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Kyoto University
Generalized Fractional Calculus of the $H$-Function

Megumi Saigo* [西郷 恵] (福岡大学 理学部)
Anatoly A. Kilbas† (ベラルーシ国立大学・ベラルーシ)

Abstract

The paper is devoted to study the generalized fractional calculus of arbitrary complex order for the $H$-function defined by the Mellin-Barnes integral

$$H_{p,q}^{m,n}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds,$$

where the function $\mathcal{H}_{p,q}^{m,n}(s)$ is a certain ratio of products of Gamma functions with the argument $s$ and the contour $\mathcal{L}$ is specially chosen. The considered generalized fractional integration and differentiation operators contain the Gauss hypergeometric function as a kernel and generalize classical fractional integrals and derivatives of Riemann-Liouville, Erdélyi-Kober type, etc. It is proved that the generalized fractional integrals and derivatives of $H$-functions are also $H$-functions but of greater order. In particular, the obtained results define more precisely and generalize known results.

1. Introduction

This paper deals with the $H$-function $H_{p,q}^{m,n}(z)$. For integers $m, n, p, q$ such that $0 \leq m \leq q$, $0 \leq n \leq p$, for $a_i, b_j \in \mathbb{C}$ with $\mathbb{C}$ the field of complex numbers and for $\alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty)$ ($i = 1, 2, \ldots, p; j = 1, 2, \ldots, q$) the $H$-function $H_{p,q}^{m,n}(z)$ is defined via a Mellin-Barnes type integral in the following way:

$$H_{p,q}^{m,n}(z) \equiv H_{p,q}^{m,n} \begin{bmatrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{bmatrix} = H_{p,q}^{m,n} \begin{bmatrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{bmatrix}$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \begin{bmatrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{bmatrix} s^{-s} ds,$$

(1.1)

* Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan
† Department of Mathematics and Mechanics, Belarusian State University, Minsk 220050, Belarus
where the contour $\mathcal{L}$ is specially chosen and

\[
\mathcal{H}_{p,q}^{m,n}(s) \equiv \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c} (\alpha_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] s = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)}, \quad (1.2)
\]

in which an empty product, if it occurs, is taken to be one. Such a function was introduced by S. Pincherle in 1888 and its theory has been developed by Mellin [10], Dixon and Ferrar [2] (see [3, §1.19] in this connection). An interest to the $H$-function arose again in 1961 when Fox [4] has investigated such a function as a symmetrical Fourier kernel. Therefore this function is sometimes called as Fox's $H$-function. The theory of this function may be found in [1], [9, Chapter 1], [17, Chapter 2] and [11, 8.8.3].

Classical Riemann-Liouville fractional calculus of real order [17, §2.2] (see (2.1) - (2.6) below) was investigated in [12] - [14], [18] and [11]. The right-sided fractional integrals and derivatives of the $H$-function (1.1) were studied in [12] - [14] and the results were presented in [18, §2.7], where the case of left-sided fractional differentiation of the $H$-function was also considered. The left-sided fractional integration of the $H$-function was given in [11, 2.25.2]. Such results for the generalized fractional calculus operators with the Gauss hypergeometric function as a kernel (see (2.7) - (2.10) below), being introduced by the first author [15], were obtained in [16].

However, some of the results obtained in [12] - [14] (cited in [18]) and [16] can be taken to be more precisely. Moreover, these results were given provided that the parameters $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0, \beta_j > 0 (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$ of the $H$-function satisfy certain conditions. These conditions were based on asymptotic behavior of $H_{p,q}^{m,n}(z)$ at zero and infinity. In [5] we extended such the known asymptotic results for the $H$-function to more wide class of parameters.

In [7], [8] we have applied the obtained asymptotic estimates in [5] to find the Riemann-Liouville fractional integrals and derivatives of any complex order of the $H$-function. In particular, we could make more precisely the known results from [12] - [14], [18] and [11].

The present paper is devoted to obtain such type results for the generalized fractional integration and differentiation operators of any complex order with the Gauss hypergeometric function as a kernel. In particular, we give more precisely some of the results from [16] and generalize the results obtained in [7], [8]. The paper is organized as follow. In Section 2 we present classical and generalized fractional calculus operators and some facts from the theory of Gauss hypergeometric function. Sections 3 and 4 contain the result from the theory of the $H$-function. The existence of $H_{p,q}^{m,n}(z)$ and its asymptotic behavior at zero and infinity is considered in Section 3 and certain reduction and differentiation properties in Section 4. Sections 5 and 6 deal with generalized fractional differentiation of the $H$-function (1.1). Sections 7 and 8 are devoted to the generalized fractional differentiation of the $H$-function. Another type of fractional integro-differentiation of the $H$-function is given in Section 9.
2. Classical and Generalized Fractional Calculus Operators

For $\alpha \in \mathbb{C}, \text{Re}(\alpha) > 0$, the Riemann-Liouville left- and right-sided fractional calculus operators are defined as follows [17, §2.3 and §2.4]:

\[
(I_{0+}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (x > 0), \tag{2.1}
\]

\[
(I_{-}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(t)dt}{(t-x)^{1-\alpha}} \quad (x > 0), \tag{2.2}
\]

and

\[
(D_{0+}^{\alpha}f)(x) = \left(\frac{d}{dx}\right)^{[\text{Re}(\alpha)]+1} \left(I_{0+}^{1-\alpha+[\text{Re}(\alpha)]}f\right)(x)
= \frac{d}{dx} \left(\frac{1}{\Gamma(1-\alpha+|\text{Re}(\alpha)|)} \int_{0}^{x} \frac{f(t)dt}{(x-t)^{1-\alpha}}\right) \quad (x > 0), \tag{2.3}
\]

\[
(D_{-}^{\alpha}f)(x) = \left(-\frac{d}{dx}\right)^{[\text{Re}(\alpha)]+1} \left(I_{-}^{1-\alpha+[\text{Re}(\alpha)]}f\right)(x)
= -\frac{d}{dx} \left(\frac{1}{\Gamma(1-\alpha+|\text{Re}(\alpha)|)} \int_{x}^{\infty} \frac{f(t)dt}{(t-x)^{1-\alpha}}\right) \quad (x > 0), \tag{2.4}
\]

