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Generalized Fractional Calculus of the $H$-Function

Megumi Saigo* [西郷 恵] （福岡大学 理学部）

Anatoly A. Kilbas† （ベラルーシ国立大学・ベラルーシ）

Abstract

The paper is devoted to study the generalized fractional calculus of arbitrary complex order for the $H$-function defined by the Mellin-Barnes integral

$$H_{p,q}^{m,n}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds,$$

where the function $\mathcal{H}_{p,q}^{m,n}(s)$ is a certain ratio of products of Gamma functions with the argument $s$ and the contour $\mathcal{L}$ is specially chosen. The considered generalized fractional integration and differentiation operators contain the Gauss hypergeometric function as a kernel and generalize classical fractional integrals and derivatives of Riemann-Liouville, Erdélyi-Kober type, etc. It is proved that the generalized fractional integrals and derivatives of $H$-functions are also $H$-functions but of greater order. In particular, the obtained results define more precisely and generalize known results.

1. Introduction

This paper deals with the $H$-function $H_{p,q}^{m,n}(z)$. For integers $m, n, p, q$ such that $0 \leq m \leq q$, $0 \leq n \leq p$, for $a_i, b_j \in \mathbb{C}$ with $\mathbb{C}$ the field of complex numbers and for $\alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty)$ ($i = 1, 2, \ldots, p; j = 1, 2, \ldots, q$) the $H$-function $H_{p,q}^{m,n}(z)$ is defined via a Mellin-Barnes type integral in the following way:

$$H_{p,q}^{m,n}(z) \equiv H_{p,q}^{m,n} \left[ \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds,$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \right] s^{-s} ds,$$

(1.1)

* Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan
† Department of Mathematics and Mechanics, Belarusian State University, Minsk 220050, Belarus
where the contour $\mathcal{L}$ is specially chosen and

$$
\mathcal{H}_{p,q}^{m,n}(s) \equiv \mathcal{H}_{p,q}^{m,n}
\begin{pmatrix}
(a_i, \alpha_i)_{1, p} \\
(b_j, \beta_j)_{1, q}
\end{pmatrix}
$$

$$
= \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)}, \quad (1.2)
$$

in which an empty product, if it occurs, is taken to be one. Such a function was introduced by S. Pincherle in 1888 and its theory has been developed by Mellin [10], Dixon and Ferrar [2] (see [3, §1.19] in this connection). An interest to the $H$-function arose again in 1961 when Fox [4] has investigated such a function as a symmetrical Fourier kernel. Therefore this function is sometimes called as Fox's $H$-function. The theory of this function may be found in [1], [9, Chapter 1], [17, Chapter 2] and [11, 8.8.3].

Classical Riemann-Liouville fractional calculus of real order [17, §2.2] (see (2.1) - (2.6) below) was investigated in [12] - [14], [18] and [11]. The right-sided fractional integrals and derivatives of the $H$-function (1.1) were studied in [12] - [14] and the results were presented in [18, §2.7], where the case of left-sided fractional differentiation of the $H$-function was also considered. The left-sided fractional integration of the $H$-function was given in [11, 2.25.2]. Such results for the generalized fractional calculus operators with the Gauss hypergeometric function as a kernel (see (2.7) - (2.10) below), being introduced by the first author [15], were obtained in [16].

However, some of the results obtained in [12] - [14] (cited in [18]) and [16] can be taken to be more precisely. Moreover, these results were given provided that the parameters $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0, \beta_j > 0 (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$ of the $H$-function satisfy certain conditions. These conditions were based on asymptotic behavior of $II_{p,q}^{m,n}(z)$ at zero and infinity. In [5] we extended such the known asymptotic results for the $H$-function to more wide class of parameters.

In [7], [8] we have applied the obtained asymptotic estimates in [5] to find the Riemann-Liouville fractional integrals and derivatives of any complex order of the $H$-function. In particular, we could make more precisely the known results from [12] - [14], [18] and [11].

The present paper is devoted to obtain such type results for the generalized fractional integration and differentiation operators of any complex order with the Gauss hypergeometric function as a kernel. In particular, we give more precisely some of the results from [16] and generalize the results obtained in [7], [8]. The paper is organized as follow. In Section 2 we present classical and generalized fractional calculus operators and some facts from the theory of Gauss hypergeometric function. Sections 3 and 4 contain the result from the theory of the $H$-function. The existence of $II_{p,q}^{m,n}(z)$ and its asymptotic behavior at zero and infinity is considered in Section 3 and certain reduction and differentiation properties in Section 4. Sections 5 and 6 deal with generalized fractional differentiation of the $H$-function (1.1). Sections 7 and 8 are devoted to the generalized fractional differentiation of the $H$-function. Another type of fractional integro-differentiation of the $H$-function is given in Section 9.
2. Classical and Generalized Fractional Calculus Operators

For \( \alpha \in \mathbb{C}, \Re(\alpha) > 0 \), the Riemann-Liouville left- and right-sided fractional calculus operators are defined as follows [17, §2.3 and §2.4]:

\[
(I_{0+}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (x > 0),
\]

\[
(I_{-}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(t)dt}{(t-x)^{1-\alpha}} \quad (x > 0),
\]

and

\[
(D_{0+}^{\alpha}f)(x) = \left( \frac{d}{dx} \right)^{[\Re(\alpha)]+1} \left( I_{0+}^{1-\alpha+\Re(\alpha)}f \right)(x)
\]

\[
= \left( \frac{d}{dx} \right)^{[\Re(\alpha)]+1} \frac{1}{\Gamma(1-\alpha + |\Re(\alpha)|)} \int_{0}^{x} \frac{f(t)dt}{(x-t)^{\alpha-|\Re(\alpha)|}} \quad (x > 0),
\]

\[
(D_{-}^{\alpha}f)(x) = \left( -\frac{d}{dx} \right)^{[\Re(\alpha)]+1} \left( I_{-}^{1-\alpha+\Re(\alpha)}f \right)(x)
\]

\[
= \left( -\frac{d}{dx} \right)^{[\Re(\alpha)]+1} \frac{1}{\Gamma(1-\alpha + |\Re(\alpha)|)} \int_{x}^{\infty} \frac{f(t)dt}{(t-x)^{\alpha-|\Re(\alpha)|}} \quad (x > 0),
\]

