<table>
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<th>Generalized Fractional Calculus of the $H$-Function (Applications of Complex Function Theory to Differential Equations)</th>
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<tr>
<td>Author(s)</td>
<td>Saigo, Megumi; Kilbas, Anatoly A.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1998), 1062: 89-107</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1998-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62401">http://hdl.handle.net/2433/62401</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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京都大学
Generalized Fractional Calculus of the \( H \)-Function

Megumi Saigo*  [西郷 恵]  (福岡大学 理学部)
Anatoly A. Kilbas†  (ベラルーシ国立大学・ベラルーシ)

Abstract

The paper is devoted to study the generalized fractional calculus of arbitrary complex order for the \( H \)-function defined by the Mellin-Barnes integral

\[
H_{p,q}^{m,n}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds,
\]

where the function \( \mathcal{H}_{p,q}^{m,n}(s) \) is a certain ratio of products of Gamma functions with the argument \( s \) and the contour \( \mathcal{L} \) is specially chosen. The considered generalized fractional integration and differentiation operators contain the Gauss hypergeometric function as a kernel and generalize classical fractional integrals and derivatives of Riemann-Liouville, Erdélyi-Kober type, etc. It is proved that the generalized fractional integrals and derivatives of \( H \)-functions are also \( H \)-functions but of greater order. In particular, the obtained results define more precisely and generalize known results.

1. Introduction

This paper deals with the \( H \)-function \( H_{p,q}^{m,n}(z) \). For integers \( m, n, p, q \) such that \( 0 \leq m \leq q, 0 \leq n \leq p, \) for \( a_i, b_j \in \mathbb{C} \) with \( \mathbb{C} \) the field of complex numbers and for \( \alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty) \) \( (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q) \) the \( H \)-function \( H_{p,q}^{m,n}(z) \) is defined via a Mellin-Barnes type integral in the following way:

\[
H_{p,q}^{m,n}(z) \equiv H_{p,q}^{m,n} \left[ \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \equiv H_{p,q}^{m,n} \left[ \begin{array}{c}
(a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\
(b_1, \beta_1), \ldots, (b_q, \beta_q)
\end{array} \right] \\
= \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] (s) z^{-s} ds,
\]

(1.1)

* Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan
† Department of Mathematics and Mechanics, Belarusian State University, Minsk 220050, Belarus
where the contour $\mathcal{L}$ is specially chosen and

$$
\mathcal{H}_{p,q}^{m,n}(s) \equiv \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)},
$$

in which an empty product, if it occurs, is taken to be one. Such a function was introduced by S. Pincherle in 1888 and its theory has been developed by Mellin [10], Dixon and Ferrar [2] (see [3, §1.19] in this connection). An interest to the $H$-function arose again in 1961 when Fox [4] has investigated such a function as a symmetrical Fourier kernel. Therefore, this function is sometimes called as Fox's $H$-function. The theory of this function may be found in [1], [9, Chapter 1], [17, Chapter 2] and [11, 8.8.3].

Classical Riemann-Liouville fractional calculus of real order [17, §2.2] (see (2.1) - (2.6) below) was investigated in [12] - [14], [18] and [11]. The right-sided fractional integrals and derivatives of the $H$-function (1.1) were studied in [12] - [14] and the results were presented in [18, §2.7], where the case of left-sided fractional differentiation of the $H$-function was also considered. The left-sided fractional integration of the $H$-function was given in [11, 2.25.2]. Such results for the generalized fractional calculus operators with the Gauss hypergeometric function as a kernel (see (2.7) - (2.10) below), being introduced by the first author [15], were obtained in [16].

However, some of the results obtained in [12] - [14] (cited in [18]) and [16] can be taken to be more precisely. Moreover, these results were given provided that the parameters $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0, \beta_j > 0$ ($i = 1, 2, \ldots; p$, $j = 1, 2, \ldots, q$) of the $H$-function satisfy certain conditions. These conditions were based on asymptotic behavior of $H_{p,q}^{m,n}(z)$ at zero and infinity. In [5] we extended such the known asymptotic results for the $H$-function to more wide class of parameters.

In [7], [8] we have applied the obtained asymptotic estimates in [5] to find the Riemann-Liouville fractional integrals and derivatives of any complex order of the $H$-function. In particular, we could make more precisely the known results from [12] - [14], [18] and [11].

The present paper is devoted to obtain such type results for the generalized fractional integration and differentiation operators of any complex order with the Gauss hypergeometric function as a kernel. In particular, we give more precisely some of the results from [16] and generalize the results obtained in [7], [8]. The paper is organized as follow. In Section 2 we present classical and generalized fractional calculus operators and some facts from the theory of Gauss hypergeometric function. Sections 3 and 4 contain the result from the theory of the $H$-function. The existence of $H_{p,q}^{m,n}(z)$ and its asymptotic behavior at zero and infinity is considered in Section 3 and certain reduction and differentiation properties in Section 4. Sections 5 and 6 deal with generalized fractional differentiation of the $H$-function (1.1). Sections 7 and 8 are devoted to the generalized fractional differentiation of the $H$-function. Another type of fractional integro-differentiation of the $H$-function is given in Section 9.
2. Classical and Generalized Fractional Calculus Operators

For $\alpha \in \mathbb{C}, \Re(\alpha) > 0$, the Riemann-Liouville left- and right-sided fractional calculus operators are defined as follow [17, §2.3 and §2.4]:

\[
(I_{0+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (x > 0),
\]

\[
(I_{-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)dt}{(t-x)^{1-\alpha}} \quad (x > 0),
\]

and

\[
(D_{0+}^\alpha f)(x) = \left(\frac{d}{dx}\right)^{\lfloor\Re(\alpha)\rfloor+1} (I_{0+}^{1-\alpha+[\Re(\alpha)]} f)(x)
\]

\[
= \left(\frac{d}{dx}\right)^{\lfloor\Re(\alpha)\rfloor+1} \frac{1}{\Gamma(1-\alpha+|\Re(\alpha)|)} \int_0^x \frac{f(t)dt}{(x-t)^{\alpha-|\Re(\alpha)|}} \quad (x > 0),
\]

\[
(D_{-}^\alpha f)(x) = \left(-\frac{d}{dx}\right)^{\lfloor\Re(\alpha)\rfloor+1} (I_{-}^{1-\alpha+[\Re(\alpha)]} f)(x)
\]