respectively, where the symbol $[\kappa]$ means the integral part of a real number $\kappa$, i.e. the largest integer not exceeding $\kappa$. In particular, for real $\alpha > 0$, the operators $D_{0+}^{\alpha}$ and $D_{-}^{\alpha}$ take more simple forms

\[
(D_{0+}^{\alpha}f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{0+}^{1-\alpha+[\alpha]}f\right)(x)
= \frac{d}{dx} \left(\frac{1}{\Gamma(1-\alpha)} \int_{0}^{x} \frac{f(t)dt}{(x-t)^{1-\alpha}}\right) \quad (x > 0), \tag{2.5}
\]

and

\[
(D_{-}^{\alpha}f)(x) = \left(-\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{-}^{1-\alpha+[\alpha]}f\right)(x)
= -\frac{d}{dx} \left(\frac{1}{\Gamma(1-\alpha)} \int_{x}^{\infty} \frac{f(t)dt}{(t-x)^{1-\alpha}}\right) \quad (x > 0), \tag{2.6}
\]

respectively, where $\{\kappa\}$ stands for the fractional part of $\kappa$, i.e. $\{\kappa\} = \kappa - [\kappa]$.

For $\alpha, \beta, \eta \in \mathbb{C}$ and $x > 0$ the generalized fractional calculus operators are defined by [15]

\[
(I_{0+}^{\alpha,\beta,\eta}f)(x) = \frac{x^{\alpha-\beta}}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} {}_{2}F_{1}\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right) f(t)dt \tag{2.7}
\]
\[
\left(I^{\alpha, \beta, \eta}_{0^+} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I^{\alpha+n, \beta-n, \eta-n}_{0^+} f\right)(x) \quad (\text{Re}(\alpha) > 0; n = [\text{Re}(\alpha)] + 1); \tag{2.8}
\]
\[
\left(I^{\alpha, \beta, \eta}_{-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} \ \left(2F_1\right) \left(\alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t}\right) f(t) dt \tag{2.9}
\]
\[
\left(D^{\alpha, \beta, \eta}_{0^+} f\right)(x) \equiv \left(I^{\alpha, \beta, \eta}_{0^+} f\right)(x) \quad (\text{Re}(\alpha) > 0; n = [\text{Re}(\alpha)] + 1); \tag{2.10}
\]
and
\[
\left(D^{\alpha, \beta, \eta}_{-} f\right)(x) \equiv \left(I^{\alpha, \beta, \eta}_{-} f\right)(x) \quad (\text{Re}(\alpha) > 0; n = [\text{Re}(\alpha)] + 1)). \tag{2.11}
\]

Here \(2F_1(a, b; c; z)\) \((a, b, c, z \in \mathbb{C})\) is the Gauss hypergeometric function defined by the series
\[
2F_1(a, b; c; z) = \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{2.13}
\]
with
\[
(a)_0 = 1, \quad (a)_k = a(a+1) \cdots (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (k \in \mathbb{N}), \tag{2.14}
\]
where \(\Gamma(z)\) is the Gamma function [3, Chapter I] and \(\mathbb{N}\) denotes the set of positive integers.

The series in (2.13) is convergent for \(|z| < 1\) and for \(|z| = 1\) with \(\text{Re}(c-a-b) > 0\), and can be analytically continued into \(\{z \in \mathbb{C} : |\arg(1-z)| < \pi\}\) (see [3, Chapter II]).

Since
\[
2F_1(0, b; c; z) = 1 \tag{2.15}
\]
for \(\beta = -\alpha\), the generalized fractional calculus operators (2.7), (2.9), (2.11) and (2.12) coincide with the Riemann-Liouville operators (2.1) - (2.4) for \(\text{Re}(\alpha) > 0\):

\[
\left(I^{\alpha, -\alpha, \eta}_{0^+} f\right)(x) = \left(I^{\alpha}_{0^+} f\right)(x), \quad \left(I^{\alpha, -\alpha, \eta}_{-} f\right)(x) = \left(I^{\alpha}_{-} f\right)(x), \tag{2.16}
\]
\[
\left(D^{\alpha, -\alpha, \eta}_{0^+} f\right)(x) = \left(D^{\alpha}_{0^+} f\right)(x), \quad \left(D^{\alpha, -\alpha, \eta}_{-} f\right)(x) = \left(D^{\alpha}_{-} f\right)(x). \tag{2.17}
\]
According to the relation [3, 2.8(4)]

\[ 2F_1(a, b; c; z) = (1 - z)^{-c}, \quad (2.18) \]

when \( \beta = 0 \) the operators (2.7) and (2.9) coincide with the Erdélyi-Kober fractional integrals [17, §18.1]:

\[
(I_{0+}^\alpha \eta f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (t-x)^{\alpha-1}t^{-\eta}f(t)dt \equiv (I_{\eta,\alpha}^+ f)(x) \quad (\alpha, \eta \in \mathbb{C}, \Re(\alpha) > 0), \tag{2.19}
\]

\[
(I_{-}^\alpha \eta f)(x) = \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1}t^{-\eta}f(t)dt \equiv (K_{\eta,\alpha}^- f)(x) \quad (\alpha, \eta \in \mathbb{C}, \Re(\alpha) > 0). \tag{2.20}
\]

Therefore the operators (2.7), (2.9) and (2.11), (2.12) are called "generalized" fractional integrals and derivatives, respectively. Moreover, the operators (2.11) and (2.12) are inverse to (2.7) and (2.9):

\[
D_{0+}^{\alpha,\beta,\eta} = (I_{0+}^{\alpha,\beta,\eta})^{-1}, \quad D_{-}^{\alpha,\beta,\eta} = (I_{-}^{\alpha,\beta,\eta})^{-1}. \tag{2.21}
\]

Fractional calculus operators (2.1), (2.3), (2.5), (2.7), (2.8), (2.11) and (2.2), (2.4), (2.6), (2.9), (2.10), (2.12) are called left-sided and right-sided, respectively [17, §2].

We give some other properties of \( 2F_1(a, b; c; z) \) [3, 2.8(46), 2.9(2), 2.10(14)] which will be used in the following calculations:

\[
2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c \neq -0, -1, -2, \ldots; \Re(c-a-b) > 0); \tag{2.22}
\]

\[
2F_1(a, b; c; z) = (1 - z)^{c-a-b} 2F_1(c-a, c-b; c; z); \tag{2.23}
\]

\[
2F_1(a, b; a+b; z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(k!)^2} [2\psi(1+k) - \psi(a+k) + \psi(b+k) - \log(1-z)](1-z)^k \quad (|\arg(z)| < \pi; a, b \neq 0, -1, -2, \ldots), \tag{2.24}
\]

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the Psi function [3, 1.7].

