varrho^* = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right], \quad (3.18) \]

and $N^*$ is the order of one of the point $b_j$ in (3.1) to which some of other poles of $\Gamma(b_j + \beta_j s)$ ($j = 1, \ldots, m$) coincide.

(ii) If $\Delta \leq 0$ or $\Delta > 0, a^* > 0$, then the $H$-function has either of the asymptotic estimates at infinity

\[ H_{p,q}^{m,n}(z) = O \left( z^{\varrho^*} \right) \quad (|z| \to \infty) \quad (3.19) \]
or
\[ H_{p,q}^{m,n}(z) = O \left( z^{|\log(z)|^N} \right) \quad (|z| \to \infty), \tag{3.20} \]

with the additional condition \( |\arg(z)| < a^* \pi/2 \) when \( \Delta > 0, a^* > 0 \). Here
\[ \varrho = \max_{1 \leq i \leq n} \left[ \frac{\Re(a_i) - 1}{\alpha_i} \right], \tag{3.21} \]

and \( N \) is the order of one of the point \( a_{ik} \) in (3.2) in which some other poles of \( \Gamma(1 - \alpha_i - \alpha_i s) \) \( (i = 1, \cdots, n) \) coincide.

4. Reduction and Differentiation Properties of the \( H \)-Function

In this and next sections we suppose that the conditions for the existence of the \( H \)-function given in Theorem A are satisfied.

The following two Lemmas which characterize symmetric and reduction properties of the \( H \)-function follow from the definition of the \( H \)-function in (1.1) - (1.2).

**Lemma 2.** The \( H \)-function (1.1) is commutative in the set of pairs \( (a_1, \alpha_1), \cdots, (a_n, \alpha_n); \)
\( (a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p); \)
\( (b_1, \beta_1), \cdots, (b_m, \beta_m) \) and in \( (b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q). \)

**Lemma 3.** If one of \( (a_i, \alpha_i) \) \( (i = 1, \cdots, n) \) is equal to one of \( (b_j, \beta_j) \) \( (j = m + 1, \cdots, q) \)
(or one of \( (a_i, \alpha_i) \) \( (i = n + 1, \cdots, p) \) is equal to one of \( (b_j, \beta_j) \) \( (j = 1, \cdots, m) \)), then the \( H \)-function reduces to the lower order one, that is, \( p, q \) and \( n \) (or \( m \)) decrease by unity. Two such results have the forms
\[ H_{p,q}^{m,n} \left[ \begin{array}{l} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q-1} \end{array} \right] = H_{p-1,q-1}^{m,n-1} \left[ \begin{array}{l} (a_i, \alpha_i)_{2,p} \\ (b_j, \beta_j)_{1,q-1} \end{array} \right] \tag{4.1} \]
provided that \( n \geq 1 \) and \( q > m, \) and
\[ H_{p,q}^{m,n} \left[ \begin{array}{l} (a_i, \alpha_i)_{1,p-1} \\ (b_j, \beta_j)_{1,q} \end{array} \right] = H_{p-1,q}^{m-1,n} \left[ \begin{array}{l} (a_i, \alpha_i)_{1,p-1} \\ (b_j, \beta_j)_{2,q} \end{array} \right] \tag{4.2} \]
provided that \( m \geq 1 \) and \( p > n. \)

The next differentiation formulae follow from the definition of the \( H \)-function given in (1.1) - (1.2) and from the functional equation for the Gamma function [3, §1.2(6)]
\[ \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}. \tag{4.3} \]
Lemma 4. There hold the following differentiation formulae for $\omega, c \in \mathbb{C}, \sigma > 0$

\[
\left( \frac{d}{dz} \right)^k \left\{ z^{\omega} \mathcal{H}_{p,q}^m \left[ \begin{array}{c} \alpha_i \\ (b_j, \beta_j) \end{array} \right] \right\} 
= z^{\omega-k} \mathcal{H}_{p+1,q+1}^{m+1} \left[ \begin{array}{c} \alpha_i \\ (b_j, \beta_j) \end{array} \right] 
\]

(4.4)

\[
\left( \frac{d}{dz} \right)^k \left\{ z^{\omega} \mathcal{H}_{p,q}^m \left[ \begin{array}{c} \alpha_i \\ (b_j, \beta_j) \end{array} \right] \right\} 
= (-1)^k z^{\omega-k} \mathcal{H}_{p+1,q+1}^{m+1} \left[ \begin{array}{c} \alpha_i \\ (b_j, \beta_j) \end{array} \right] 
\]