\[
= \left(-\frac{d}{dx}\right)^{\lfloor\Re(\alpha)\rfloor+1} \frac{1}{\Gamma(1-\alpha+|\Re(\alpha)|)} \int_x^\infty \frac{f(t)dt}{(t-x)^{\alpha-|\Re(\alpha)|}} \quad (x > 0),
\]

respectively, where the symbol $[\kappa]$ means the integral part of a real number $\kappa$, i.e. the largest integer not exceeding $\kappa$. In particular, for real $\alpha > 0$, the operators $D_{0+}^\alpha$ and $D_{-}^\alpha$ take more simple forms

\[
(D_{0+}^\alpha f)(x) = \left(\frac{d}{dx}\right)^{\{\alpha\}+1} (I_{0+}^{1-\{\alpha\}} f)(x)
\]

\[
= \left(\frac{d}{dx}\right)^{\{\alpha\}+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_0^x \frac{f(t)dt}{(x-t)^{\{\alpha\}}} \quad (x > 0),
\]

and

\[
(D_{-}^\alpha f)(x) = \left(-\frac{d}{dx}\right)^{\{\alpha\}+1} (I_{-}^{1-\{\alpha\}} f)(x)
\]

\[
= \left(-\frac{d}{dx}\right)^{\{\alpha\}+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_x^\infty \frac{f(t)dt}{(t-x)^{\{\alpha\}}} \quad (x > 0),
\]

respectively, where $\{\kappa\}$ stands for the fractional part of $\kappa$, i.e. $\{\kappa\} = \kappa - [\kappa]$.

For $\alpha, \beta, \eta \in \mathbb{C}$ and $x > 0$ the generalized fractional calculus operators are defined by [15]

\[
(I_{0+}^{\alpha, \beta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \binom{\alpha}{\alpha, -\beta; \alpha; 1 - \frac{t}{x}} f(t)dt
\]
\[ (I_{0+}^{0,\beta,\eta} f)(x) = \left( \frac{d}{dx} \right)^n (I_{0+}^{0+n,\beta-n,\eta-n} f)(x) \quad (\text{Re}(\alpha) > 0); \]

\[ (I_{0+}^{0,\beta,\eta} f)(x) = \frac{1}{\Gamma(n)} \int_x^{\infty} (t-x)^{\alpha-1} (t-x)^{-\beta} \quad _2F_1 \left( \alpha + \beta, -\eta; \alpha + \beta + \eta - 1; \frac{x}{t} \right) f(t) \, dt \quad (\text{Re}(\alpha) > 0); \]

\[ (I_{-}^{\alpha,\beta,\eta} f)(x) = \left( -\frac{d}{dx} \right)^n (I_{-}^{\alpha+n,\beta-n,\eta-n} f)(x) \quad (\text{Re}(\alpha) \leq 0; n = \lceil \text{Re}(\alpha) \rceil + 1); \]

\[ (D_{0+}^{\alpha,\beta,\eta} f)(x) \equiv (I_{0+}^{\alpha,-\beta,\alpha+\eta} f)(x) = \left( \frac{d}{dx} \right)^n (I_{0+}^{\alpha+n,\beta-n,\eta-n} f)(x) \quad (\text{Re}(\alpha) \leq 0; n = \lceil \text{Re}(\alpha) \rceil + 1); \]

and

\[ (D_{0+}^{\alpha,\beta,\eta} f)(x) \equiv (I_{0+}^{\alpha,-\beta,\alpha+\eta} f)(x) \]

\[ = \left( -\frac{d}{dx} \right)^n (I_{0+}^{\alpha+n,\beta-n,\eta-n} f)(x) \quad (\text{Re}(\alpha) > 0; n = \lceil \text{Re}(\alpha) \rceil + 1); \]

\[ (D_{-}^{\alpha,\beta,\eta} f)(x) \equiv (I_{-}^{\alpha,-\beta,\alpha+\eta} f)(x) \]

\[ = \left( -\frac{d}{dx} \right)^n (I_{-}^{\alpha+n,\beta-n,\eta-n} f)(x) \quad (\text{Re}(\alpha) > 0; n = \lceil \text{Re}(\alpha) \rceil + 1)). \]

Here \(_2F_1(a, b; c; z)\) \((a, b, c, z \in \mathbb{C})\) is the Gauss hypergeometric function defined by the series

\[ _2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \quad (\text{Re}(\alpha) > 0); \]

with

\[ (a)_0 = 1, \quad (a)_k = a(a+1) \cdots (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (k \in \mathbb{N}), \]

where \(\Gamma(z)\) is the Gamma function \([3, \text{Chapter I}]\) and \(\mathbb{N}\) denotes the set of positive integers.

The series in (2.13) is convergent for \(|z| < 1\) and for \(|z| = 1\) with \(\text{Re}(c - a - b) > 0\), and can be analytically continued into \(\{ z \in \mathbb{C} : |\arg(1 - z)| < \pi \} \) (see \([3, \text{Chapter II}]\)).

Since

\[ _2F_1(0, b; c; z) = 1 \]

for \(\beta = -\alpha\), the generalized fractional calculus operators (2.7), (2.9), (2.11) and (2.12) coincide with the Riemann-Liouville operators (2.1) - (2.4) for \(\text{Re}(\alpha) > 0\):

\[ (I_{0+}^{\alpha,-\alpha,\eta} f)(x) = (I_{0+}^{\alpha} f)(x), \quad (I_{0+}^{\alpha,-\alpha,\eta} f)(x) = (I_{0}^{\alpha} f)(x), \]

\[ (D_{0+}^{\alpha,-\alpha,\eta} f)(x) = (D_{0+}^{\alpha} f)(x), \quad (D_{0+}^{\alpha,-\alpha,\eta} f)(x) = (D_{0}^{\alpha} f)(x). \]
According to the relation [3, 2.8(4)]

\[ 2F_1(a, b; c; z) = (1 - z)^{-b}, \]  

(2.18)

when \( \beta = 0 \) the operators (2.7) and (2.9) coincide with the Erdélyi-Kober fractional integrals [17, §18.1]:

\[
\left(I_{0+}^{\alpha, \beta, \eta} f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} t^{-\eta} f(t) dt \equiv \left(I_{0+}^{\alpha, \beta, \eta} f\right)(x) \quad (\alpha, \eta \in \mathbb{C}, \text{Re}(\alpha) > 0),
\]