Formulas (2.22) - (2.24) mean the following asymptotic behavior of \( 2F_1(a, b; c; z) \) at the point \( z = 1 \).

**Lemma 1.** For \( a, b, c \in \mathbb{C} \) with \( \Re(c) > 0 \) and \( z \in \mathbb{C} \), there hold the following asymptotic relations near \( z = 1 \):

\[
2F_1(a, b; c; z) = O(1) \quad (z \to 1-) \tag{2.25}
\]

for \( \Re(c-a-b) > 0 \);

\[
2F_1(a, b; c; z) = O \left( (1-z)^{c-a-b} \right) \quad (z \to 1-) \tag{2.26}
\]

for \( \Re(c-a-b) < 0 \); and

\[
2F_1(a, b; c; z) = O \left( \log(1-z) \right) \quad (z \to 1-) \tag{2.27}
\]
for \( c - a - b = 0, \ a, b \neq 0, -1, -2, \ldots \) and \( |\arg(z)| < \pi \).

3. Existence and Asymptotic Behavior of the \( H \)-Function

We shall consider the \( H \)-function (1.1) provided that the poles

\[
b_{jl} = \frac{-b_{j} - l}{\beta_{j}} \quad (j = 1, \ldots, m; l \in \mathbb{N}_{0})
\]

of the Gamma functions \( \Gamma(b_{j} + \beta_{j}s) \) and that

\[
a_{ik} = \frac{1 - \alpha_{i} + k}{\alpha_{i}} \quad (i = 1, \ldots, n; k \in \mathbb{N}_{0})
\]

of \( \Gamma(1 - \alpha_{i} - \alpha_{i}s) \) do not coincide:

\[
\alpha_{i}(b_{j} + l) \neq \beta_{j}(a_{i} - k - 1) \quad (i = 1, \ldots, n; j = 1, \ldots, m; k, l \in \mathbb{N}_{0}),
\]

where \( \mathbb{N}_{0} = \mathbb{N} \cup \{0\} \). \( \mathcal{L} \) in (1.1) is the infinite contour splitting poles \( b_{jl} \) in (3.1) to the left and poles \( a_{ik} \) in (3.2) to the right of \( \mathcal{L} \) and has one of the following forms:

(i) \( \mathcal{L} = \mathcal{L}_{-\infty} \) is a left loop situated in a horizontal strip starting at the point \( -\infty + i\varphi_{1} \) and terminating at the point \( -\infty + i\varphi_{2} \) with \( -\infty < \varphi_{1} < \varphi_{2} < +\infty \);

(ii) \( \mathcal{L} = \mathcal{L}_{+\infty} \) is a right loop situated in a horizontal strip starting at the point \( +\infty + i\varphi_{1} \) and terminating at the point \( +\infty + i\varphi_{2} \) with \( -\infty < \varphi_{1} < \varphi_{2} < +\infty \).

(iii) \( \mathcal{L} = \mathcal{L}_{i\gamma \infty} \) is a contour starting at the point \( \gamma + i\infty \) and terminating at the point \( \gamma + i\infty \) with \( \gamma \in \mathbb{R} = (-\infty, +\infty) \).

The properties of the \( H \)-function \( H^{m,n}_{p,q}(z) \) depend on the numbers \( a^*, \Delta, \delta \) and \( \mu \) which are expressed via \( p, q, \alpha_{i}, \alpha_{i} (i = 1, 2, \ldots, p) \) and \( b_{j}, \beta_{j} (j = 1, 2, \ldots, q) \) by the following relations:

\[
a^* = \sum_{i=1}^{n} \alpha_{i} - \sum_{i=n+1}^{p} \alpha_{i} + \sum_{j=1}^{m} \beta_{j} - \sum_{j=m+1}^{q} \beta_{j},
\]

\[
\Delta = \sum_{j=1}^{q} \beta_{j} - \sum_{i=1}^{p} \alpha_{i},
\]

\[
\delta = \prod_{i=1}^{p} \alpha_{i}^{-\alpha_{i}} \prod_{j=1}^{q} \beta_{j}^{\beta_{j}},
\]

\[
\mu = \sum_{j=1}^{q} b_{j} - \sum_{i=1}^{p} a_{i} + \frac{p - q}{2}.
\]

Here an empty sum in (3.4), (3.5), (3.7) and an empty product in (3.6), if they occur, are taken to be zero and one, respectively.

The existence of the \( H \)-function is given by the following result [6].
Theorem A. Let \( a^*, \Delta, \delta \) and \( \mu \) be given by (3.4)-(3.7). Then the \( H \)-function \( H_{p,q}^{m,n}(z) \) defined by (1.1) and (1.2) makes sense in the following cases:

\[
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{-\infty}, \quad \Delta > 0, \quad z \neq 0; \\
\mathcal{L} &= \mathcal{L}_{-\infty}, \quad \Delta = 0, \quad 0 < |z| < \delta; \\
\mathcal{L} &= \mathcal{L}_{-\infty}, \quad \Delta = 0, \quad \Re(\mu) < -1, \quad |z| = \delta; \\
\mathcal{L} &= \mathcal{L}_{+\infty}, \quad \Delta < 0, \quad z \neq 0; \\
\mathcal{L} &= \mathcal{L}_{+\infty}, \quad \Delta = 0, \quad |z| > \delta; \\
\mathcal{L} &= \mathcal{L}_{\gamma\infty}, \quad \Delta = 0, \quad \Re(\mu) < -1, \quad |z| = \delta; \\
\mathcal{L} &= \mathcal{L}_{\gamma\infty}, \quad a^* > 0, \quad |\arg z| < \frac{a^*\pi}{2}, \quad z \neq 0; \\
\mathcal{L} &= \mathcal{L}_{\gamma\infty}, \quad a^* = 0, \quad \Delta \gamma + \Re(\mu) < -1, \quad \arg z = 0, \quad z \neq 0.
\end{aligned}
\] (3.8) - (3.15)

Remark 1. The results of Theorem A in the cases (3.10), (3.13) and (3.15) are more precisely than those in [11, §8.3.1].