respectively, where the symbol \([\kappa]\) means the integral part of a real number \( \kappa \), i.e., the largest integer not exceeding \( \kappa \). In particular, for real \( \alpha > 0 \), the operators \( D_{0+}^{\alpha} \) and \( D_{-}^{\alpha} \) take more simple forms

\[
(D_{0+}^{\alpha}f)(x) = \left( \frac{d}{dx} \right)^{[\alpha]+1} \left( I_{0+}^{1-\alpha+\Re(\alpha)}f \right)(x)
\]

\[
= \left( \frac{d}{dx} \right)^{[\alpha]+1} \frac{1}{\Gamma(1-\alpha)} \int_{0}^{x} \frac{f(t)dt}{(x-t)^{\alpha}} \quad (x > 0),
\]

and

\[
(D_{-}^{\alpha}f)(x) = \left( -\frac{d}{dx} \right)^{[\alpha]+1} \left( I_{-}^{1-\alpha+\Re(\alpha)}f \right)(x)
\]

\[
= \left( -\frac{d}{dx} \right)^{[\alpha]+1} \frac{1}{\Gamma(1-\alpha)} \int_{x}^{\infty} \frac{f(t)dt}{(t-x)^{\alpha}} \quad (x > 0),
\]

respectively, where \( \{\kappa\} \) stands for the fractional part of \( \kappa \), i.e., \( \{\kappa\} = \kappa - [\kappa] \).

For \( \alpha, \beta, \eta \in \mathbb{C} \) and \( x > 0 \) the generalized fractional calculus operators are defined by [15]

\[
(I_{0+}^{\alpha,\beta,\eta}f)(x) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1-\beta} {}_{2}F_{1} \left( \alpha+\beta, -\eta; \alpha; 1 - \frac{t}{x} \right) f(t)dt \quad (x > 0),
\]
\[ (I_{0+}^{\alpha,\beta,\eta}f)(x) = \left( \frac{d}{dx} \right)^n \left( I_{0+}^{\alpha+n,\beta-n,\eta-n}f \right)(x) \quad (\text{Re}(\alpha) > 0); \quad (\text{Re}(\alpha) \leq 0; n = [\text{Re}(\alpha)] + 1); \quad (2.8) \]

\[ (I_{-}^{\alpha,\beta,\eta}f)(x) = \left( -\frac{d}{dx} \right)^n \left( I_{-}^{\alpha+n,\beta-n,\eta-n}f \right)(x) \quad (\text{Re}(\alpha) \leq 0; n = [\text{Re}(\alpha)] + 1); \quad (2.10) \]

and

\[ (D_{0+}^{\alpha,\beta,\eta}f)(x) \equiv (I_{0+}^{-\alpha,-\beta,\eta}f)(x) \quad (\text{Re}(\alpha) > 0; n = [\text{Re}(\alpha)] + 1); \quad (2.11) \]

\[ (D_{-}^{\alpha,\beta,\eta}f)(x) \equiv (I_{-}^{-\alpha,-\beta,\eta}f)(x) \quad (\text{Re}(\alpha) > 0; n = [\text{Re}(\alpha)] + 1)). \quad (2.12) \]

Here \( _2F_1(a, b; c; z) \) \((a, b, c, z \in \mathbb{C})\) is the Gauss hypergeometric function defined by the series

\[ _2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!} \quad (2.13) \]

with

\[ (a)_0 = 1, \quad (a)_k = a(a+1) \cdots (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (k \in \mathbb{N}), \quad (2.14) \]

where \( \Gamma(z) \) is the Gamma function \([3, \text{Chapter I}]\) and \( \mathbb{N} \) denotes the set of positive integers.

The series in (2.13) is convergent for \(|z| < 1\) and for \(|z| = 1\) with \(\text{Re}(c - a - b) > 0\), and can be analytically continued into \( \{z \in \mathbb{C} : \arg(1-z) < \pi\} \) \((\text{see } [3, \text{Chapter II}])\). Since

\[ _2F_1(0, b; c; z) = 1 \quad (2.15) \]

for \(\beta = -\alpha\), the generalized fractional calculus operators (2.7), (2.9), (2.11) and (2.12) coincide with the Riemann-Liouville operators (2.1) - (2.4) for \(\text{Re}(\alpha) > 0\):

\[ \left( I_{0+}^{\alpha,-\beta,\eta}f \right)(x) = \left( I_{0+}^{\alpha}f \right)(x), \quad \left( I_{-}^{\alpha,-\beta,\eta}f \right)(x) = \left( I_{-}^{\alpha}f \right)(x), \quad (2.16) \]

\[ \left( D_{0+}^{\alpha,-\beta,\eta}f \right)(x) = \left( D_{0+}^{\alpha}f \right)(x), \quad \left( D_{-}^{\alpha,-\beta,\eta}f \right)(x) = \left( D_{-}^{\alpha}f \right)(x). \quad (2.17) \]
According to the relation [3, 2.8(4)]

\[ 2F_1(a, b; c; z) = (1 - z)^{-b}, \]  

(2.18)

when \( \beta = 0 \) the operators (2.7) and (2.9) coincide with the Erdélyi-Kober fractional integrals [17, §18.1]:

\[ (I_{0+}^{\alpha, \beta, \eta} f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{-\eta} f(t) dt \equiv (I_{\eta, \alpha}^+ f)(x) \quad (\alpha, \eta \in \mathbb{C}, \Re(\alpha) > 0), \]  

(2.19)

\[ (I_{-\eta, \alpha}^- f)(x) = \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\eta} f(t) dt \equiv (K_{\eta, \alpha}^- f)(x) \quad (\alpha, \eta \in \mathbb{C}, \Re(\alpha) > 0). \]  

(2.20)

Therefore the operators (2.7), (2.9) and (2.11), (2.12) are called ”generalized” fractional integrals and derivatives, respectively. Moreover, the operators (2.11) and (2.12) are inverse to (2.7) and (2.9):

\[ D_{0+}^{\alpha, \beta, \eta} = (I_{0+}^{\alpha, \beta, \eta})^{-1}, \quad D_{-}^{\alpha, \beta, \eta} = (I_{-\eta, \alpha}^-)^{-1}. \]  

(2.21)

Fractional calculus operators (2.1), (2.3), (2.5), (2.7), (2.8), (2.11) and (2.2), (2.4), (2.6), (2.9), (2.10), (2.12) are called left-sided and right-sided, respectively [17, §2].