(2.19)

\[
\left(I_{-}^{\alpha, \beta, \eta} f\right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} f(t) dt \equiv \left(I_{-}^{\alpha, \beta, \eta} f\right)(x) \quad (\alpha, \eta \in \mathbb{C}, \text{Re}(\alpha) > 0).
\]

(2.20)

Therefore the operators (2.7), (2.9) and (2.11), (2.12) are called ”generalized” fractional integrals and derivatives, respectively. Moreover, the operators (2.11) and (2.12) are inverse to (2.7) and (2.9):

\[ D_{0+}^{\alpha, \beta, \eta} = \left(I_{0+}^{\alpha, \beta, \eta}\right)^{-1}, \quad D_{-}^{\alpha, \beta, \eta} = \left(I_{-}^{\alpha, \beta, \eta}\right)^{-1}. \]  

(2.21)

Fractional calculus operators (2.1), (2.3), (2.5), (2.7), (2.8), (2.11) and (2.2), (2.4), (2.6), (2.9), (2.10), (2.12) are called left-sided and right-sided, respectively [17, §2].

We give some other properties of \( 2F_1(a, b; c; z) \) [3, 2.8(46), 2.9(2), 2.10(14)] which will be used in the following calculations:

\[ 2F_1(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \quad (c \neq 0, -1, -2, \ldots; \text{Re}(c-a-b) > 0); \]  

(2.22)

\[ 2F_1(a, b; c; z) = (1-z)^{c-a-b} 2F_1(c-a, c-b; c; z); \]  

(2.23)

\[
2F_1(a, b; a+b; z) = \frac{\Gamma(a+b)}{\Gamma(a) \Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(k!)^2} [2\psi(1+k) - \psi(a+k) + \psi(b+k)] (-\log(1-z))(1-z)^k \quad (|\arg(z)| < \pi; a,b \neq 0,-1,-2,\ldots),
\]

(2.24)

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the Psi function [3, 1.7].

Formulas (2.22) - (2.24) mean the following asymptotic behavior of \( 2F_1(a, b; c; z) \) at the point \( z = 1 \).

**Lemma 1.** For \( a, b, c \in \mathbb{C} \) with \( \text{Re}(c) > 0 \) and \( z \in \mathbb{C} \), there hold the following asymptotic relations near \( z = 1 \):

\[ 2F_1(a, b; c; z) = O(1) \quad (z \to 1-) \]  

(2.25)

for \( \text{Re}(c-a-b) > 0 \);

\[ 2F_1(a, b; c; z) = O\left((1-z)^{c-a-b}\right) \quad (z \to 1-) \]  

(2.26)

for \( \text{Re}(c-a-b) < 0 \); and

\[ 2F_1(a, b; c; z) = O\left(\log(1-z)\right) \quad (z \to 1-) \]  

(2.27)
for $c - a - b = 0$, $a, b \neq 0, -1, -2, \ldots$ and $|\arg(z)| < \pi$.

3. Existence and Asymptotic Behavior of the $H$-Function

We shall consider the $H$-function (1.1) provided that the poles

$$b_{jl} = \frac{-b_j - l}{\beta_j} \quad (j = 1, \ldots, m; l \in \mathbb{N}_0)$$

(3.1)

of the Gamma functions $\Gamma(b_j + \beta_j s)$ and that

$$a_{ik} = \frac{1 - a_i + k}{\alpha_i} \quad (i = 1, \ldots, n; k \in \mathbb{N}_0)$$

(3.2)

of $\Gamma(1 - a_i - \alpha_i s)$ do not coincide:

$$\alpha_i(b_j + l) \neq \beta_j(a_i - k - 1) \quad (i = 1, \ldots, n; j = 1, \ldots, m; k, l \in \mathbb{N}_0),$$

(3.3)

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathcal{L}$ in (1.1) is the infinite contour splitting poles $b_{jl}$ in (3.1) to the left and poles $a_{ik}$ in (3.2) to the right of $\mathcal{L}$ and has one of the following forms:

(i) $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty + i\varphi_1$ and terminating at the point $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$;

(ii) $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at the point $+\infty + i\varphi_1$ and terminating at the point $+\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$.

(iii) $\mathcal{L} = \mathcal{L}_{\gamma i\infty}$ is a contour starting at the point $\gamma - i\infty$ and terminating at the point $\gamma + i\infty$ with $\gamma \in \mathbb{R} = (-\infty, +\infty)$.

The properties of the $H$-function $H_{\rho,q}^{m,n}(z)$ depend on the numbers $a^*, \Delta, \delta$ and $\mu$ which are expressed via $p, q, a_i, \alpha_i$ ($i = 1, 2, \ldots, p$) and $b_j, \beta_j$ ($j = 1, 2, \ldots, q$) by the following relations:

$$a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j,$$

(3.4)

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i,$$

(3.5)

$$\delta = \prod_{i=1}^{p} a_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j},$$

(3.6)

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p - q}{2}.$$  

(3.7)

Here an empty sum in (3.4), (3.5), (3.7) and an empty product in (3.6), if they occur, are taken to be zero and one, respectively.

The existence of the $H$-function is given by the following result [6].
Theorem A. Let $a^*$, $\Delta$, $\delta$ and $\mu$ be given by (3.4) - (3.7). Then the $H$-function $H_{p,q}^{m,n}(z)$ defined by (1.1) and (1.2) makes sense in the following cases:

\begin{align*}
\mathcal{L} &= \mathcal{L}_{-\infty}, \quad \Delta > 0, \quad z \neq 0; & \quad (3.8) \\
\mathcal{L} &= \mathcal{L}_{-\infty}, \quad \Delta = 0, \quad 0 < |z| < \delta; & \quad (3.9) \\
\mathcal{L} &= \mathcal{L}_{-\infty}, \quad \Delta = 0, \quad \text{Re}(\mu) < -1, \quad |z| = \delta; & \quad (3.10) \\
\mathcal{L} &= \mathcal{L}_{+\infty}, \quad \Delta < 0, \quad z \neq 0; & \quad (3.11) \\
\mathcal{L} &= \mathcal{L}_{+\infty}, \quad \Delta = 0, \quad |z| > \delta; & \quad (3.12) \\
\mathcal{L} &= \mathcal{L}_{\gamma\infty}, \quad a^* > 0, \quad |\arg z| < \frac{a^* \pi}{2}, \quad z \neq 0; & \quad (3.13) \\
\mathcal{L} &= \mathcal{L}_{\gamma\infty}, \quad a^* = 0, \quad \Delta \gamma + \text{Re}(\mu) < -1, \quad \arg z = 0, \quad z \neq 0. & \quad (3.14)
\end{align*}

Remark 1. The results of Theorem A in the cases (3.10), (3.13) and (3.15) are more precisely than those in [11, §8.3.1].