(4.5)

5. Left-Sided Generalized Fractional Integration of the $H$-Function

In the following sections we treat the $H$-function (1.1) - (1.2) with $\mathcal{L} = \mathcal{L}_{\gamma h}$ and under the assumptions $a^* > 0$ or $a^* = 0, \Delta \gamma + \text{Re}(\mu) < -1$ for $a^*, \Delta, \mu$ being given by (3.4), (3.5), (3.7).

Here we consider the left-sided generalized fractional integration $I_{0+}^{a,b,\eta}$ defined by (2.7).

**Theorem 1.** Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0, \text{Re}(\beta) \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \ (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

\[
\sigma \leq \min_{1 \leq i \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min\{0, \text{Re}(\eta - \beta)\} + 1 > 0, 
\]

(5.1)

\[
\sigma \gamma < \text{Re}(\omega) + \min\{0, \text{Re}(\eta - \beta)\} + 1. 
\]

(5.2)

Then the generalized fractional integral $I_{0+}^{a,b,\eta}$ of the $H$-function (1.1) exists and the following relation holds:

\[
\left( I_{0+}^{a,b,\eta} t^{\omega} \mathcal{H}_{p,q}^m \left[ \begin{array}{c} \alpha_i \\ (b_j, \beta_j) \end{array} \right] \right) (x) 
= x^{\omega-\beta} \mathcal{H}_{p+2,q+2}^{m+2} \left[ \begin{array}{c} \alpha_i \\ (b_j, \beta_j) \end{array} \right] 
\]

(5.3)

**Proof.** By (2.7) we have

\[
\left( I_{0+}^{a,b,\eta} t^{\omega} \mathcal{H}_{p,q}^m \left[ \begin{array}{c} \alpha_i \\ (b_j, \beta_j) \end{array} \right] \right) (x) 
= x^{-\alpha-\beta} \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}t^\omega \mathcal{H}_{p,q}^m \left[ \begin{array}{c} \alpha_i \\ (b_j, \beta_j) \end{array} \right] 
\]

(5.4)
According to (2.25), (2.26), (3.16) and (3.17), the integrand in (5.4) for any \( x > 0 \) has the asymptotic estimate at zero

\[
(x - t)^{-\alpha - \beta} t^{\sigma - 1} 2F1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] 
= O \left( t^{\omega + \sigma \alpha^* + \min(0, \Re(\eta - \beta))} \right) \quad (t \to +0)
\]
or

\[
(x - t)^{-\alpha - \beta} t^{\sigma - 1} 2F1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] 
= O \left( t^{\omega + \sigma \alpha^* + \min(0, \Re(\eta - \beta))[\log(t)]^N} \right) \quad (t \to +0).
\]

Here \( \alpha^* \) is given by (3.18) and \( N^* \) is indicated in Theorem B(i). Therefore the condition (5.1) ensures the existence of the integral (5.4).

Applying (1.2), making the change of variable \( t = x \tau \), changing the order of integration and taking into account the formula [11, §2.21.11]

\[
\int_0^x t^{\alpha - 1} (x - t)^{\omega - 1} 2F1 \left( a, b; c; 1 - \frac{t}{x} \right) dt = \frac{\Gamma(c)\Gamma(a)\Gamma(\alpha + c - a - b)}{\Gamma(\alpha + c - a)\Gamma(\alpha + c - b)} x^{\alpha + c - 1} \quad (5.5)
\]

\[\begin{align*}
(a, b, c, \alpha & \in \mathbb{C}, \Re(\alpha) > 0, \Re(c) > 0, \Re(\alpha + c - a - b) > 0),
\end{align*}\]

we obtain

\[
\left( I_{0+}^{\alpha, \beta, \omega} t^\sigma H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right) (x)
\]

\[
= \frac{x^{-\alpha - \beta}}{\Gamma(a)} \int_0^x (x - t)^{-\alpha - 1} t^{\omega} 2F1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] dt
\]

\[
= \frac{x^{-\alpha - \beta}}{2\pi i \Gamma(a)} \int_\mathbb{C} \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] s ds \int_0^x (x - t)^{-\alpha - 1} t^{\omega - \sigma s} 2F1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) dt
\]

\[
= \frac{x^{\omega - \beta}}{2\pi i} \int_\mathbb{C} \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] s \frac{\Gamma(1 + \omega - s\sigma)\Gamma(1 + \omega - \beta + \eta - s\sigma)}{\Gamma(1 + \omega - \beta - s\sigma)\Gamma(1 + \omega + \alpha + \eta - s\sigma)} x^{-\sigma s} ds. (5.6)
\]