The next statement being followed from the results in [5] characterizes the asymptotic behavior of the \( H \)-function at zero and infinity.

Theorem B. Let \( a^* \) and \( \Delta \) be given by (3.4) and (3.5) and let conditions in (3.3) be satisfied.

(i) If \( \Delta \geq 0 \) or \( \Delta < 0, a^* > 0 \), then the \( H \)-function has either of the asymptotic estimates at zero

\[
H_{p,q}^{m,n}(z) = O \left( z^{\varrho^*} \right) \quad (|z| \to 0)
\] (3.16)

or

\[
H_{p,q}^{m,n}(z) = O \left( z^{\varrho^*} |\log(z)|^{N^*} \right) \quad (|z| \to 0),
\] (3.17)

with the additional condition \(|\arg(z)| < a^* \pi/2\) when \( \Delta < 0, a^* > 0 \). Here

\[
\varrho^* = \min_{1 \leq j \leq m} \left[ \frac{\Re(b_j)}{\beta_j} \right],
\] (3.18)

and \( N^* \) is the order of one of the point \( b_j \) in (3.1) to which some other poles of \( \Gamma(b_j + \beta_j s) \) \((j = 1, \ldots, m)\) coincide.

(ii) If \( \Delta \leq 0 \) or \( \Delta > 0, a^* > 0 \), then the \( H \)-function has either of the asymptotic estimates at infinity

\[
H_{p,q}^{m,n}(z) = O \left( z^\varrho \right) \quad (|z| \to \infty)
\] (3.19)
or

\[ H_{p,q}^{m,n}(z) = O \left( z^\varrho |\log(z)|^N \right) \quad (|z| \to \infty), \]  
(3.20)

with the additional condition \(|\arg(z)| < \alpha \pi / 2\) when \(\Delta > 0, \alpha > 0\). Here

\[ \varrho = \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right], \]  
(3.21)

and \(N\) is the order of one of the point \(a_{ik}\) in (3.2) in which some other poles of \(\Gamma(1 - a_i - \alpha_i)\) \((i = 1, \cdots, n)\) coincide.

4. Reduction and Differentiation Properties of the \(H\)-Function

In this and next sections we suppose that the conditions for the existence of the \(H\)-function given in Theorem A are satisfied.

The following two Lemmas which characterize symmetric and reduction properties of the \(H\)-function follow from the definition of the \(H\)-function in (1.1) - (1.2).

**Lemma 2.** The \(H\)-function (1.1) is commutative in the set of pairs \((a_1, \alpha_1), \cdots, (a_n, \alpha_n)\); in \((a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p)\); in \((b_1, \beta_1), \cdots, (b_m, \beta_m)\) and in \((b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q)\).

**Lemma 3.** If one of \((a_i, \alpha_i)\) \((i = 1, \cdots, n)\) is equal to one of \((b_j, \beta_j)\) \((j = m + 1, \cdots, q)\) (or one of \((a_i, \alpha_i)\) \((i = n + 1, \cdots, p)\) is equal to one of \((b_j, \beta_j)\) \((j = 1, \cdots, m))\), then the \(H\)-function reduces to the lower order one, that is, \(p, q\) and \(n\) (or \(m\)) decrease by unity. Two such results have the forms

\[ H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q-1}, (a_1, \alpha_1) \end{array} \right] = H_{p-1,q-1}^{m-1,n-1} \left[ \begin{array}{c} (a_i, \alpha_i)_{2,p} \\ (b_j, \beta_j)_{1,q-1} \end{array} \right] \]  
(4.1)

provided that \(n \geq 1\) and \(q > m\), and

\[ H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p-1}, (b_1, \beta_1) \\ (b_j, \beta_j)_{1,q} \end{array} \right] = H_{p-1,q}^{m-1,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p-1} \\ (b_j, \beta_j)_{2,q} \end{array} \right] \]  
(4.2)

provided that \(m \geq 1\) and \(p > n\).

The next differentiation formulae follow from the definition of the \(H\)-function given in (1.1) - (1.2) and from the functional equation for the Gamma function [3, §1.2(6)]

\[ \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}. \]  
(4.3)
Lemma 4. There hold the following differentiation formulae for \( \omega, c \in \mathbb{C}, \sigma > 0 \)
\[
\left( \frac{d}{dz} \right)^k \left\{ z^\omega I_{p,q}^{m,n} \left[ \begin{array}{c} cz \\ (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right\} \\
= z^{\omega-k} I_{p+1,q+1}^{m+1,n+1} \left[ \begin{array}{c} cz \\ (\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (k - \omega, \sigma) \end{array} \right],
\]
(4.4)
\[
\left( \frac{d}{dz} \right)^k \left\{ z^\omega I_{p,q}^{m,n} \left[ \begin{array}{c} cz \\ (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right\} \\
= (-1)^k z^{-\omega-k} I_{p+1,q+1}^{m+1,n+1} \left[ \begin{array}{c} cz \\ (a_i, \alpha_i)_{1,p}, (\omega, \sigma) \\ (b_j, \beta_j)_{1,q}, (k - \omega, \sigma) \end{array} \right].
\]
(4.5)

5. Left-Sided Generalized Fractional Integration of the \( H \)-Function

In the following sections we treat the \( H \)-function (1.1) - (1.2) with \( \mathcal{S} = \mathcal{S}_{\gamma_{\infty}} \) and under the assumptions \( a^* > 0 \) or \( a^* = 0, \Delta \gamma \neq \text{Re}(\mu) < -1 \) for \( \alpha^*, \Delta, \mu \) being given by (3.4), (3.5), (3.7).

Here we consider the left-sided generalized fractional integration \( I_{0+}^{\alpha, \beta, \eta} \) defined by (2.7).