The next statement being followed from the results in [5] characterizes the asymptotic behavior of the $H$-function at zero and infinity.

Theorem B. Let $a^*$ and $\Delta$ be given by (3.1) and (3.5) and let conditions in (3.3) be satisfied.

(i) If $\Delta \geq 0$ or $\Delta < 0, a^* > 0$, then the $H$-function has either of the asymptotic estimates at zero

\[ H_{p,q}^{m,n}(z) = O\left(z^{\varrho^*}\right) \quad (|z| \to 0) \]  

or

\[ H_{p,q}^{m,n}(z) = O\left(z^{\varrho^*} |\log(z)|^{N^*}\right) \quad (|z| \to 0), \]  

with the additional condition $|\arg(z)| < a^* \pi/2$ when $\Delta < 0, a^* > 0$. Here

\[ \varrho^* = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right], \]  

and $N^*$ is the order of one of the point $b_j$ in (3.1) to which some other poles of $\Gamma(b_j + \beta_j s)$ ($j = 1, \ldots, m$) coincide.

(ii) If $\Delta \leq 0$ or $\Delta > 0, a^* > 0$, then the $H$-function has either of the asymptotic estimates at infinity

\[ H_{p,q}^{m,n}(z) = O\left(z^{\varrho}\right) \quad (|z| \to \infty) \]  

\[ \varrho = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right], \]  

and $N^*$ is the order of one of the point $b_j$ in (3.1) to which some other poles of $\Gamma(b_j + \beta_j s)$ ($j = 1, \ldots, m$) coincide.
or

\[ H_{p,q}^{m,n}(z) = O \left( z^{\epsilon} |\log(z)|^{N} \right) \quad (|z| \to \infty), \tag{3.20} \]

with the additional condition \(|\arg(z)| < a^{*}\pi/2\) when \(\Delta > 0, a^{*} > 0\). Here

\[ \epsilon = \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_{i}) - 1}{\alpha_{i}} \right], \tag{3.21} \]

and \(N\) is the order of one of the point \(a_{ik}\) in (3.2) in which some other poles of \(\Gamma(1 - a_{i} - \alpha_{i}s)\) \((i = 1, \cdots, n)\) coincide.

### 4. Reduction and Differentiation Properties of the \(H\)-Function

In this and next sections we suppose that the conditions for the existence of the \(H\)-function given in Theorem A are satisfied.

The following two Lemmas which characterize symmetric and reduction properties of the \(H\)-function follow from the definition of the \(H\)-function in (1.1) - (1.2).

**Lemma 2.** The \(H\)-function (1.1) is commutative in the set of pairs \((a_{1}, \alpha_{1}), \cdots, (a_{n}, \alpha_{n});\)
\(\text{in } (a_{n+1}, \alpha_{n+1}), \cdots, (a_{p}, \alpha_{p});\)
\(\text{in } (b_{1}, \beta_{1}), \cdots, (b_{m}, \beta_{m})\) and in \((b_{m+1}, \beta_{m+1}), \cdots, (b_{q}, \beta_{q}).\)

**Lemma 3.** If one of \((a_{i}, \alpha_{i})\) \((i = 1, \cdots, n)\) is equal to one of \((b_{j}, \beta_{j})\) \((j = m + 1, \cdots, q)\)
(or one of \((a_{i}, \alpha_{i})\) \((i = n + 1, \cdots, p)\) is equal to one of \((b_{j}, \beta_{j})\) \((j = 1, \cdots, m))\),
then the \(H\)-function reduces to the lower order one, that is, \(p, q \text{ and } n \text{ or } m\) decrease by unity. Two such results have the forms

\[ H_{p,q}^{m,n} \left[ z \begin{array}{c}(a_{i}, \alpha_{i})_{1,p} \\
(b_{j}, \beta_{j})_{1,q-1}, (a_{1}, \alpha_{1}) \end{array} \right] = H_{p-1,q-1}^{m-1,n-1} \left[ z \begin{array}{c}(a_{i}, \alpha_{i})_{2,p} \\
(b_{j}, \beta_{j})_{1,q-1} \end{array} \right] \tag{4.1} \]

provided that \(n \geq 1\) and \(q > m, \text{ and} \)

\[ H_{p,q}^{m,n} \left[ z \begin{array}{c}(a_{i}, \alpha_{i})_{1,p-1}, (b_{1}, \beta_{1}) \end{array} \right] = H_{p-1,q-1}^{m-1,n} \left[ z \begin{array}{c}(a_{i}, \alpha_{i})_{1,p-1} \\
(b_{j}, \beta_{j})_{2,q} \end{array} \right] \tag{4.2} \]

provided that \(m \geq 1\) and \(p > n.\)

The next differentiation formulae follow from the definition of the \(H\)-function given in (1.1) - (1.2) and from the functional equation for the Gamma function [3, §1.2(6)]

\[ \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}. \tag{4.3} \]
Lemma 4. There hold the following differentiation formulae for $\omega, \epsilon \in \mathbb{C}, \sigma > 0$

\[
\left( \frac{d}{dz} \right)^{k} \left( z^{\omega} H_{p,q}^{m,n} \left[ \begin{array}{c} \epsilon \\ z \end{array} \middle| \begin{array}{c} (a_{i,\alpha_{i}})_{1_{p}} \\ (b_{j,\beta_{j}})_{1_{q}} \end{array} \right] \right) = z^{\omega-k} H_{p+1,q+1}^{m+1,n+1} \left[ \begin{array}{c} -\omega, \sigma \\ (a_{i,\alpha_{i}})_{1_{p}} \\ (b_{j,\beta_{j}})_{1_{q}} \end{array} \right].
\]

(4.4)

\[
\left( \frac{d}{dz} \right)^{k} \left( z^{\omega} H_{p,q}^{m,n} \left[ \begin{array}{c} \epsilon \\ z \end{array} \middle| \begin{array}{c} (a_{i,\alpha_{i}})_{1_{p}} \\ (b_{j,\beta_{j}})_{1_{q}} \end{array} \right] \right) = (-1)^{k} z^{\omega-k} H_{p+1,q+1}^{m+1,n+1} \left[ \begin{array}{c} (a_{i,\alpha_{i}})_{1_{p}} \\ (b_{j,\beta_{j}})_{1_{q}} \end{array} \middle| \begin{array}{c} -\omega, \sigma \\ (k-\omega, \sigma) \end{array} \right].
\]

(4.5)

5. Left-Sided Generalized Fractional Integration of the $H$-Function

In the following sections we treat the $H$-function (1.1) - (1.2) with $\mathcal{L} = \mathcal{L}_{\tau_{\omega}}$ and under the assumptions $\alpha^{*} > 0$, $\Delta \gamma + \text{Re}(\mu) < -1$ for $\alpha^{*}, \Delta, \mu$ being given by (3.4), (3.5), (3.7).