We give some other properties of \( 2F_1(a, b; c; z) \) [3, 2.8(46), 2.9(2), 2.10(14)] which will be used in the following calculations:

\[ 2F_1(a, b; c; 1) = \Gamma(c)\Gamma(c-a-b) / \Gamma(1)\Gamma(c-b) \quad (c \neq 0, -1, -2, \cdots; \Re(c-a-b) > 0); \]  

(2.22)

\[ 2F_1(a, b; c; z) = (1 - z)^{c-a-b} 2F_1(c - a, c - b; c; z); \]  

(2.23)

\[ 2F_1(a, b; a + b; z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^\infty \frac{(a)_k(b)_k}{(k!)^2} [2^\psi(1+k) - \psi(a+k) + \psi(b+k) - \log(1-z)(1-z)^k \quad (|\arg(z)| < \pi; a, b \neq 0, -1, -2, \cdots), \]  

(2.24)

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the Psi function [3, 1.7].

Formulas (2.22) - (2.24) mean the following asymptotic behavior of \( 2F_1(a, b; c; z) \) at the point \( z = 1 \):

**Lemma 1.** For \( a, b, c \in \mathbb{C} \) with \( \Re(c) > 0 \) and \( z \in \mathbb{C} \), there hold the following asymptotic relations near \( z = 1 \):

\[ 2F_1(a, b; c; z) = O(1) \quad (z \to 1-) \]  

(2.25)

for \( \Re(c-a-b) > 0 \); and

\[ 2F_1(a, b; c; z) = O((1-z)^{c-a-b}) \quad (z \to 1-) \]  

(2.26)

for \( \Re(c-a-b) < 0 \); and

\[ 2F_1(a, b; c; z) = O(\log(1-z)) \quad (z \to 1-) \]  

(2.27)
for \( c - a - b = 0, \ a, b \neq 0, -1, -2, \cdots \) and \(|\arg(z)| < \pi \).

3. Existence and Asymptotic Behavior of the \( H \)-Function

We shall consider the \( H \)-function (1.1) provided that the poles

\[
b_{jl} = \frac{-b_{j} - l}{\beta_{j}} \quad (j = 1, \cdots, m; l \in \mathbb{N}_{0})
\]

(3.1)
of the Gamma functions \( \Gamma(b_{j} + \beta_{j}s) \) and that

\[
a_{ik} = \frac{1 - a_{i} + k}{\alpha_{i}} \quad (i = 1, \cdots, n; k \in \mathbb{N}_{0})
\]

(3.2)
of \( \Gamma(1 - a_{i} - \alpha_{i}s) \) do not coincide:

\[
\alpha_{i}(b_{j} + l) \neq \beta_{j}(a_{i} - k - 1) \quad (i = 1, \cdots, n; j = 1, \cdots, m; k, l \in \mathbb{N}_{0}),
\]

(3.3)
where \( \mathbb{N}_{0} = \mathbb{N} \cup \{0\} \). \( \mathcal{L} \) in (1.1) is the infinite contour splitting poles \( b_{jl} \) in (3.1) to the left and poles \( a_{ik} \) in (3.2) to the right of \( \mathcal{L} \) and has one of the following forms:

(i) \( \mathcal{L} = \mathcal{L}_{-\infty} \) is a left loop situated in a horizontal strip starting at the point \(-\infty + i\varphi_{1}\) and terminating at the point \(-\infty + i\varphi_{2}\) with \(-\infty < \varphi_{1} < \varphi_{2} < +\infty\);

(ii) \( \mathcal{L} = \mathcal{L}_{+\infty} \) is a right loop situated in a horizontal strip starting at the point \(+\infty + i\varphi_{1}\) and terminating at the point \(+\infty + i\varphi_{2}\) with \(-\infty < \varphi_{1} < \varphi_{2} < +\infty\);

(iii) \( \mathcal{L} = \mathcal{L}_{i\gamma\infty} \) is a contour starting at the point \(\gamma - i\infty\) and terminating at the point \(\gamma + i\infty\) with \(\gamma \in \mathbb{R} = (-\infty, +\infty)\).

The properties of the \( H \)-function \( H_{p,q}^{m,n}(z) \) depend on the numbers \( a^{*}, \Delta, \delta \) and \( \mu \) which are expressed via \( p, q, a_{i}, a_{i} (i = 1, 2, \cdots, p) \) and \( b_{j}, \beta_{j} (j = 1, 2, \cdots, q) \) by the following relations:

\[
a^{*} = \sum_{i=1}^{n} \alpha_{i} - \sum_{i=n+1}^{p} \alpha_{i} + \sum_{j=1}^{m} \beta_{j} - \sum_{j=m+1}^{q} \beta_{j},
\]

(3.4)
\[
\Delta = \sum_{j=1}^{q} \beta_{j} - \sum_{i=1}^{p} \alpha_{i},
\]

(3.5)
\[
\delta = \prod_{i=1}^{p} \alpha_{i}^{-\alpha_{i}} \prod_{j=1}^{q} \beta_{j}^{\beta_{j}},
\]

(3.6)
\[
\mu = \sum_{j=1}^{q} b_{j} - \sum_{i=1}^{p} a_{i} + \frac{p - q}{2}.
\]

(3.7)
Here an empty sum in (3.4), (3.5), (3.7) and an empty product in (3.6), if they occur, are taken to be zero and one, respectively.

The existence of the \( H \)-function is given by the following result [6].
Theorem A. Let $a^*$, $\Delta$, $\delta$ and $\mu$ be given by (3.4) - (3.7). Then the $H$-function $H_{p,q}^{m,n}(z)$ defined by (1.1) and (1.2) makes sense in the following cases:

\begin{align}
\mathcal{L} = \mathcal{L}_{-\infty}, & \quad \Delta > 0, \quad z \neq 0; \\
\mathcal{L} = \mathcal{L}_{-\infty}, & \quad \Delta = 0, \quad 0 < |z| < \delta; \\
\mathcal{L} = \mathcal{L}_{-\infty}, & \quad \Delta = 0, \quad \text{Re}(\mu) < -1, \quad |z| = \delta; \\
\mathcal{L} = \mathcal{L}_{+\infty}, & \quad \Delta < 0, \quad z \neq 0; \\
\mathcal{L} = \mathcal{L}_{+\infty}, & \quad \Delta = 0, \quad |z| > \delta; \\
\mathcal{L} = \mathcal{L}_{\gamma \infty}, & \quad a^* > 0, \quad |\arg z| < \frac{a^* \pi}{2}, \quad z \neq 0; \\
\mathcal{L} = \mathcal{L}_{\gamma \infty}, & \quad a^* = 0, \quad \Delta \gamma + \text{Re}(\mu) < -1, \quad \arg z = 0, \quad z \neq 0.
\end{align}

Remark 1. The results of Theorem A in the cases (3.10), (3.13) and (3.15) are more precisely than those in [11, §8.3.1].