Theorem 1. Let \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \text{Re}(\beta) \neq \text{Re}(\eta) \). Let the constants
\( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 (i = 1, \ldots, p; j = 1, \ldots, q) \) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy
\[
\sigma \min_{1 \leq i \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1 > 0,
\]
(5.1)
\[
\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1.
\]
(5.2)
The generalized fractional integral \( I_{0+}^{\alpha, \beta, \eta} \) of the \( H \)-function (1.1) exists and the following relation holds:
\[
\left( I_{0+}^{\alpha, \beta, \eta} t^\omega I_{p,q}^{m,n} \left[ \begin{array}{c} t^\sigma \\ (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right)(x) \\
= x^{\omega-\beta} I_{p+2,q+2}^{m+2,n+2} \left[ x^\sigma \left| \begin{array}{c} (\omega, \sigma), (\omega + \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (\omega + \beta, \sigma), (\omega - \alpha - \eta, \sigma) \end{array} \right] \right].
\]
(5.3)

Proof. By (2.7) we have
\[
\left( I_{0+}^{\alpha, \beta, \eta} t^\omega I_{p,q}^{m,n} \left[ \begin{array}{c} t^\sigma \\ (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right)(x) \\
= \frac{x^{\omega-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\omega 2F1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) I_{p,q}^{m,n} \left[ t^\sigma \\ (b_j, \beta_j)_{1,q} \right] dt.
\]
According to (2.25), (2.26), (3.16) and (3.17), the integrand in (5.4) for any $x > 0$ has the asymptotic estimate at zero

$$(x-t)^{-\alpha-1}t^\omega 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) H_{p,q}^{m,n} \left[ I^{\sigma} \mid (a_i, \alpha_i)_{1,p} \right.$$

$$(b_j, \beta_j)_{1,q} \left. \right] = O \left( t^{\omega + \sigma \varphi + \min\{0, \Re(\eta - \beta)\} \right) \quad (t \to +0)$$

or

$$(x-t)^{-\alpha-1}t^\omega 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) H_{p,q}^{m,n} \left[ I^{\sigma} \mid (a_i, \alpha_i)_{1,p} \right.$$

$$(b_j, \beta_j)_{1,q} \left. \right] = O \left( t^{\omega + \varphi + \min\{0, \Re(\eta - \beta)\} |\log(t)|^{N^*} \right) \quad (t \to +0).$$

Here $\varphi^*$ is given by (3.18) and $N^*$ is indicated in Theorem B(i). Therefore the condition (5.1) ensures the existence of the integral (5.4).

Applying (1.2), making the change of variable $t = x\tau$, changing the order of integration and taking into account the formula [11, §2.21.11]

$$\int_0^x t^{\alpha-1}(x-t)^{-\alpha} 2F_1 \left( a, b; c; 1 - \frac{t}{x} \right) dt = \frac{\Gamma(c)\Gamma(\alpha+c-a-b)}{\Gamma(\alpha+c-a)\Gamma(\alpha+c-b)} x^{\alpha+c-1}$$

$$(a, b, c, \alpha \in \mathbb{C}, \Re(\alpha) > 0, \Re(c) > 0, \Re(\alpha+c-a-b) > 0),$$

we obtain

$$\left( I_0^{\alpha,\beta,\omega,t_{p+1},m,n} \left[ I^{\sigma} \mid (a_i, \alpha_i)_{1,p} \right. \right.$$

$$(b_j, \beta_j)_{1,q} \left. \right] \right)(x)$$

$$= \frac{x^{\omega-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}t^\omega 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) H_{p,q}^{m,n} \left[ I^{\sigma} \mid (a_i, \alpha_i)_{1,p} \right.$$

$$(b_j, \beta_j)_{1,q} \left. \right] \, dt$$

$$= \frac{x^{\omega-\beta}}{2\pi i \Gamma(\alpha)} \int_{\mathcal{L}} \{ a_i, \alpha_i \}_{1,p} \right] \, ds \int_0^x (x-t)^{\alpha-1}t^{\omega-\sigma s} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) dt$$

$$= \frac{x^{\omega-\beta}}{2\pi i} \int_{\mathcal{L}} \{ a_i, \alpha_i \}_{1,p} \right] \, ds \Gamma(1 + \omega - s\sigma)\Gamma(1 + \omega - \beta + \eta - s\sigma)$$

$$\Gamma(1 + \omega - \beta - s\sigma)\Gamma(1 + \omega + \alpha + \eta - s\sigma) \, x^{-s\sigma} ds. \quad (5.6)$$

We note that since $\mathcal{L} = \mathcal{L}_{\gamma \infty}$, $\Re(s) = \gamma$ and therefore the condition (5.2) ensures the existence of the Mellin-Barnes integral above. Hence in view of (1.2)

$$\left( I_0^{\alpha,\beta,\omega,t_{p+1},m,n} \left[ I^{\sigma} \mid (a_i, \alpha_i)_{1,p} \right. \right.$$

$$(b_j, \beta_j)_{1,q} \left. \right] \right)(x)$$

$$= x^{\omega-\beta} H_{p+1,q}^{m,n+2} \left[ I^{\sigma} \mid (-\omega, \sigma), (-\omega + \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \right.$$

$$(b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \left. \right]. \quad (5.7)$$

and in accordance with (1.1) we obtain (5.3) which completes the proof of Theorem 1.
Corollary 1.1. Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy
\[
\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + 1 > 0, \quad (5.8)
\]
\[
\sigma \gamma < \text{Re}(\omega) + 1. \quad (5.9)
\]
Then the Riemann-Liouville fractional integral $I_0^\alpha$ of the $H$-function (1.1) exists and the following relation holds:
\[
\left( I_{0+}^{\alpha_1} \omega H_{m,n+1}^{\eta, \beta, \eta_1} \left[ \begin{array}{c} \omega \\ \eta \end{array} \right] \right)(x) = x^{\omega + \alpha_1} H_{m,n+1}^{\eta, \beta, \eta_1} \left[ \begin{array}{c} \omega \\ \eta \end{array} \right] \left[ \begin{array}{c} (-\omega, \sigma), (a_i, \alpha_i), (b_j, \beta_j) \\ (-\omega - \alpha, \sigma) \end{array} \right]. \quad (5.10)
\]

Corollary 1.2. Let $\alpha, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy
\[
\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\eta)] + 1 > 0, \quad (5.11)
\]
\[
\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\eta)] + 1. \quad (5.12)
\]
Then the Erdélyi-Kober fractional integral $I_+^{\alpha, \eta}$ of the $H$-function (1.1) exists and the following relation holds:
\[
\left( I_{+}^{\alpha_1, \eta_1} \omega H_{m,n+1}^{\eta, \beta, \eta_1} \left[ \begin{array}{c} \omega \\ \eta \end{array} \right] \right)(x) = x^{\omega + \alpha_1} H_{m,n+1}^{\eta, \beta, \eta_1} \left[ \begin{array}{c} \omega \\ \eta \end{array} \right] \left[ \begin{array}{c} (-\omega - \eta, \sigma), (a_i, \alpha_i), (b_j, \beta_j) \\ (-\omega - \alpha - \eta, \sigma) \end{array} \right]. \quad (5.13)
\]

Remark 2. In the case $\alpha^* > 0, \Delta \geq 0$ the relation (5.3) was indicated in [16, (4.2)], but in the assumptions of the result the condition (5.2) of Theorem 1 should be added.