Here we consider the left-sided generalized fractional integration $I_{0}^{\alpha,\beta,\eta}$ defined by (2.7).

**Theorem 1.** Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0, \text{Re}(\beta) \neq \text{Re}(\eta)$. Let the constants $a_{i}, b_{j} \in \mathbb{C}, \alpha_{i}, \beta_{j} > 0 (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

\[
\sigma \min_{1 \leq i \leq m} \frac{\text{Re}(b_{j})}{\beta_{j}} + \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1 > 0,
\]

(5.1)

\[
\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1.
\]

(5.2)

Then the generalized fractional integral $I_{0}^{\alpha,\beta,\eta}$ of the $H$-function (1.1) exists and the following relation holds:

\[
\left( I_{0}^{\alpha,\beta,\eta} \right)^{\omega} H_{p,q}^{m,n} \left[ \begin{array}{c} \epsilon \\ z \end{array} \middle| \begin{array}{c} (a_{i,\alpha_{i}})_{1_{p}} \\ (b_{j,\beta_{j}})_{1_{q}} \end{array} \right](x) = x^{\omega-\beta} H_{p+2q+2}^{m+2,n+2} \left[ \begin{array}{c} (-\omega, \sigma), (-\omega+\beta-\eta, \sigma), (a_{i,\alpha_{i}})_{1_{p}} \\ (b_{j,\beta_{j}})_{1_{q}}, (-\omega+\beta, \sigma), (-\omega-\alpha-\eta, \sigma) \end{array} \right].
\]

(5.3)

**Proof.** By (2.7) we have

\[
\left( I_{0}^{\alpha,\beta,\eta} \right)^{\omega} H_{p,q}^{m,n} \left[ \begin{array}{c} \epsilon \\ z \end{array} \middle| \begin{array}{c} (a_{i,\alpha_{i}})_{1_{p}} \\ (b_{j,\beta_{j}})_{1_{q}} \end{array} \right](x) = x^{\omega-\beta} \int_{0}^{x} (x-t)^{\alpha-1} t^{\omega} \left( H_{p+2q+2}^{m+2,n+2} \left[ \begin{array}{c} (a_{i,\alpha_{i}})_{1_{p}} \\ (b_{j,\beta_{j}})_{1_{q}} \end{array} \right] \right) dt.
\]

(5.4)
According to (2.25), (2.26), (3.16) and (3.17), the integrand in (5.4) for any $x > 0$ has the asymptotic estimate at zero
\[
(x - t)^{\alpha-1}t^\omega 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) H_{p,q}^{m,n} \left[ l^\sigma \begin{pmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{pmatrix} \right]
= O \left( t^{\omega^* + \min\{0, \Re(\eta - \beta)\}} \right) \quad (t \to +0)
\]
or
\[
= O \left( t^{\omega^* + \min\{0, \Re(\eta - \beta)\}} \right) \quad (t \to +0).
\]
Here $\omega^*$ is given by (3.18) and $N^*$ is indicated in Theorem B(i). Therefore the condition (5.1) ensures the existence of the integral (5.4).

Applying (1.2), making the change of variable $t = x\tau$, changing the order of integration and taking into account the formula [11, §2.21.11]
\[
\int_0^x (x - t)^{\alpha-1}t^\omega 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) dt = \frac{\Gamma(c)\Gamma(\alpha)\Gamma(\alpha + c - a - b)}{\Gamma(\alpha + c - a)\Gamma(\alpha + c - b)} x^{\alpha+c-1}
\]
\[
(a, b, c, \alpha \in \mathbb{C}, \Re(\alpha) > 0, \Re(c) > 0, \Re(\alpha + c - a - b) > 0),
\]
we obtain
\[
\left( I_{0+}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n} \left[ l^\sigma \begin{pmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{pmatrix} \right] \right)(x)
= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1}t^\omega 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) H_{p,q}^{m,n} \left[ l^\sigma \begin{pmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{pmatrix} \right] dt
\]
\[
= \frac{x^{-\alpha-\beta}}{2\pi i \Gamma(\alpha)} \int_L \mathcal{J}(x) \left[ \begin{pmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{pmatrix} s \right] ds \int_0^x (x - t)^{\alpha-1}t^{\omega-\sigma s} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) dt
\]
\[
= \frac{x^{\omega-\beta}}{2\pi i} \int_L \mathcal{J}(x) \left[ \begin{pmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{pmatrix} s \right] \frac{\Gamma(s + \omega - \sigma \alpha)\Gamma(s + \omega - \beta + \eta - \sigma s)}{\Gamma(1 + \omega + \beta - \sigma s)\Gamma(1 + \omega - \beta - \sigma s)} x^{-\sigma s} ds.
\]
We note that since $\mathcal{L} = \mathcal{L}_{\gamma = \infty}$, $\Re(s) = \gamma$ and therefore the condition (5.2) ensures the existence of the Mellin-Barnes integral above. Hence in view of (1.2)
\[
\left( I_{0+}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n} \left[ l^\sigma \begin{pmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{pmatrix} \right] \right)(x)
= \frac{x^{\omega-\beta}}{H_{p,q}^{m,n+2}} \left[ x^\sigma \begin{pmatrix} (-\omega, \sigma), (-\omega + \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \end{pmatrix} \right].
\]
and in accordance with (1.1) we obtain (5.3) which completes the proof of Theorem 1.
Corollary 1.1. Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \ldots, p; j = 1, \ldots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy
\[
\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + 1 > 0, \tag{5.8}
\]
\[
\sigma \gamma < \text{Re}(\omega) + 1. \tag{5.9}
\]
Then the Riemann-Liouville fractional integral $I_{0+}^\alpha$ of the $H$-function (1.1) exists and the following relation holds:
\[
\left( \int_{0+}^\alpha t^\omega H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c}(a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right] \right)(x) = 2^{\omega+\alpha} I_{p+1,q+1}^\alpha \left[ t^\sigma \left| \begin{array}{c}(-\omega, \sigma), (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q}, (-\omega - \alpha, \sigma) \end{array} \right] \right]. \tag{5.10}
\]