We note that since \( \mathcal{L} = \mathcal{L}_{\gamma, \infty} \), \( \Re(s) = \gamma \) and therefore the condition (5.2) ensures the existence of the Mellin-Barnes integral above. Hence in view of (1.2)

\[
\left( I_{0+}^{\alpha, \beta, \eta} t^\sigma H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right) (x)
\]

\[
x^{\omega - \beta} H_{p+2,q+2}^{m,n+1,2} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] (\omega, \sigma), (-\omega + \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\
(-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \right] . (5.7)
\]

and in accordance with (1.1) we obtain (5.3) which completes the proof of Theorem 1.
Corollary 1.1. Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy
\[ \sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + 1 > 0, \] (5.8)
\[ \sigma \gamma < \text{Re}(\omega) + 1. \] (5.9)
Then the Riemann-Liouville fractional integral $I_{0+}^\sigma$ of the $H$-function (1.1) exists and the following relation holds:
\[ \left( I_{0+}^\sigma \int_{\eta} H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right)(x) = x^{\omega+\alpha} H_{p+1,q+1}^{m,n+1} \left[ \begin{array}{c} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \alpha, \sigma) \end{array} \right]. \] (5.10)

Corollary 1.2. Let $\alpha, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy
\[ \sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \text{min}[0, \text{Re}(\eta)] + 1 > 0, \] (5.11)
\[ \sigma \gamma < \text{Re}(\omega) + \text{min}[0, \text{Re}(\eta)] + 1. \] (5.12)
Then the Erdélyi-Kober fractional integral $I_{\eta,\alpha}^+ \frac{\omega}{H}$ of the $H$-function (1.1) exists and the following relation holds:
\[ \left( I_{\eta,\alpha}^+ \int_{\eta} H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right)(x) = x^{\omega} H_{p+1,q+1}^{m,n+1} \left[ \begin{array}{c} (-\omega - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \alpha - \eta, \sigma) \end{array} \right]. \] (5.13)

Remark 2. In the case $\alpha^* > 0, \Delta \geq 0$ the relation (5.3) was indicated in [16, (4.2)], but in the assumptions of the result the condition (5.2) of Theorem 1 should be added.

Remark 3. Corollary 1.1 coincides with Theorem 1 in [7]. For real $\alpha > 0$ and $\alpha^* > 0$ the relation (5.10) was indicated in [11, 25.2.2], but the conditions of its validity have to be also corrected according to (5.8) and (5.9).

6. Right-Sided Generalized Fractional Integration of the $H$-Function

In this section we consider the right-sided generalized fractional integration $I_{-}^{\alpha,\beta,\eta}$ defined by (2.9).
Theorem 2. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\Re(\alpha) > 0, \Re(\beta) \neq \Re(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$
\sigma \max_{1 \leq i \leq n} \left[ \frac{\Re(a_i) - 1}{\alpha_i} \right] + \Re(\omega) < \min[\Re(\beta), \Re(\eta)], \quad (6.1)
$$

$$
\sigma \gamma > \Re(\omega) - \min[\Re(\beta), \Re(\eta)]. \quad (6.2)
$$

Then the generalized fractional integral $I_{\gamma, \eta}^{\alpha, \beta, \omega}$ of the $H$-function (1.1) exists and the following relation holds:

$$
\left( I_{\gamma, \eta}^{\alpha, \beta, \omega} H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right) (x)
$$

$$
= x^{\omega-\beta} H_{p,q}^{m+2,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\ (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right] \right]. \quad (6.3)
$$

Proof. By (2.9) we have

$$
\left( I_{\gamma, \eta}^{\alpha, \beta, \omega} H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right) (x)
$$

$$
= \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{\omega-\alpha-\beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right] dt. \quad (6.4)
$$