Remark 3. Corollary 1.1 coincides with Theorem 1 in [7]. For real $\alpha > 0$ and $\alpha^* > 0$ the relation (5.10) was indicated in [11, 25.2.2], but the conditions of its validity have to be also corrected according to (5.8) and (5.9).

6. Right-Sided Generalized Fractional Integration of the $H$-Function

In this section we consider the right-sided generalized fractional integration $I_{+}^{\alpha, \beta, \eta}$ defined by (2.9).
Theorem 2. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0, \text{Re}(\beta) \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy
\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \min[\text{Re}(\beta), \text{Re}(\eta)], \quad (6.1)
\]
\[
\sigma \gamma > \text{Re}(\omega) - \min[\text{Re}(\beta), \text{Re}(\eta)]. \quad (6.2)
\]
Then the generalized fractional integral $I_{-}^{\alpha, \beta, \eta} H_{p,q}^{m,n}$ of the $H$-function (1.1) exists and the following relation holds:
\[
I_{-}^{\alpha, \beta, \eta} H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] (x) = x^{\omega-\beta} H_{p,q}^{m,n+2} \left[ x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\ (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right. \right]. \quad (6.3)
\]

Proof. By (2.9) we have
\[
I_{-}^{\alpha, \beta, \eta} H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] (x) = \frac{1}{\Gamma(\alpha)} \int_{1/x}^\infty (t-x)^{\alpha-1} t^{\omega-\alpha-\beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] dt. \quad (6.4)
\]
Due to (2.25), (2.26), (3.19) and (3.20), the integrand in (6.4) for any $x > 0$ has the asymptotic at infinity
\[
(t-x)^{\alpha-1} t^{\omega-\alpha-\beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] = O \left( t^{\omega-\min[\text{Re}(\beta), \text{Re}(\eta)]-1} \sigma^\theta \right) \quad (t \to + \infty)
\]
or
\[
= O \left( t^{\omega-\min[\text{Re}(\beta), \text{Re}(\eta)]-1} \sigma^\theta \log(t)^N \right) \quad (t \to + \infty).
\]
Here $\theta$ is given by (3.21) and $N$ is indicated in Theorem B(ii). Therefore the condition (6.1) ensures the existence of the integral (6.4). Applying (1.2), making the change $t = 1/\tau$ and using (5.5), we obtain
\[
I_{-}^{\alpha, \beta, \eta} H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \left( \frac{1}{x} \right) = \frac{1}{\Gamma(\alpha)} \int_{1/x}^\infty (t-\frac{1}{x})^{\alpha-1} t^{\omega-\alpha-\beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{1}{tx} \right) H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] dt.
\]
\[
\frac{x^{1-\alpha}}{2\pi i\Gamma(\alpha)} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] x^{\sigma s} ds \\
\cdot \int_{0}^{x} (x - \tau)^{\alpha-1} \tau^{\omega - 1 + \beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x} \right) d\tau \\
= \frac{x^{-\omega + \beta}}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \Gamma(-\omega + \beta + \sigma s) \Gamma(-\omega + \eta + \sigma s) \Gamma(-\omega + \alpha + \beta + \eta + \sigma s) x^{\sigma s} ds.
\]
(6.5)

Since \( \mathcal{L} = \xi_{\gamma, \infty} \), \( \text{Re}(s) = \gamma \) and therefore the condition (6.2) guarantees the existence of the Mellin-Barnes integral above. Replacing in (6.5) \( x \) by \( 1/x \), we obtain (6.3).

**Corollary 2.1.** Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \( (i = 1, \cdots, p; j = 1, \cdots, q) \) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) + \text{Re}(\alpha) < 0,
\]
(6.6)

\[
\sigma \gamma > \text{Re}(\omega) + \text{Re}(\alpha).
\]
(6.7)

Then the Riemann-Liouville fractional integral \( I_\omega^\alpha \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( I_\omega^\alpha \right)^\omega H_{p,q}^{m,n} \left[ \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] (x) = x^{\omega + \alpha} H_{p+1,q+1}^{m+1,n} \left[ \begin{array}{c}
x^\sigma \\
(a_i, \alpha_i)_{1,p}, (-\omega, \sigma) \\
(-\omega - \alpha, \sigma), (b_j, \beta_j)_{1,q}
\end{array} \right].
\]
(6.8)

**Corollary 2.2.** Let \( \alpha, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \( (i = 1, \cdots, p; j = 1, \cdots, q) \) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \text{Re}(\eta),
\]
(6.9)

\[
\sigma \gamma > \text{Re}(\omega) - \text{Re}(\eta).
\]
(6.10)

Then the Erdélyi-Kober fractional integral \( K_{\eta, \alpha}^\omega \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( K_{\eta, \alpha}^\omega \right)^\omega H_{p,q}^{m,n} \left[ \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] (x) = x^{\omega + \eta + \alpha} H_{p+1,q+1}^{m+1,n+1} \left[ \begin{array}{c}
x^\sigma \\
(a_i, \alpha_i)_{1,p}, (-\omega + \eta + \alpha, \sigma) \\
(-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q}
\end{array} \right].
\]
(6.11)

**Remark 4.** In the case \( \alpha^* > 0, \Delta \geq 0 \) the relation of the form (6.3) was indicated in [16, (4.3)]. But it includes a mistake and should be replaced by (6.3) with the conditions (6.1) and (6.2).
Remark 5. Corollary 2.1 coincides with Theorem 2 in [7]. For real $\alpha > 0$ and $\alpha^* > 0$ the relation (6.8) was indicated in [18, (2.5)], but the conditions of its validity have to be also corrected in accordance with (6.6) and (6.7).

7. Left-Sided Generalized Fractional Differentiation of the $H$-Function

Now we treat the left-sided generalized fractional derivative $D^\alpha_{0+} \omega_1$ given by (2.11).