The next statement being followed from the results in [5] characterizes the asymptotic behavior of the $H$-function at zero and infinity.

Theorem B. Let $a^*$ and $\Delta$ be given by (3.4) and (3.5) and let conditions in (3.3) be satisfied.

(i) If $\Delta \geq 0$ or $\Delta < 0, a^* > 0$, then the $H$-function has either of the asymptotic estimates at zero

\[ H_{p,q}^{m,n}(z) = O\left(z^{\varrho^*}\right) \quad (|z| \to 0) \]  

or

\[ H_{p,q}^{m,n}(z) = O\left(z^{\varrho^*} |\log(z)|^{N^*}\right) \quad (|z| \to 0), \]  

with the additional condition $|\arg(z)| < a^* \pi/2$ when $\Delta < 0, a^* > 0$. Here

\[ \varrho^* = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_{j})}{\beta_{j}} \right], \]  

and $N^*$ is the order of one of the point $b_{ji}$ in (3.1) to which some other poles of $\Gamma(b_{j}+\beta_{j}s)$ ($j = 1, \cdots, m$) coincide.

(ii) If $\Delta \leq 0$ or $\Delta > 0, a^* > 0$, then the $H$-function has either of the asymptotic estimates at infinity

\[ H_{p,q}^{m,n}(z) = O\left(z^{\varrho^*}\right) \quad (|z| \to \infty) \]  

and

\[ H_{p,q}^{m,n}(z) = O\left(z^{\varrho^*} \right) \quad (|z| \to \infty). \]
or
\[
H_{p,q}^{m,n}(z) = O \left( z^\alpha |\log(z)|^N \right) \quad (|z| \to \infty),
\]
with the additional condition \(|\arg(z)| < \alpha \pi/2\) when \(\Delta > 0, \alpha > 0\). Here
\[
\varrho = \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right],
\]
and \(N\) is the order of one of the point \(a_{ik}\) in (3.2) in which some other poles of \(\Gamma(1 - a_i - \alpha_i s)\) \((i = 1, \ldots, n)\) coincide.

4. Reduction and Differentiation Properties of the \(H\)-Function

In this and next sections we suppose that the conditions for the existence of the \(H\)-function given in Theorem A are satisfied.

The following two Lemmas which characterize symmetric and reduction properties of the \(H\)-function follow from the definition of the \(H\)-function in (1.1) - (1.2).

**Lemma 2.** The \(H\)-function (1.1) is commutative in the set of pairs \((a_1, \alpha_1), \ldots, (a_n, \alpha_n);\)
in \((a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p);\) in \((b_1, \beta_1), \ldots, (b_m, \beta_m)\) and in \((b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q).\)

**Lemma 3.** If one of \((a_i, \alpha_i)\) \((i = 1, \ldots, n)\) is equal to one of \((b_j, \beta_j)\) \((j = m + 1, \ldots, q)\)
or one of \((a_i, \alpha_i)\) \((i = n + 1, \ldots, p)\) is equal to one of \((b_j, \beta_j)\) \((j = 1, \ldots, m)\), then the \(H\)-function reduces to the lower order one, that is, \(p, q\) and \(n\) (or \(m\)) decrease by unity. Two such results have the forms
\[
H_{p,q}^{m,n} \left[ \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q-1}, (a_1, \alpha_1)\end{array} \right] = H_{p-1,q-1}^{m,n-1} \left[ \begin{array}{c}
(a_i, \alpha_i)_{2,p} \\
(b_j, \beta_j)_{1,q-1}\end{array} \right]
\]
provided that \(n \geq 1\) and \(q > m,\) and
\[
H_{p,q}^{m,n} \left[ \begin{array}{c}
(a_i, \alpha_i)_{1,p-1}, (b_1, \beta_1) \\
(b_j, \beta_j)_{1,q}\end{array} \right] = H_{p-1,q-1}^{m-1,n} \left[ \begin{array}{c}
(a_i, \alpha_i)_{1,p-1} \\
(b_j, \beta_j)_{2,q}\end{array} \right]
\]
provided that \(m \geq 1\) and \(p > n.\)

The next differentiation formulae follow from the definition of the \(H\)-function given in (1.1) - (1.2) and from the functional equation for the Gamma function [3, §1.2(6)]
\[
\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}. \tag{4.3}
\]
Lemma 4. There hold the following differentiation formulae for $\omega, c \in \mathbb{C}, \sigma > 0$

\[
\left( \frac{d}{dz} \right)^{k} \left\{ z^{\omega} I_{p,q}^{m,n} \left[ cz^{\sigma} \begin{array}{c} (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q} \end{array} \right] \right\} = z^{\omega-k} I_{p,q}^{m,n+1} \left[ cz^{\sigma} \begin{array}{c} (-\omega, \sigma), (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q}, (k - \omega, \sigma) \end{array} \right],
\]

(4.4)

\[
\left( \frac{d}{dz} \right)^{k} \left\{ z^{\omega} I_{p,q}^{m,n} \left[ cz^{\sigma} \begin{array}{c} (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q} \end{array} \right] \right\} = (-1)^{k} z^{\omega-k} I_{p,q}^{m,n+1} \left[ cz^{\sigma} \begin{array}{c} (a_{i}, \alpha_{i})_{1,p}, (-\omega, \sigma) \\ (k - \omega, \sigma), (b_{j}, \beta_{j})_{1,q} \end{array} \right].
\]

(4.5)

5. Left-Sided Generalized Fractional Integration of the $H$-Function

In the following sections we treat the $H$-function (1.1) - (1.2) with $\mathfrak{L} = \mathfrak{L}_{\gamma_{00}}$ and under the assumptions $a^{*} > 0$ or $a^{*} = 0, \Delta \gamma + \Re(\mu) < -1$ for $a^{*}, \Delta, \mu$ being given by (3.4), (3.5), (3.7).

Here we consider the left-sided generalized fractional integration $I_{0+}^{\alpha, \beta, \eta}$ defined by (2.7).