Corollary 1.2. Let $\alpha, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \ldots, p; j = 1, \ldots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy
\[
\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\eta)] + 1 > 0, \tag{5.11}
\]
\[
\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\eta)] + 1. \tag{5.12}
\]
Then the Erdélyi-Kober fractional integral $I_{\eta,\alpha}^+$ of the $H$-function (1.1) exists and the following relation holds:
\[
\left( \int_{\eta,\alpha}^+ t^\omega H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c}(a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{array} \right] \right)(x) = 2^{\omega+\alpha} I_{p+1,q+1}^\alpha \left[ t^\sigma \left| \begin{array}{c}(-\omega - \eta, \sigma), (a_i, \alpha_i)_{1, p} \\ (b_j, \beta_j)_{1, q}, (-\omega - \alpha - \eta, \sigma) \end{array} \right] \right]. \tag{5.13}
\]

Remark 2. In the case $\alpha^* > 0, \Delta \geq 0$ the relation (5.3) was indicated in [16, (4.2)], but in the assumptions of the result the condition (5.2) of Theorem 1 should be added.

Remark 3. Corollary 1.1 coincides with Theorem 1 in [7]. For real $\alpha > 0$ and $\alpha^* > 0$ the relation (5.10) was indicated in [11, 25.2.2], but the conditions of its validity have to be also corrected according to (5.8) and (5.9).

6. Right-Sided Generalized Fractional Integration of the $H$-Function

In this section we consider the right-sided generalized fractional integration $I_{-}^{\alpha,\beta,\eta}$ defined by (2.9).
Theorem 2. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0, \text{Re}(\beta) \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \ldots, p; j = 1, \ldots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \min[\text{Re}(\beta), \text{Re}(\eta)],
$$

(6.1)

and

$$
\sigma \gamma > \text{Re}(\omega) - \min[\text{Re}(\beta), \text{Re}(\eta)].
$$

(6.2)

Then the generalized fractional integral $I^{\alpha, \beta, \eta}_{\omega} t^m$ of the II-function (1.1) exists and the following relation holds:

$$
\left( I^{\alpha, \beta, \eta}_{\omega} \Gamma_{p,q}^{m,n} \left[ \Gamma^\sigma \left( (a_i, \alpha_i)_{1,p} \right) \right] \right) (x)
$$

$$
= x^{\omega-\beta} \Gamma_{p,q}^{m+2,n} \left[ \Gamma^\sigma \left( (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,p} \right) \right].
$$

(6.3)

Proof. By (2.9) we have

$$
\left( I^{\alpha, \beta, \eta}_{\omega} \Gamma_{p,q}^{m,n} \left[ \Gamma^\sigma \left( (a_i, \alpha_i)_{1,p} \right) \right] \right) (x)
$$

$$
= \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} t^{\omega-\alpha-\beta} \Gamma_{p,q}^{m,n} \left[ \Gamma^\sigma \left( (a_i, \alpha_i)_{1,p} \right) \right] \left( \frac{1}{t} \right) dt.
$$

(6.4)

Due to (2.25), (2.26), (3.19) and (3.20), the integrand in (6.4) for any $x > 0$ has the asymptotic at infinity

$$(t-x)^{\alpha-1} t^{\omega-\alpha-\beta} \Gamma_{p,q}^{m,n} \left[ \Gamma^\sigma \left( (a_i, \alpha_i)_{1,p} \right) \right] = O\left( t^{\omega-\min[\text{Re}(\beta), \text{Re}(\eta)]+1+s} \right) \quad (t \to +\infty)$$

or

$$= O\left( t^{\omega-\min[\text{Re}(\beta), \text{Re}(\eta)]+1+s}\log(t)^N \right) \quad (t \to +\infty).$$

Here $\varrho$ is given by (3.21) and $N$ is indicated in Theorem B(ii). Therefore the condition (6.1) ensures the existence of the integral (6.4). Applying (1.2), making the change $t = 1/\tau$ and using (5.5), we obtain

$$
\left( I^{\alpha, \beta, \eta}_{\omega} \Gamma_{p,q}^{m,n} \left[ \Gamma^\sigma \left( (a_i, \alpha_i)_{1,p} \right) \right] \right) \left( \frac{1}{x} \right)
$$

$$
= \frac{1}{\Gamma(\alpha)} \int_{1/x}^{\infty} \left( t - \frac{1}{x} \right)^{\alpha-1} t^{\omega-\alpha-\beta} \Gamma_{p,q}^{m,n} \left[ \Gamma^\sigma \left( (a_i, \alpha_i)_{1,p} \right) \right] dt.
$$
Since $\mathcal{L} = \mathcal{L}_{\tau^{\mathrm{I}}}$, $\Re(s) = \gamma$ and therefore the condition (6.2) guarantees the existence of the Mellin-Barnes integral above. Replacing in (6.5) $x$ by $1/x$, we obtain (6.3).