Due to (2.25), (2.26), (3.19) and (3.20), the integrand in (6.4) for any $x > 0$ has the asymptotic at infinity

$$
(t-x)^{\alpha-1} t^{\omega-\alpha-\beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right]
$$

$$
= O \left( t^{\omega-\min[\Re(\beta), \Re(\eta)]-1+\sigma \rho} \right) \quad (t \to +\infty)
$$

or

$$
= O \left( t^{\omega-\min[\Re(\beta), \Re(\eta)]-1+\sigma \rho \log(t)}^{N} \right) \quad (t \to +\infty).
$$

Here $\rho$ is given by (3.21) and $N$ is indicated in Theorem B(ii). Therefore the condition (6.1) ensures the existence of the integral (6.4). Applying (1.2), making the change $t = 1/\tau$ and using (5.5), we obtain

$$
\left( I_{\gamma, \eta}^{\alpha, \beta, \omega} H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right) \left( \frac{1}{\tau} \right)
$$

$$
= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \left( \frac{1}{\tau} - \frac{1}{x} \right)^{\alpha-1} t^{\omega-\alpha-\beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{1}{\tau x} \right) H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right] dt
$$
\[
\frac{x^{1-\alpha}}{2\pi i\Gamma(\alpha)} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{l}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] s^\sigma ds
\]
\[
\cdot \int_0^\infty (x - \tau)^{\alpha - 1 - \omega} \tau^{-1} \Gamma(\alpha) \Gamma(\omega + \beta + \sigma s) \Gamma(\omega + \eta + \sigma s) \tau^\sigma ds.
\]
\[
= \frac{x^{-\omega + \beta}}{2\pi i \Gamma(\alpha)} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{l}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \Gamma(\omega + \beta + \sigma s) \Gamma(\omega + \eta + \sigma s) \tau^\sigma ds. \quad (6.5)
\]

Since \( \mathcal{L} = \mathcal{L}_{\gamma, \infty}, \) \( \text{Re}(s) = \gamma \) and therefore the condition (6.2) guarantees the existence of the Mellin-Barnes integral above. Replacing in (6.5) \( x \) by \( 1/x \), we obtain (6.3).

**Corollary 2.1.** Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \cdots, p; j = 1, \cdots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) + \text{Re}(\alpha) < 0,
\]
\[
\sigma \gamma > \text{Re}(\omega) + \text{Re}(\alpha). \quad (6.6)
\]

Then the Riemann-Liouville fractional integral \( I^\alpha_0 \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( I^\alpha_0 t^\omega H_{p,q}^{m,n} \left[ \begin{array}{l}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \right)(x) = x^{\omega + \alpha} H_{p+1,q+1}^{m+1,n+1} \left[ \begin{array}{l}
(a_i, \alpha_i)_{1,p} \\
(-\omega, \sigma)
\end{array} \right] \left( -\omega - \alpha, \sigma, (b_j, \beta_j)_{1,q} \right). \quad (6.8)
\]

**Corollary 2.2.** Let \( \alpha, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \cdots, p; j = 1, \cdots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \text{Re}(\eta),
\]
\[
\sigma \gamma > \text{Re}(\omega) - \text{Re}(\eta). \quad (6.9)
\]

Then the Erdélyi-Kober fractional integral \( K_{\eta, \alpha}^\omega \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( K_{\eta, \alpha}^\omega H_{p,q}^{m,n} \left[ \begin{array}{l}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \right)(x) = x^{\omega + \alpha} H_{p+1,q+1}^{m+1,n+1} \left[ \begin{array}{l}
(a_i, \alpha_i)_{1,p} \\
(-\omega + \eta + \alpha, \sigma)
\end{array} \right] \left( -\omega + \eta, \sigma, (b_j, \beta_j)_{1,q} \right). \quad (6.11)
\]

**Remark 4.** In the case \( \alpha^* > 0, \Delta \geq 0 \) the relation of the form (6.3) was indicated in [16, (4.3)]. But it includes a mistake and should be replaced by (6.3) with the conditions (6.1) and (6.2).
Remark 5. Corollary 2.1 coincides with Theorem 2 in [7]. For real \( \alpha > 0 \) and \( \alpha^* > 0 \) the relation (6.8) was indicated in [18, (2.5)], but the conditions of its validity have to be also corrected in accordance with (6.6) and (6.7).