Theorem 1. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\Re(\alpha) > 0, \Re(\beta) \neq \Re(\eta)$. Let the constants $a_{i}, b_{j} \in \mathbb{C}, \alpha_{i}, \beta_{j} > 0 (i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

\[
\sigma \min_{1 \leq j \leq m} \left[ \frac{\Re(b_{j})}{\beta_{j}} \right] + \Re(\omega) + \min[0, \Re(\eta - \beta)] + 1 > 0,
\]

(5.1)

\[
\sigma \gamma < \Re(\omega) + \min[0, \Re(\eta - \beta)] + 1.
\]

(5.2)

Then the generalized fractional integral $I_{0+}^{\alpha, \beta, \eta}$ of the $H$-function (1.1) exists and the following relation holds:

\[
\left( I_{0+}^{\alpha, \beta, \eta} \int_{0}^{\omega} I_{p,q}^{m,n} \left[ t^{\sigma} \begin{array}{c} (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q} \end{array} \right] \right)(x) = x^{\omega-\beta} I_{p,q}^{m,n+2} \left[ x^{\sigma} \begin{array}{c} (-\omega, \sigma), (\alpha + \beta - \eta, \sigma), (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q}, (-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \end{array} \right].
\]

(5.3)

Proof. By (2.7) we have

\[
\left( I_{0+}^{\alpha, \beta, \eta} \int_{0}^{\omega} I_{p,q}^{m,n} \left[ t^{\sigma} \begin{array}{c} (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q} \end{array} \right] \right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{0}^{x} (x - t)^{\alpha-1} t^{\omega} 2F_{1} \left( \alpha + \beta, -\eta; 1 - \frac{t}{x} \right) I_{p,q}^{m,n} \left[ t^{\sigma} \begin{array}{c} (a_{i}, \alpha_{i})_{1,p} \\ (b_{j}, \beta_{j})_{1,q} \end{array} \right] dt.
\]

(5.4)
According to (2.25), (2.26), (3.16) and (3.17), the integrand in (5.4) for any $x > 0$ has the asymptotic estimate at zero

$$(x-t)^{\alpha-1}t^\omega \frac{2 F_1}{(a, b; c; 1 - \frac{t}{x})} H_{\pi, q}^{m, n} \left[ t^\sigma \left| \begin{array}{c} (a, \alpha)_{1,p} \\ (b, \beta)_{1,q} \end{array} \right] \right]$$

$$= O \left(t^{\omega + \sigma \phi + \min(0, \Re(\eta - \beta))}\right) \quad (t \to +0)$$
or

$$= O \left(t^{\omega + \sigma \phi + \min(0, \Re(\eta - \beta))} \left[\log(t)\right] N^*\right) \quad (t \to +0).$$

Here $\phi$ is given by (3.18) and $N^*$ is indicated in Theorem B(i). Therefore the condition (5.1) ensures the existence of the integral (5.4).

Applying (1.2), making the change of variable $t = x \tau$, changing the order of integration and taking into account the formula [11, §2.21.11]

$$\int_0^x (x-t)^{\alpha-1}t^\omega \frac{2 F_1}{(a, b; c; 1 - \frac{t}{x})} dt = \frac{\Gamma(c)\Gamma(a)\Gamma(a + c - a - b)}{\Gamma(a + c - a)\Gamma(a + c - b)} x^{\alpha+c-1}$$

we obtain

$$(I_{0+}^{0,\alpha, \eta, \omega} t^{m.n} p.q \left[ t^\sigma \left| \begin{array}{c} (a, \alpha)_{1,p} \\ (b, \beta)_{1,q} \end{array} \right] \right])$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma(a)} \int_0^x (x-t)^{\alpha-1}t^\omega \frac{2 F_1}{(a, b; c; 1 - \frac{t}{x})} H_{\pi, q}^{m, n} \left[ t^\sigma \left| \begin{array}{c} (a, \alpha)_{1,p} \\ (b, \beta)_{1,q} \end{array} \right] \right] dt$$

$$= \frac{x^{-\alpha-\beta}}{2\pi i} \int_L \mathcal{L}_{p,q}^{m,n} \left[ \begin{array}{c} (a, \alpha)_{1,p} \\ (b, \beta)_{1,q} \end{array} \right] s ds \int_0^x (x-t)^{\alpha-1}t^{\omega-s\beta} x^{\alpha+c-1} 2 F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) dt$$

We note that since $\mathcal{L} = \mathcal{L}_{\gamma, \gamma}$, $\Re(s) = \gamma$ and therefore the condition (5.2) ensures the existence of the Mellin-Barnes integral above. Hence in view of (1.2)

$$\left(I_{0+}^{0,\alpha, \eta, \omega} t^{m.n} p.q \left[ t^\sigma \left| \begin{array}{c} (a, \alpha)_{1,p} \\ (b, \beta)_{1,q} \end{array} \right] \right]\right)$$

$$= \frac{x^\omega - \beta}{2\pi i} \int_L \mathcal{L}_{p,q}^{m,n} \left[ \begin{array}{c} (a, \alpha)_{1,p} \\ (b, \beta)_{1,q} \end{array} \right] s \frac{\Gamma(1 + \omega - s\sigma)\Gamma(1 + \omega - \beta + \eta - s\sigma)}{\Gamma(1 + \omega - \beta - s\sigma)\Gamma(1 + \omega + \alpha - \eta - s\sigma)} x^{-s\beta} ds,$$ 

and in accordance with (1.1) we obtain (5.3) which completes the proof of Theorem 1.
Corollary 1.1. Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \cdots, p; j = 1, \cdots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + 1 > 0, \quad (5.8)$$

$$\sigma \gamma < \text{Re}(\omega) + 1. \quad (5.9)$$

Then the Riemann-Liouville fractional integral $I_{0+}^\alpha$ of the $H$-function (1.1) exists and the following relation holds:

$$\left( \int_{0+}^\alpha t^\omega H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right)(x) = 2^{\omega+\alpha} I_{p+1,q+1}^\alpha \left[ \begin{array}{c} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \alpha, \sigma) \end{array} \right]. \quad (5.10)$$

Corollary 1.2. Let $\alpha, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \cdots, p; j = 1, \cdots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\eta)] + 1 > 0, \quad (5.11)$$

$$\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\eta)] + 1. \quad (5.12)$$

Then the Erdélyi-Kober fractional integral $I_{\eta,\alpha}^+$ of the $H$-function (1.1) exists and the following relation holds:

$$\left( I_{\eta,\alpha}^+ t^\omega H_{p,q}^{m,n} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right)(x) = 2^{\omega} I_{p+1,q+1}^{\omega+\alpha} \left[ \begin{array}{c} (-\omega - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \alpha - \eta, \sigma) \end{array} \right]. \quad (5.13)$$

Remark 2. In the case $a^* > 0, \Delta \geq 0$ the relation (5.3) was indicated in [16, (4.2)], but in the assumptions of the result the condition (5.2) of Theorem 1 should be added.