Theorem 3. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0, \text{Re}(\alpha + \beta + \eta) \neq 0$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \ldots, p; j = 1, \ldots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$
\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) \geq \text{Re}(\alpha + \beta + \eta)] + 1 > 0, \quad (7.1)
$$

$$
\sigma \gamma < \text{Re}(\omega) + \text{Re}(\alpha + \beta + \eta)] + 1. \quad (7.2)
$$

Then the generalized fractional derivative $D^\alpha_{0+} \omega_1$ of the $H$-function (1.1) exists and the following relation holds:

$$
\left( D^\alpha_{0+} \omega_1 \right)^{m,n,p,q} \left[ t^{\sigma} \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \right) (x)
$$

$$
= x^{\omega + \beta} \left( D^\alpha_{0+} \omega_1 \right)^{m,n,p+2,q+2} \left[ t^{\sigma} \left| \begin{array}{c}
(-\omega, \sigma), (-\omega - \eta - \alpha, \beta, \gamma), (a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}, (-\omega - \beta, \sigma), (-\omega - \eta, \sigma)
\end{array} \right] \right) . \quad (7.3)
$$

Proof. Let $n \in [\text{Re}(\alpha)] + 1$. From (2.11) we have

$$
\left( D^\alpha_{0+} \omega_1 \right)^{m,n,p,q} \left[ t^{\sigma} \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \right) (x)
$$

$$
= \left( \frac{d}{dx} \right)^n \left( D^{\alpha + \beta, \eta} \omega \right)^{m,n,p+2,q+2} \left[ t^{\sigma} \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \right) (x), \quad (7.4)
$$

which exists according to Theorem 1 with $\alpha, \beta$ and $\eta$ being replaced by $-\alpha + n, -\beta - n$ and $\alpha + \eta - n$, respectively. Then we find

$$
\left( D^\alpha_{0+} \omega_1 \right)^{m,n,p,q} \left[ t^{\sigma} \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \right) (x)
$$

$$
= \left( \frac{d}{dx} \right)^n x^{\omega + \beta + n} \left( D^{\alpha + \beta, \eta} \omega \right)^{m,n+2,p+2,q+2} \left[ t^{\sigma} \left| \begin{array}{c}
(-\omega, \sigma), (-\omega - \alpha - \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}, (-\omega - \beta - n, \sigma), (-\omega - \eta, \sigma)
\end{array} \right] \right) . \quad (7.5)
$$
Taking into account the differentiation formula (4.4) we have

\[
\begin{align*}
\left( D_{0+}^{\alpha, \beta, \eta, \omega} I_{p,q}^{m,n} \left[ t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right] \right)(x) &= x^{\omega + \beta} H_{pq}^{m,n} + \sum_{j=1}^{q} \left[ (-\omega - \beta - n, \sigma), (-\omega, \sigma), \cdots, \left( \omega, \sigma \right), (-\omega - \eta, \sigma), (-\omega - \sigma, \beta, \omega) \right] \left( a_i, \alpha_i \right)_{1,p} \\
&= x^{\omega + \beta} H_{pq}^{m,n+3} \left[ x^\sigma \begin{bmatrix} (-\omega - \beta - n, \sigma), (-\omega - \beta - \eta, \sigma), (-\omega - \sigma, \beta, \omega) \end{bmatrix} \right],
\end{align*}
\]

(7.6)

and Lemma 2 and the reduction relation (4.1) imply (7.3), which completes the proof of theorem.

**Corollary 3.1.** Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \cdots, p; j = 1, \cdots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy the conditions in (5.8) and (5.9). Then the Riemann-Liouville fractional derivative \( D_{0+}^{\alpha, \beta, \eta} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\begin{align*}
\left( D_{0+}^{\alpha, \beta, \eta, \omega} I_{p,q}^{m,n} \left[ t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right] \right)(x) &= x^{\omega - \alpha} H_{p+1,q+1}^{m,n+1} \left[ x^\sigma \begin{bmatrix} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right] \\
&= (-1)^{\text{Re}(\alpha)} x^{\omega + \beta} H_{p+2,q+2}^{m+2,n+2} \left[ x^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma) \\ (-\omega - \beta, \sigma), (-\omega + \alpha + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{bmatrix} \right].
\end{align*}
\]

(8.3)

**Remark 6.** For real \( \alpha > 0 \) and \( \alpha^* > 0 \) the relation (7.3) was given in [18, (2.7.13)], but the conditions of its validity have to be corrected in accordance with (7.1) and (7.2).

**Remark 7.** Corollary 3.1 coincides with Theorem 3 in [7].

8. **Right-Sided Generalized Fractional Differentiation of the \( H \)-Function**

Here we deal with the right-sided generalized fractional derivative \( D_{-}^{\alpha, \beta, \eta} \) given by (2.12).

**Theorem 4.** Let \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \text{Re}(\alpha + \beta + \eta) + [\text{Re}(\alpha)] + 1 \neq 0 \). Let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \cdots, p; j = 1, \cdots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) + \max[\text{Re}(\beta) + [\text{Re}(\alpha)] + 1, -\text{Re}(\alpha + \eta)] < 0, \quad (8.1)
\]

\[
\sigma \gamma > \text{Re}(\omega) + \max[\text{Re}(\beta) + [\text{Re}(\alpha)] + 1, -\text{Re}(\alpha + \eta)]. \quad (8.2)
\]

Then the generalized fractional derivative \( D_{-}^{\alpha, \beta, \eta} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\begin{align*}
\left( D_{-}^{\alpha, \beta, \eta, \omega} I_{p,q}^{m,n} \left[ t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right] \right)(x) &= (-1)^{\text{Re}(\alpha)} x^{\omega + \beta} H_{p+2,q+2}^{m+2,n+2} \left[ x^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma) \\ (-\omega - \beta, \sigma), (-\omega + \alpha + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{bmatrix} \right].
\end{align*}
\]

(8.3)
Proof. Let \( n = [\text{Re}(\alpha)] + 1 \). Owing to (2.12) we have

\[
\left( D_{-}^{\alpha, \beta, \eta} I_{p,q}^{m,n} \right) \left( t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right) (x) = \left( -\frac{d}{dx} \right)^n I_{-\alpha+n, -\beta-n, \alpha+\eta} I_{p,q}^{m,n} \left( t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right) (x),
\]

which exists according to Theorem 2 with \( \alpha, \beta \) and \( \eta \) being replaced by \(-\alpha + n, -\beta - n \) and \( \alpha + \eta \), respectively. Then applying the differentiation formula (4.5), similarly to (7.5), (7.6), we find in view of the reduction formula (1.2) that

\[
\left( D_{-}^{\alpha, \beta, \eta} I_{p,q}^{m,n} \right) \left( t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right) (x) = (-1)^n x^{\omega+\beta} I_{p+3,q+3}^{m+3,n} \left[ x^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma) \\ (-\omega - \beta - n, \sigma), (-\omega + \alpha + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{bmatrix} \right],
\]

which implies the formula (8.3).