7. Left-Sided Generalized Fractional Differentiation of the \( H \)-Function

Now we treat the left-sided generalized fractional derivative \( \mathcal{D}_{0+}^{\alpha,\beta,\eta} \) given by (2.11).

Theorem 3. Let \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \text{Re}(\alpha + \beta + \eta) \neq 0 \). Let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \ldots, p; j = 1, \ldots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\alpha + \beta + \eta)] + 1 > 0, \tag{7.1}
\]

\[
\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\alpha + \beta + \eta)] + 1. \tag{7.2}
\]

Then the generalized fractional derivative \( \mathcal{D}_{0+}^{\alpha,\beta,\eta} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( \mathcal{D}_{0+}^{\alpha,\beta,\eta} \right)^{m,n} \left[ t^n \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right| \right](x) = x^{\omega+\beta} \left[ \begin{array}{c}
(-\omega, \sigma), (-\omega - \eta - \alpha - \beta, \sigma), (a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}, (-\omega - \beta, \sigma), (-\omega - \eta, \sigma)
\end{array} \right]. \tag{7.3}
\]

Proof. Let \( n = [\text{Re}(\alpha)] + 1 \). From (2.11) we have

\[
\left( \mathcal{D}_{0+}^{\alpha,\beta,\eta} \right)^{m,n} \left[ t^n \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right| \right](x) = \left( \frac{d}{dx} \right)^n \left( \mathcal{I}_{0+}^{\alpha+n-n, \beta-n, \alpha+n-n} \left[ t^n \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right| \right](x) \right), \tag{7.4}
\]

which exists according to Theorem 1 with \( \alpha, \beta \) and \( \eta \) being replaced by \( -\alpha + n, -\beta - n \) and \( \alpha + \eta - n \), respectively. Then we find

\[
\left( \mathcal{D}_{0+}^{\alpha,\beta,\eta} \right)^{m,n} \left[ t^n \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right| \right](x) = \left( \frac{d}{dx} \right)^n x^{\omega+\beta+n} \left[ \begin{array}{c}
(-\omega, \sigma), (-\omega - \alpha - \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}, (-\omega - \beta - n, \sigma), (-\omega - \eta, \sigma)
\end{array} \right]. \tag{7.5}
\]
Taking into account the differentiation formula (4.1) we have
\[
(D_{0+}^{\alpha,\beta,\eta} t^\omega H_{m,n}^{p,q} \left[ t^\sigma \middle| (a_i, \alpha_i)_{1,p} \right] (x))
\]
\[
= x^{\omega+\beta} H_{m,n+3}^{p+3,q+3} \left[ x^\sigma \middle| (-\omega - \beta - n, \sigma), (-\omega, \sigma), (-\omega - \alpha - \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \right] (b_j, \beta_j)_{1,q}, \tag{7.6}
\]
and Lemma 2 and the reduction relation (4.1) imply (7.3), which completes the proof of theorem.

**Corollary 3.1.** Let \( \alpha \in \mathbb{C} \) with \( \mathrm{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \( (i = 1, \ldots, p; j = 1, \ldots, q) \) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy the conditions in (5.8) and (5.9). Then the Riemann-Liouville fractional derivative \( D_{0+}^\alpha \) of the \( H \)-function (1.1) exists and the following relation holds:
\[
(D_{0+}^{\alpha,\beta,\eta} t^\omega H_{m,n}^{p,q} \left[ t^\sigma \middle| (a_i, \alpha_i)_{1,p} \right] (x)) = x^{\omega-\alpha} H_{m,n+1}^{p+1,q+1} \left[ x^\sigma \middle| (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \right] (b_j, \beta_j)_{1,q}, \tag{7.7}
\]

**Remark 6.** For real \( \alpha > 0 \) and \( \alpha' > 0 \) the relation (7.3) was given in [18, (2.7.13)], but the conditions of its validity have to be corrected in accordance with (7.1) and (7.2).

**Remark 7.** Corollary 3.1 coincides with Theorem 3 in [7].

### 8. Right-Sided Generalized Fractional Differentiation of the \( H \)-Function

Here we deal with the right-sided generalized fractional derivative \( D_{-}^{\alpha,\beta,\eta} \) given by (2.12).