Remark 3. Corollary 1.1 coincides with Theorem 1 in [7]. For real $\alpha > 0$ and $a^* > 0$ the relation (5.10) was indicated in [11, 2.25.2.2], but the conditions of its validity have to be also corrected according to (5.8) and (5.9).

6. Right-Sided Generalized Fractional Integration of the $H$-Function

In this section we consider the right-sided generalized fractional integration $I_{-}^{\alpha,\beta,\eta}$ defined by (2.9).
Theorem 2. Let \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \text{Re}(\beta) \neq \text{Re}(\eta) \). Let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \cdots, p; j = 1, \cdots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy
\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] > \text{Re}(\omega) < \min[\text{Re}(\beta), \text{Re}(\eta)],
\]
\[
\sigma \gamma > \text{Re}(\omega) - \min[\text{Re}(\beta), \text{Re}(\eta)].
\]
Then the generalized fractional integral \( I_{-}^{\alpha, \beta, \eta} \) of the \( H \)-function (1.1) exists and the following relation holds:
\[
\left( I_{-}^{\alpha, \beta, \eta} t^\omega H_{p,q}^{m,n} \left[ \sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right)(x)
\]
\[
= x^{\omega-\beta} H_{p+2,q+2}^m \left[ \sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\ (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right. \right].
\]

Proof. By (2.9) we have
\[
\left( I_{-}^{\alpha, \beta, \eta} t^\omega H_{p,q}^{m,n} \left[ \sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right)(x)
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} t^{\omega-\alpha-\beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) H_{p,q}^{m,n} \left[ \sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] dt.
\]
Due to (2.25), (2.26), (3.19) and (3.20), the integrand in (6.4) for any \( x > 0 \) has the asymptotic at infinity
\[
(t-x)^{\alpha-1} t^{\omega-\alpha-\beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) H_{p,q}^{m,n} \left[ \sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right]
\]
\[
= O \left( t^{\omega-\min[\text{Re}(\beta), \text{Re}(\eta)]-1+\sigma_0} \right) \quad (t \to +\infty)
\]
or
\[
= O \left( t^{\omega-\min[\text{Re}(\beta), \text{Re}(\eta)]-1+\sigma_0[\log(t)]^N} \right) \quad (t \to +\infty).
\]
Here \( \varphi \) is given by (3.21) and \( N \) is indicated in Theorem B(ii). Therefore the condition (6.1) ensures the existence of the integral (6.4). Applying (1.2), making the change \( t = 1/\tau \) and using (5.5), we obtain
\[
\left( I_{-}^{\alpha, \beta, \eta} t^\omega H_{p,q}^{m,n} \left[ \sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right) \left( \frac{1}{x} \right)
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_{1/x}^{\infty} \left( t - \frac{1}{x} \right)^{\alpha-1} t^{\omega-\alpha-\beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{1}{tx} \right) H_{p,q}^{m,n} \left[ \sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] dt.
\]
Since $\mathcal{L} = \mathcal{L}_{\gamma\infty}$, $\text{Re}(s) = \gamma$ and therefore the condition (6.2) guarantees the existence of the Mellin-Barnes integral above. Replacing in (6.5) $x$ by $1/x$, we obtain (6.3).

**Corollary 2.1.** Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \ (i = 1, \ldots, p; j = 1, \ldots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) + \text{Re}(\alpha) < 0, \tag{6.6}
\]

\[
\sigma \gamma > \text{Re}(\omega) + \text{Re}(\alpha). \tag{6.7}
\]

Then the Riemann-Liouville fractional integral $I^\alpha_0$ of the $H$-function (1.1) exists and the following relation holds:

\[
\left( I^\alpha_0 \frac{\omega}{1} H_{m,n}^{p,q} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right) (x) = x^{\omega+\alpha} H_{p+1,q+1}^{m+1,n+1} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \tag{6.8}
\]

**Corollary 2.2.** Let $\alpha, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \ (i = 1, \ldots, p; j = 1, \ldots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \text{Re}(\eta), \tag{6.9}
\]

\[
\sigma \gamma > \text{Re}(\omega) - \text{Re}(\eta). \tag{6.10}
\]

Then the Erdélyi-Kober fractional integral $K_{\eta,\alpha}^\omega$ of the $H$-function (1.1) exists and the following relation holds:

\[
\left( K_{\eta,\alpha}^\omega \frac{\omega}{1} H_{m,n}^{p,q} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right) (x) = x^\omega H_{p+1,q+1}^{m+1,n+1} \left[ \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \tag{6.11}
\]

**Remark 4.** In the case $a^* > 0, \Delta \geq 0$ the relation of the form (6.3) was indicated in [16, (4.3)]. But it includes a mistake and should be replaced by (6.3) with the conditions (6.1) and (6.2).
Remark 5. Corollary 2.1 coincides with Theorem 2 in [7]. For real $\alpha > 0$ and $\alpha^* > 0$ the relation (6.8) was indicated in [18, (2.5)], but the conditions of its validity have to be also corrected in accordance with (6.6) and (6.7).

7. Left-Sided Generalized Fractional Differentiation of the $H$-Function

Now we treat the left-sided generalized fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ given by (2.11).