**Corollary 2.1.** Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \ (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

\[
\sigma = \max_{1 \leq i \leq n} \left[ \frac{\Re(a_i) - 1}{\alpha_i} \right] + \Re(\omega) + \Re(\alpha) < 0, \tag{6.6}
\]

\[
\sigma \gamma > \Re(\omega) + \Re(\alpha). \tag{6.7}
\]

Then the Riemann-Liouville fractional integral $I^{\alpha}_{\Gamma, \eta} \Gamma^{m,n}_{p,q}(x)$ of the II-function (1.1) exists and the following relation holds:

\[
\left[ I^{\alpha}_{\Gamma, \eta} \Gamma^{m,n}_{p,q} \right](x) = x^{\omega + \alpha} \Gamma^{m,n+1}_{p+1,q+1} \left[ x^{\omega} \left| (a_i, \alpha_i)_1, p; b_j, \beta_j \right)_1, q \right]. \tag{6.8}
\]

**Corollary 2.2.** Let $\alpha, \eta \in \mathbb{C}$ with $\Re(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \ (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

\[
\sigma = \max_{1 \leq i \leq n} \left[ \frac{\Re(a_i) - 1}{\alpha_i} \right] + \Re(\omega) < \Re(\eta), \tag{6.9}
\]

\[
\sigma \gamma > \Re(\omega) - \Re(\eta). \tag{6.10}
\]

Then the Erdélyi-Kober fractional integral $K^{-\alpha}_{\eta, \omega} \Gamma^{m,n}_{p,q}(x)$ of the H-function (1.1) exists and the following relation holds:

\[
\left[ K^{-\alpha}_{\eta, \omega} \Gamma^{m,n}_{p,q} \right](x) = x^{\omega + \alpha} \Gamma^{m,n+1}_{p+1,q+1} \left[ x^{\omega} \left| (a_i, \alpha_i)_1, p; b_j, \beta_j \right)_1, q \right]. \tag{6.11}
\]

**Remark 4.** In the case $a^* > 0, \Delta \geq 0$ the relation of the form (6.3) was indicated in [16, 4.3]. But it includes a mistake and should be replaced by (6.3) with the conditions (6.1) and (6.2).
Remark 5. Corollary 2.1 coincides with Theorem 2 in [7]. For real $\alpha > 0$ and $\alpha^* > 0$ the relation (6.8) was indicated in [18, (2.5)], but the conditions of its validity have to be also corrected in accordance with (6.6) and (6.7).

7. Left-Sided Generalized Fractional Differentiation of the $H$-Function

Now we treat the left-sided generalized fractional derivative $D_{0+}^{\alpha, \beta, \eta}$ given by (2.11).

Theorem 3. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0, \text{Re}(\alpha + \beta + \eta) \neq 0$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \cdots, p; j = 1, \cdots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \min_{1 \leq i \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\alpha + \beta + \eta)] + 1 > 0,$$

$$\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\alpha + \beta + \eta)] + 1.$$  \hspace{1cm} (7.1)\hspace{1cm} (7.2)

Then the generalized fractional derivative $D_{0+}^{\alpha, \beta, \eta}$ of the $H$-function (1.1) exists and the following relation holds:

$$\left( D_{0+}^{\alpha, \beta, \eta} \omega H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right)(x) = x^{\omega + \beta} H_{p+2,q+2}^{m,n+2} \left[ x^\sigma \left| \begin{array}{c} (-\omega, \sigma), (-\omega - \eta - \alpha - \beta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \beta, \sigma), (-\omega - \eta, \sigma) \end{array} \right. \right].$$ \hspace{1cm} (7.3)

Proof. Let $n = [\text{Re}(\alpha)] + 1$. From (2.11) we have

$$\left( D_{0+}^{\alpha, \beta, \eta} \omega H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right)(x) = (\frac{d}{dx})^n \left( I_{0+}^{-\alpha+n-\beta-n, \alpha+n-\eta} t^\omega H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right)(x),$$ \hspace{1cm} (7.4)

which exists according to Theorem 1 with $\alpha, \beta$ and $\eta$ being replaced by $-\alpha + n, -\beta - n$ and $\alpha + \eta - n$, respectively. Then we find

$$\left( D_{0+}^{\alpha, \beta, \eta} \omega H_{p,q}^{m,n} \left[ t^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right)(x) = (\frac{d}{dx})^n \left( x^{\omega + \beta + n} H_{p+2,q+2}^{m,n+2} \left[ x^\sigma \left| \begin{array}{c} (-\omega, \sigma), (-\omega - \alpha - \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \beta - n, \sigma), (-\omega - \eta, \sigma) \end{array} \right. \right] \right).$$ \hspace{1cm} (7.5)
Taking into account the differentiation formula (4.1) we have

\[
(D_{0+}^{\alpha,\beta,\eta}t^\omega H_{p,q}^{m,n} \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right| (x)) = x^{\omega+\beta} H_{p,q}^{m,n} + \iota 3 + 3 + 3 \quad (7.6)
\]

and Lemma 2 and the reduction relation (4.1) imply (7.3), which completes the proof of theorem.

**Corollary 3.1.** Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \cdots, p; j = 1, \cdots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy the conditions in (5.8) and (5.9). Then the Riemann-Liouville fractional derivative \( D_{0+}^{\alpha,\beta,\eta} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
(D_{0+}^{\alpha,\beta,\eta}t^\omega H_{p,q}^{m,n} \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right| (x)) = x^{\omega+\alpha} H_{p,q}^{m,n+1} \left| \begin{array}{c}
(-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right| (x) \quad (7.7)
\]

**Remark 6.** For real \( \alpha > 0 \) and \( \alpha' > 0 \) the relation (7.3) was given in [18, (2.7.13)], but the conditions of its validity have to be corrected in accordance with (7.1) and (7.2).

**Remark 7.** Corollary 3.1 coincides with Theorem 3 in [7].

8. **Right-Sided Generalized Fractional Differentiation of the \( H \)-Function**

Here we deal with the right-sided generalized fractional derivative \( D_{-}^{\alpha,\beta,\eta} \) given by (2.12).

**Theorem 4.** Let \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \text{Re}(\alpha + \beta + \eta) + [\text{Re}(\alpha)] + 1 \neq 0 \). Let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \cdots, p; j = 1, \cdots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) + \max \{\text{Re}(\beta) + [\text{Re}(\alpha)] + 1, -\text{Re}(\alpha + \eta)\} < 0, \quad (8.1)
\]

\[
\sigma \gamma > \text{Re}(\omega) + \max \{\text{Re}(\beta) + [\text{Re}(\alpha)] + 1, -\text{Re}(\alpha + \eta)\}. \quad (8.2)
\]

Then the generalized fractional derivative \( D_{-}^{\alpha,\beta,\eta} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
(D_{-}^{\alpha,\beta,\eta}t^\omega H_{p,q}^{m,n} \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right| (x)) = (-1)^{|\text{Re}(\alpha)| + 1} x^{\omega+\beta} H_{p,q+2}^{m+2,n+2} \left| \begin{array}{c}
(a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma) \\
(b_j, \beta_j)_{1,q}
\end{array} \right| (x) \quad (8.3)
\]
Proof. Let \( n = [\text{Re}(\alpha)] + 1 \). Owing to (2.12) we have

\[
(D_{-}^{\alpha, \beta} I_{p,q}^{m,n} \left[ t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right](x) = \left( -\frac{d}{dx} \right)^n I_{-\alpha+n, -\beta-n, \alpha+\eta} I_{p,q}^{m,n} \left[ t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right](x), \tag{8.4}
\]

which exists according to Theorem 2 with \( \alpha, \beta \) and \( \eta \) being replaced by \(-\alpha + n, -\beta - n \) and \( \alpha + \eta \), respectively. Then applying the differentiation formula (4.5), similarly to (7.5), (7.6), we find in view of the reduction formula (1.2) that

\[
(D_{-}^{\alpha, \beta} I_{p,q}^{m,n} \left[ t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right](x) = (-1)^n x^{\omega+\beta} I_{p+q}^{m+3,n} \left[ x^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right]
\]

which implies the formula (8.3).