**Theorem 4.** Let \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\alpha + \beta + \eta) + [\mathrm{Re}(\alpha)] + 1 \neq 0 \). Let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \( (i = 1, \ldots, p; j = 1, \ldots, q) \) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy
\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\mathrm{Re}(a_i) - 1}{\alpha_i} \right] + \mathrm{Re}(\omega) + \max[\mathrm{Re}(\beta) + [\mathrm{Re}(\alpha)] + 1, -\mathrm{Re}(\alpha + \eta)] < 0, \tag{8.1}
\]
\[
\sigma \gamma > \mathrm{Re}(\omega) + \max[\mathrm{Re}(\beta) + [\mathrm{Re}(\alpha)] + 1, -\mathrm{Re}(\alpha + \eta)]. \tag{8.2}
\]

Then the generalized fractional derivative \( D_{-}^{\alpha,\beta,\eta} \) of the \( H \)-function (1.1) exists and the following relation holds:
\[
(D_{-}^{\alpha,\beta,\eta} t^\omega H_{m,n}^{p,q} \left[ t^\sigma \middle| (a_i, \alpha_i)_{1,p} \right] (x))
\]
\[
= (-1)^{[\mathrm{Re}(\alpha)] + 1} x^{\omega+\beta} H_{m+2,n}^{p+2,q+2} \left[ x^\sigma \middle| (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma) \right. \]
\[ (-\omega - \beta, \sigma), (-\omega + \alpha + \eta, \sigma), (b_j, \beta_j)_{1,q} \left. \right] \tag{8.3}
\]
Proof. Let \( n = \lfloor \text{Re}(\alpha) \rfloor + 1 \). Owing to (2.12) we have

\[
\left( D^{-n} a^n \mathcal{H}^m p_{,q} \right) \left( \begin{array}{c|c}
(a, \alpha)_{1,p} \\
(b, \beta)_{1,q}
\end{array} \right) (x) = \left( \frac{-d}{dx} \right)^n \left( \begin{array}{c|c}
(a, \alpha)_{1,p} \\
(b, \beta)_{1,q}
\end{array} \right) (x),
\]

which exists according to Theorem 2 with \( \alpha, \beta \) and \( \eta \) being replaced by \(-\alpha + n, -\beta - n\) and \( \alpha + \eta \), respectively. Then applying the differentiation formula (4.5), similarly to (7.5), (7.6), we find in view of the reduction formula (1.2) that

\[
\left( D^{-n} a^n \mathcal{H}^m p_{,q} \right) \left( \begin{array}{c|c}
(a, \alpha)_{1,p} \\
(b, \beta)_{1,q}
\end{array} \right) (x) = (-1)^n a\mathcal{H}^m p_{,q+1} \left( \begin{array}{c|c}
(a, \alpha)_{1,p} \\
(b, \beta)_{1,q}
\end{array} \right) (x),
\]

which implies the formula (8.3).

**Corollary 4.1.** Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \ldots, p; j = 1, \ldots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) - \{\text{Re}(\alpha)\} + 1 < 0, \quad (8.5)
\]

\[
\sigma \gamma + \text{Re}(\omega) - \{\text{Re}(\alpha)\} + 1 > 0. \quad (8.6)
\]

Then the Riemann-Liouville fractional derivative \( D^\alpha \) of the \( \mathcal{H} \)-function (1.1) exists and there holds the relation:

\[
\left( D^\alpha a^n \mathcal{H}^m p_{,q} \right) \left( \begin{array}{c|c}
(a, \alpha)_{1,p} \\
(b, \beta)_{1,q}
\end{array} \right) (x) = (-1)^{\text{Re}(\alpha)+1} a\mathcal{H}^{m+1} p_{,q+1} \left( \begin{array}{c|c}
(a, \alpha)_{1,p} \\
(-\omega, \sigma) \\
(b, \beta)_{1,q}
\end{array} \right). \quad (8.7)
\]

**Remark 8.** The relation of the form (8.7) with real \( \alpha > 0 \) and \( \alpha > 0 \) was proved in [13, formula (14a)] (see also [12], [14, (2.2)] and [18, (2.7.9)]). But such a formula contains mistakes and should be replaced by (8.7) with the condition (8.5) and (8.6).
Remark 9. When $\alpha = k \in \mathbb{N}$, the relations (7.7) and (8.7) coincide with (4.4) and (4.5), respectively.