Theorem 3. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0, \text{Re}(\alpha + \beta + \eta) \neq 0$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ ($i = 1, \ldots, p; j = 1, \ldots, q$) and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$\sigma \min_{1 \leq j \leq m} \frac{\text{Re}(b_j)}{\beta_j} + \text{Re}(\omega) + \min[0, \text{Re}(\alpha + \beta + \eta)] + 1 > 0, \quad (7.1)$$

$$\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\alpha + \beta + \eta)] + 1. \quad (7.2)$$

Then the generalized fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ of the $H$-function (1.1) exists and the following relation holds:

$$\left( D_{0+}^{\alpha,\beta,\eta} \omega H_{p,q}^{m,n} \left[ x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right)(x)$$

$$= x^{\omega + \beta} H_{p+2,q+2}^{m,n+2} \left[ x^\sigma \left| \begin{array}{c} (-\omega, \sigma), (-\omega - \eta - \alpha - \beta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \beta, \sigma), (-\omega - \eta, \sigma) \end{array} \right. \right]. \quad (7.3)$$

Proof. Let $n = [\text{Re}(\alpha)] + 1$. From (2.11) we have

$$\left( D_{0+}^{\alpha,\beta,\eta} \omega H_{p,q}^{m,n} \left[ x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right)(x)$$

$$= \left( \frac{d}{dx} \right)^n \left( I_{0+}^{\alpha+n,-\beta-n,\alpha+n-\eta-n} \omega H_{p,q}^{m,n} \left[ x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right)(x), \quad (7.4)$$

which exists according to Theorem 1 with $\alpha, \beta$ and $\eta$ being replaced by $-\alpha + n, -\beta - n$ and $\alpha + \eta - n$, respectively. Then we find

$$\left( D_{0+}^{\alpha,\beta,\eta} \omega H_{p,q}^{m,n} \left[ x^\sigma \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right. \right] \right)(x)$$

$$= \left( \frac{d}{dx} \right)^n x^{\omega + \beta + n} H_{p+2,q+2}^{m,n+2} \left[ x^\sigma \left| \begin{array}{c} (-\omega, \sigma), (-\omega - \alpha - \beta - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \beta - n, \sigma), (-\omega - \eta, \sigma) \end{array} \right. \right]. \quad (7.5)$$
Taking into account the differentiation formula (4.1) we have

\[
\left( D_{0}^{\alpha, \beta, \eta} \right) \left( \frac{t^\omega}{p^q} \left| \frac{m,n}{1} \right| \left( a_i, \alpha_i \right)_{1,p} \frac{b_j, \beta_j}{1,q} \right) (x) = x^{\omega+\beta} H_{pq}^{m,n} + \iota 3. + \iota 3 + \iota 3 \left| \frac{\sigma}{(\omega, \beta)} \right| \frac{1}{\iota} \frac{\sigma}{(\omega-\alpha-\beta-\eta, \sigma)} \frac{a_i, \alpha_i}{1,p} \frac{b_j, \beta_j}{1,q} \right) \right],
\]

(7.6)

and Lemma 2 and the reduction relation (4.1) imply (7.3), which completes the proof of theorem.

**Corollary 3.1.** Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \ldots, p; j = 1, \ldots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy the conditions in (5.8) and (5.9). Then the Riemann-Liouville fractional derivative \( D_{0+}^{\alpha} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( D_{0+}^{\alpha} \right) \left( \frac{t^\omega}{p^q} \left| \frac{m,n}{1} \right| \left( a_i, \alpha_i \right)_{1,p} \frac{b_j, \beta_j}{1,q} \right) (x) = x^{\omega-\alpha} H_{pq}^{m,n+1} + \iota 3. + \iota 3 \left| \frac{\sigma}{(\omega, \beta)} \right| \frac{1}{\iota} \frac{\sigma}{(\omega-\alpha, \sigma)} \frac{a_i, \alpha_i}{1,p} \frac{b_j, \beta_j}{1,q} \right) \right].
\]

(7.7)

**Remark 6.** For real \( \alpha > 0 \) and \( \alpha' > 0 \) the relation (7.3) was given in [18, (2.7.13)], but the conditions of its validity have to be corrected in accordance with (7.1) and (7.2).

**Remark 7.** Corollary 3.1 coincides with Theorem 3 in [7].

### 8. Right-Sided Generalized Fractional Differentiation of the \( H \)-Function

Here we deal with the right-sided generalized fractional derivative \( D_{-}^{\alpha, \beta, \eta} \) given by (2.12).

**Theorem 4.** Let \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \text{Re}(\alpha + \beta + \eta) + |\text{Re}(\alpha)| + 1 \neq 0 \). Let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \ldots, p; j = 1, \ldots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) + \max[\text{Re}(\beta) + |\text{Re}(\alpha)| + 1, -\text{Re}(\alpha + \eta)] < 0,
\]

(8.1)

\[
\sigma \gamma > \text{Re}(\omega) + \max[\text{Re}(\beta) + |\text{Re}(\alpha)| + 1, -\text{Re}(\alpha + \eta)].
\]

(8.2)

Then the generalized fractional derivative \( D_{-}^{\alpha, \beta, \eta} \) of the \( H \)-function (1.1) exists and the following relation holds:

\[
\left( D_{-}^{\alpha, \beta, \eta} \right) \left( \frac{t^\omega}{p^q} \left| \frac{m,n}{1} \right| \left( a_i, \alpha_i \right)_{1,p} \frac{b_j, \beta_j}{1,q} \right) (x) = (-1)^{|\text{Re}(\alpha)|+1} x^{\omega+\beta} H_{pq}^{m+2,n+2} + \iota 3. + \iota 3 \left| \frac{\sigma}{(\omega, \beta)} \right| \frac{1}{\iota} \frac{\sigma}{(\omega-\alpha-\beta-\eta, \sigma)} \frac{(a_i, \alpha_i)}{1,p} \frac{(b_j, \beta_j)}{1,q} \right) \right].
\]

(8.3)
Proof. Let \( n = \text{Re}(\alpha) + 1 \). Owing to (2.12) we have

\[
\left( D_{-}^{\alpha, \beta, \eta} I_{p,q}^{m,n} \left[ t^\sigma \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \right)(x)
= \left( -\frac{d}{dx} \right)^n \left( I_{-\alpha+n-\beta-n, \alpha+n+\eta} I_{p,q}^{m,n} \left[ t^\sigma \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \right)(x), \tag{8.4}
\]

which exists according to Theorem 2 with \( \alpha, \beta \) and \( \eta \) being replaced by \(-\alpha + n, -\beta - n\) and \( \alpha + \eta \), respectively. Then applying the differentiation formula (4.5), similarly to (7.5), (7.6), we find in view of the reduction formula (1.2) that

\[
\left( D_{-}^{\alpha, \beta, \eta} I_{p,q}^{m,n} \left[ t^\sigma \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \right)(x)
= \left( -1 \right)^n x^{\omega+\beta+m+3} \left[ x^\sigma \begin{array}{c}
(a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega - \beta + \eta, \sigma) \\
(-\omega - \beta - n, \sigma), (-\omega + \alpha + \eta, \sigma), (b_j, \beta_j)_{1,q}
\end{array} \right],
\]

which implies the formula (8.3).