**Corollary 4.1.** Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \ldots, p; j = 1, \ldots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) - \{\text{Re}(\alpha)\} + 1 < 0, \tag{8.5}
\]

\[
\sigma \gamma + \text{Re}(\omega) - \{\text{Re}(\alpha)\} + 1 > 0. \tag{8.6}
\]

Then the Riemann-Liouville fractional derivative \( D_{-}^\alpha \) of the \( II \)-function (1.1) exists and there holds the relation:

\[
(D_{-}^\alpha I_{p,q}^{m,n} \left[ t^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right](x) = (-1)^{\text{Re}(\alpha)+1} x^{\omega-\alpha} I_{p+q+1}^{m+1,n} \left[ x^\sigma \begin{bmatrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{bmatrix} \right]. \tag{8.7}
\]

**Remark 8.** The relation of the form (8.7) with real \( \alpha > 0 \) and \( \alpha^* > 0 \) was proved in [13, formula (14a)] (see also [12], [14, (2.2)] and [18, (2.7.9)]). But such a formula contains mistakes and should be replaced by (8.7) with the condition (8.5) and (8.6).
Remark 9. When $\alpha = k \in \mathbb{N}$, the relations (7.7) and (8.7) coincide with (4.4) and (4.5), respectively.

9. Generalized Fractional Integro-Differentiation of the $H$-Function

Here we investigate the generalized fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,n}$ and $I_{-}^{\alpha,\beta,n}$ given by (2.8) and (2.10). The following statements are proved similarly to Theorems 3 and 4 by using the relations (2.8) and (2.10), Theorems 1 and 2, and the properties of the $H$-function in Sections 3 and 4.

Theorem 5. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) \leq 0, \text{Re}(\beta) \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \ldots, p; j = 1, \ldots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$
\sigma \min_{1 \leq i \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1 > 0, \quad (9.1)
$$

$$
\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1. \quad (9.2)
$$

Then the generalized fractional integro-differentiation $I_{0+}^{\alpha,\beta,n}$ of the $H$-function (1.1) exists and there holds the relation

$$
\left( I_{0+}^{\alpha,\beta,n} \omega I_{p,q}^{m,n} \left[ \Gamma^\sigma \left( \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right) \right] \right)(x) = x^{\omega - \beta} I_{p+2,q+2}^{m,n+2} \left[ x^{\sigma} \left( \begin{array}{c} (-\omega, \sigma), (-\omega - \eta + \beta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega + \alpha - \eta, \sigma) \end{array} \right) \right]. \quad (9.3)
$$

Theorem 6. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) \leq 0, \text{Re}(\beta) + [\text{Re}(\alpha)] - 1 \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \ldots, p; j = 1, \ldots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \min[\text{Re}(\beta) - \text{Re}(\alpha)] - 1, \text{Re}(\eta)], \quad (9.4)
$$

$$
\sigma \gamma > \text{Re}(\omega) - \min[\text{Re}(\beta) - \text{Re}(\alpha)] - 1, \text{Re}(\eta)]. \quad (9.5)
$$

Then the generalized fractional integro-differentiation $I_{-}^{\alpha,\beta,n}$ of the $H$-function (1.1) exists and there holds the relation

$$
\left( I_{-}^{\alpha,\beta,n} \omega I_{p,q}^{m,n} \left[ \Gamma^\sigma \left( \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right) \right] \right)(x) = x^{\omega - \beta} I_{p+2,q+2}^{m,n+2} \left[ x^{\sigma} \left( \begin{array}{c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\ (b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega + \eta, \sigma) \end{array} \right) \right]. \quad (9.6)
$$
Remark 10. The relation (9.3) with $\alpha^* > 0, \Delta \geq 0$ was indicated in [16, (4.2)], but conditions of its validity have to be corrected in accordance with (9.1) and (9.2).

Remark 11. The relations (9.3) and (9.6) for the fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,n}$ and $I_{n}^{\alpha,\beta,n}$, defined in (2.8) and (2.10) for $\alpha \in \mathbb{C}, \Re(\alpha) \leq 0$ coincide with that (5.3) and (6.3) for the fractional integration operators $I_{0+}^{\alpha,\beta,n}$ and $I_{n}^{\alpha,\beta,n}$, defined in (2.7) and (2.9) for $\alpha \in \mathbb{C}, \Re(\alpha) > 0$. Though the conditions for validity of (5.3) and (9.3) in Theorems 1 and 5 have the same form, that of (6.3) and (9.6) presented in Theorems 2 and 4 are slightly different.

In conclusion we note that, as it was mentioned in Remarks 2, 4 and 10, the relations (5.3), (6.3) and (9.3) for generalized calculus operator $I_{0+}^{\alpha,\beta,n}$ were already known in the case $\alpha^* > 0, \Delta \geq 0$. Further, Remarks 3,5,6 and 8 indicate that the relations (5.10) and (6.8) for the Riemann-Liouville fractional integrals $I_{\alpha}^{\lambda}, I_{\alpha}^{\beta}$ and (7.3) and (8.7) for the fractional derivative $D_{0+}^{\alpha}$, in the case real $\alpha > 0$ and $\alpha^* > 0$ were established. However, the $H$-function's asymptotic estimates (3.16), (3.17) at zero and (3.19), (3.20) at infinity allow us to prove such results under more general assumptions $\alpha^* > 0$ and $\alpha^* = 0, \Delta\gamma + \Re(\mu) < -1$.

References