9. Generalized Fractional Integro-Differentiation of the $H$-Function

Here we investigate the generalized fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{-}^{\alpha,\beta,\eta}$ given by (2.8) and (2.10). The following statements are proved similarly to Theorems 3 and 4 by using the relations (2.8) and (2.10), Theorems 1 and 2, and the properties of the $H$-function in Sections 3 and 4.

Theorem 5. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) \leq 0, \text{Re}(\beta) \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \cdots, p; j = 1, \cdots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min \{0, \text{Re}(\eta - \beta)\} + 1 > 0, \quad (9.1)$$

$$\sigma \gamma < \text{Re}(\omega) + \min \{0, \text{Re}(\eta - \beta)\} + 1. \quad (9.2)$$

Then the generalized fractional integro-differentiation $I_{0+}^{\alpha,\beta,\eta}$ of the $H$-function (1.1) exists and there holds the relation

$$I_{0+}^{\alpha,\beta,\eta} \omega H_{p, q}^{m, n} \left[ \begin{array}{c|c} \langle a_i, \alpha_i \rangle_{1,p} \\ \hline \langle b_j, \beta_j \rangle_{1,q} \end{array} \right] (x) = x^{\omega - \beta} H_{p+2,q+2}^{m,n+2} \left[ x^{\sigma} \begin{array}{c|c} (-\omega, \sigma), (-\omega - \eta + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \\ \hline (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right]. \quad (9.3)$$

Theorem 6. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) \leq 0, \text{Re}(\beta) + [\text{Re}(\alpha)] - 1 \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \cdots, p; j = 1, \cdots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \min \{\text{Re}(\beta) - [\text{Re}(\alpha)] - 1, \text{Re}(\eta)\}, \quad (9.4)$$

$$\sigma \gamma > \text{Re}(\omega) - \min \{\text{Re}(\beta) - [\text{Re}(\alpha)] - 1, \text{Re}(\eta)\}. \quad (9.5)$$

Then the generalized fractional integro-differentiation $I_{-}^{\alpha,\beta,\eta}$ of the $H$-function (1.1) exists and there holds the relation

$$I_{-}^{\alpha,\beta,\eta} \omega H_{p, q}^{m, n} \left[ \begin{array}{c|c} \langle a_i, \alpha_i \rangle_{1,p} \\ \hline \langle b_j, \beta_j \rangle_{1,q} \end{array} \right] (x) = x^{\omega - \beta} H_{p+2,q+2}^{m,n+2} \left[ x^{\sigma} \begin{array}{c|c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\ \hline (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right]. \quad (9.6)$$
Remark 10. The relation (9.3) with $a^* > 0, \Delta \geq 0$ was indicated in [16, (4.2)], but conditions of its validity have to be corrected in accordance with (9.1) and (9.2).

Remark 11. The relations (9.3) and (9.6) for the fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{1}^{\alpha,\beta,\eta}$, defined in (2.8) and (2.10) for $\alpha \in \mathbb{C}, \Re(\alpha) \leq 0$ coincide with that (5.3) and (6.3) for the fractional integration operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{1}^{\alpha,\beta,\eta}$, defined in (2.7) and (2.9) for $\alpha \in \mathbb{C}, \Re(\alpha) > 0$. Though the conditions for validity of (5.3) and (9.3) in Theorems 1 and 5 have the same form, that of (6.3) and (9.6) presented in Theorems 2 and 4 are slightly different.

In conclusion we note that, as it was mentioned in Remarks 2, 4 and 10, the relations (5.3), (6.3) and (9.3) for generalized calculus operator $I_{0+}^{\alpha,\beta,\eta}$ were already known in the case $a^* > 0, \Delta \geq 0$. Further, Remarks 3, 5, 6 and 8 indicate that the relations (5.10) and (6.8) for the Riemann-Liouville fractional integrals $I_{0+}^{\alpha,\beta}$, $I_{1}^{\alpha,\beta}$ and (7.3) and (8.7) for the fractional derivative $D_0^\alpha$, in the case real $\alpha > 0$ and $a^* > 0$ were established. However, the $II$-function’s asymptotic estimates (3.16), (3.17) at zero and (3.19), (3.20) at infinity allow us to prove such results under more general assumptions $a^* > 0$ and $a^* = 0, \Delta \gamma + \Re(\mu) < -1$.

References