Corollary 4.1. Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \cdots, p; j = 1, \cdots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \frac{\text{Re}(a_i) - 1}{\alpha_i} + \text{Re}(\omega) - \{\text{Re}(\alpha)\} + 1 < 0, \tag{8.5}
\]

\[
\sigma \gamma + \text{Re}(\omega) - \{\text{Re}(\alpha)\} + 1 > 0. \tag{8.6}
\]

Then the Riemann-Liouville fractional derivative \( D_{-}^{\alpha} \) of the II-function (1.1) exists and there holds the relation:

\[
\left( D_{-}^{\alpha, \beta} I_{p,q}^{m,n} \right) \left( t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right) (x) = (-1)^{\text{Re}(\alpha)+1} x^{-\alpha} I_{p+1,q+1}^{m+1,n} \left[ x^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma) \\ (-\omega + \alpha, \sigma), (b_j, \beta_j)_{1,q} \end{bmatrix} \right]. \tag{8.7}
\]

Remark 8. The relation of the form (8.7) with real \( \alpha > 0 \) and \( \alpha^* > 0 \) was proved in [13, formula (14a)] (see also [12], [14, (2.2)] and [18, (2.7.9)]). But such a formula contains mistakes and should be replaced by (8.7) with the condition (8.5) and (8.6).
Remark 9. When $\alpha = k \in \mathbb{N}$, the relations (7.7) and (8.7) coincide with (4.4) and (4.5), respectively.

9. Generalized Fractional Integro-Differentiation of the $H$-Function

Here we investigate the generalized fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{-}^{\alpha,\beta,\eta}$ given by (2.8) and (2.10). The following statements are proved similarly to Theorems 3 and 4 by using the relations (2.8) and (2.10), Theorems 1 and 2, and the properties of the $H$-function in Sections 3 and 4.

Theorem 5. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) \leq 0, \text{Re}(\beta) \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1 > 0,$$

$$\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1.$$  (9.1, 9.2)

Then the generalized fractional integro-differentiation $I_{0+}^{\alpha,\beta,\eta}$ of the $H$-function (1.1) exists and there holds the relation

$$I_{0+}^{\alpha,\beta,\eta} \omega \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c} \sigma \\ \left( a_i, \alpha_i \right)_{1,p} \\ \left( b_j, \beta_j \right)_{1,q} \end{array} \right] (x) = x^{\omega-\beta} \mathcal{H}_{p+2,q+2}^{m,n+2} \left[ \begin{array}{c} x^{\sigma} \\ \left( a_i, \alpha_i \right)_{1,p} \\ \left( b_j, \beta_j \right)_{1,q} \end{array} \right] \left( -\omega, \sigma, -\omega - \eta + \beta, \sigma, (a_i, \alpha_i)_{1,p}, (-\omega + \beta, \sigma), (\omega - \alpha - \eta, \sigma) \right).$$  (9.3)

Theorem 6. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) \leq 0, \text{Re}(\beta) + |\text{Re}(\alpha)| - 1 \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \min[\text{Re}(\beta) - |\text{Re}(\alpha)| - 1, \text{Re}(\eta)],$$

$$\sigma \gamma > \text{Re}(\omega) - \min[\text{Re}(\beta) - |\text{Re}(\alpha)| - 1, \text{Re}(\eta)].$$  (9.4, 9.5)

Then the generalized fractional integro-differentiation $I_{-}^{\alpha,\beta,\eta}$ of the $H$-function (1.1) exists and there holds the relation

$$I_{-}^{\alpha,\beta,\eta} \omega \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c} \sigma \\ \left( a_i, \alpha_i \right)_{1,p} \\ \left( b_j, \beta_j \right)_{1,q} \end{array} \right] (x) = x^{\omega-\beta} \mathcal{H}_{p+2,q+2}^{m,n+2} \left[ \begin{array}{c} x^{\sigma} \\ \left( a_i, \alpha_i \right)_{1,p} \end{array} \right] \left( -\omega + \alpha + \beta + \eta, \sigma, (\omega - \alpha, \beta, \sigma), (\omega + \eta, \sigma) \right).$$  (9.6)
Remark 10. The relation (9.3) with $a^* > 0, \Delta \geq 0$ was indicated in [16, (4.2)], but conditions of its validity have to be corrected in accordance with (9.1) and (9.2).

Remark 11. The relations (9.3) and (9.6) for the fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{\alpha}^{\alpha,\beta,\eta}$, defined in (2.8) and (2.10) for $\alpha \in \mathbb{C}, \Re(\alpha) \leq 0$ coincide with that (5.3) and (6.3) for the fractional integration operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{\alpha}^{\alpha,\beta,\eta}$, defined in (2.7) and (2.9) for $\alpha \in \mathbb{C}, \Re(\alpha) > 0$. Though the conditions for validity of (5.3) and (9.3) in Theorems 1 and 5 have the same form, that of (6.3) and (9.6) presented in Theorems 2 and 4 are slightly different.

In conclusion we note that, as it was mentioned in Remarks 2, 4 and 10, the relations (5.3), (6.3) and (9.3) for generalized calculus operator $I_{0+}^{\alpha,\beta,\eta}$ were already known in the case $a^* > 0, \Delta \geq 0$. Further, Remarks 3, 5, 6 and 8 indicate that the relations (5.10) and (6.8) for the Riemann-Liouville fractional integrals $I_{0+}^{C}, I_{\alpha}^{C}$ and (7.3) and (8.7) for the fractional derivative $D_{0+}^\alpha$, in the case real $\alpha > 0$ and $a^* > 0$ were established. However, the $H$-function’s asymptotic estimates (3.16), (3.17) at zero and (3.19), (3.20) at infinity allow us to prove such results under more general assumptions $a^* > 0$ and $a^* = 0, \Delta \gamma + \Re(\mu) < -1$.

References