Corollary 4.1. Let \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), and let the constants \( a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0 \) \((i = 1, \ldots, p; j = 1, \ldots, q)\) and \( \omega \in \mathbb{C}, \sigma > 0 \) satisfy

\[
\sigma \max_{1 \leq i \leq n} \frac{\text{Re}(a_i) - 1}{\alpha_i} + \text{Re}(\omega) - \{\text{Re}(\alpha)\} + 1 < 0, \tag{8.5}
\]

\[
\sigma \gamma + \text{Re}(\omega) - \{\text{Re}(\alpha)\} + 1 > 0. \tag{8.6}
\]

Then the Riemann-Liouville fractional derivative \( D^\alpha \) of the \( II \)-function (1.1) exists and there holds the relation:

\[
\left( D_{-}^{\alpha, \omega} I_{p,q}^{m,n} \left[ t^\sigma \begin{array}{c}
(a_i, \alpha_i)_{1,p} \\
(b_j, \beta_j)_{1,q}
\end{array} \right] \right)(x)
= (-1)^{\text{Re}(\alpha)+1} x^{\omega-\alpha} I_{p+1,q+1}^{m+3,n} \left[ x^\sigma \begin{array}{c}
(a_i, \alpha_i)_{1,p}, (-\omega, \sigma) \\
(-\omega + \alpha, \sigma), (b_j, \beta_j)_{1,q}
\end{array} \right]. \tag{8.7}
\]

Remark 8. The relation of the form (8.7) with real \( \alpha > 0 \) and \( \alpha^* > 0 \) was proved in [13, formula (14a)] (see also [12], [14, (2.2)] and [18, (2.7.9)]). But such a formula contains mistakes and should be replaced by (8.7) with the condition (8.5) and (8.6).
Remark 9. When $\alpha = k \in \mathbb{N}$, the relations (7.7) and (8.7) coincide with (4.4) and (4.5), respectively.

9. Generalized Fractional Integro-Differentiation of the $H$-Function

Here we investigate the generalized fractional integro-differentiation operators $I_{0+}^{\alpha,\beta, \eta}$ and $I_{-}^{\alpha,\beta, \eta}$ given by (2.8) and (2.10). The following statements are proved similarly to Theorems 3 and 4 by using the relations (2.8) and (2.10), Theorems 1 and 2, and the properties of the $H$-function in Sections 3 and 4.

Theorem 5. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) \leq 0, \text{Re}(\beta) \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \ldots, p; j = 1, \ldots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$
\sigma \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right] + \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1 > 0,
$$

(9.1)

$$
\sigma \gamma < \text{Re}(\omega) + \min[0, \text{Re}(\eta - \beta)] + 1.
$$

(9.2)

Then the generalized fractional integro-differentiation $I_{0+}^{\alpha,\beta, \eta}$ of the $H$-function (1.1) exists and there holds the relation

$$
\left( I_{0+}^{\alpha,\beta, \eta} \omega H_{m,n}^{p,q} \left[ \begin{array}{c} \sigma^{(a_i, \alpha_i)_{1,p}} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right)(x)
= x^{\omega - \beta} H_{m,n+2}^{p+2,q+2} \left[ x^\sigma \begin{array}{c} (-\omega, \sigma), (-\omega - \eta + \beta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \end{array} \right].
$$

(9.3)

Theorem 6. Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) \leq 0, \text{Re}(\beta) + [\text{Re}(\alpha)] - 1 \neq \text{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \ldots, p; j = 1, \ldots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy

$$
\sigma \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right] + \text{Re}(\omega) < \min[\text{Re}(\beta) - [\text{Re}(\alpha)] - 1, \text{Re}(\eta)],
$$

(9.4)

$$
\sigma \gamma > \text{Re}(\omega) - \min[\text{Re}(\beta) - [\text{Re}(\alpha)] - 1, \text{Re}(\eta)].
$$

(9.5)

Then the generalized fractional integro-differentiation $I_{-}^{\alpha,\beta, \eta}$ of the $H$-function (1.1) exists and there holds the relation

$$
\left( I_{-}^{\alpha,\beta, \eta} \omega H_{m,n}^{p,q} \left[ \begin{array}{c} \sigma^{(a_i, \alpha_i)_{1,p}} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \right)(x)
= x^{\omega - \beta} H_{m+2,n}^{p+2,q+2} \left[ x^\sigma \begin{array}{c} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\ (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{array} \right].
$$

(9.6)
Remark 10. The relation (9.3) with \( a^* > 0, \Delta \geq 0 \) was indicated in [16, (4.2)], but conditions of its validity have to be corrected in accordance with (9.1) and (9.2).

Remark 11. The relations (9.3) and (9.6) for the fractional integro-differentiation operators \( I_{0+}^\alpha, \beta, \eta \) and \( I_{-}^\alpha, \beta, \eta \), defined in (2.8) and (2.10) for \( \alpha \in \mathbb{C}, \Re(\alpha) \leq 0 \) coincide with that (5.3) and (6.3) for the fractional integration operators \( \mathcal{I}_0^\alpha, \beta, \eta \) and \( \mathcal{I}_{-}^\alpha, \beta, \eta \), defined in (2.7) and (2.9) for \( \alpha \in \mathbb{C}, \Re(\alpha) > 0 \). Though the conditions for validity of (5.3) and (9.3) in Theorems 1 and 5 have the same form, that of (6.3) and (9.6) presented in Theorems 2 and 4 are slightly different.

In conclusion we note that, as it was mentioned in Remarks 2, 4 and 10, the relations (5.3), (6.3) and (9.3) for generalized calculus operator \( \mathcal{I}_0^\alpha, \beta, \eta \) were already known in the case \( a^* > 0, \Delta \geq 0 \). Further, Remarks 3, 5, 6 and 8 indicate that the relations (5.10) and (6.8) for the Riemann-Liouville fractional integrals \( \mathcal{I}_0^\alpha, \beta, \eta \) and \( \mathcal{I}_{-}^\alpha, \beta, \eta \), and (8.7) for the fractional derivative \( D_0^\alpha, \beta, \eta \), in the case \( \Re(\alpha) > 0 \) and \( a^* > 0 \) were established. However, the II-function’s asymptotic estimates (3.16), (3.17) at zero and (3.19), (3.20) at infinity allow us to prove such results under more general assumptions \( a^* > 0 \) and \( a^* = 0, \Delta \gamma + \Re(\mu) < -1 \).

References


