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UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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Abstract

In this work some integral operators are studied and the authors determine conditions for the univalence of these integral operators.

1 INTRODUCTION

Let $A$ be the class of the functions $f$ which are analytic in the unit disc $U = \left\{ z \in \mathbb{C} ; |z| < 1 \right\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $S$ the class of the function $f \in A$ which are univalent in $U$.

Many authors studied the problem of integral operators which preserve the class $S$. In this sense an important result is due to J. Pfaltzgraff [4].

THEOREM A [4]. If $f$ is univalent in $U$, $\alpha$ a complex number and $|\alpha| \leq \frac{1}{4}$, then the function

$$G_\alpha(z) = \int_{0}^{z} \left[ f'(\xi) \right]^\alpha d\xi$$

(1)

is univalent in $U$.

THEOREM B [3]. If the function $g \in S$ and $\alpha$ is a complex number, $|\alpha| \leq \frac{1}{4n}$, then the function defined by

$$G_{\alpha,n}(z) = \int_{0}^{z} \left[ g'(u^n) \right]^\alpha du$$

(2)

is univalent in $U$ for all positive integer $n$.

2 PRELIMINARY RESULTS

We will need the following theorems in this work.

THEOREM C [2]. Let $\alpha$ be a complex number, $\text{Re} \alpha > 0$ and $f \in A$. If
\[ \frac{1 - |z|^2 \Re \alpha}{\Re \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \]  
for all \( z \in U \), then for any complex number \( \beta, \Re \beta \geq \Re \alpha \) the function 
\[ F_\beta(z) = \left[ \beta \int_0^z u^{\beta-1} f'(u)du \right]^{\frac{1}{\beta}} \]
is in the class \( S \).

**THEOREM D [1].** If the function \( g \) is regular in \( U \) and \( |g(z)| < 1 \) in \( U \), then for all \( \xi \in U \) and \( z \in U \) the following inequalities hold 
\[ \left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \overline{z}\xi} \right| \]  
and 
\[ |g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}, \]
the equalities hold only in the case \( g(z) = \epsilon \frac{\ast + u}{1 + \overline{u}\ast} \) \( \ast \) where \( |\epsilon| = 1 \) and \( |u| < 1 \).

**REMARK [1].** For \( z = 0 \), from inequality (5) 
\[ \left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)} \right| \leq |\xi| \]  
and, hence 
\[ |g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}. \]
Considering \( g(0) = a \) and \( \xi = z \), 
\[ |g(z)| \leq \frac{|z| + |a|}{1 + |a||z|}. \]
for all \( z \in U \).

**LEMMA SCHWARZ [1].** If the function \( g \) is regular in \( U \), \( g(0) = 0 \) and \( |g(z)| \leq 1 \) for all \( z \in U \), then the following inequalities hold 
\[ |g(z)| \leq |z| \]  
for all \( z \in U \), and \( |g'(0)| \leq 1 \), the equalities (in inequality (10) for \( z \neq 0 \)) hold only in the case \( g(z) = \epsilon z \), where \( |\epsilon| = 1 \).

### 3 MAIN RESULTS

**THEOREM 1.** Let \( \alpha, \gamma \) be complex numbers, \( \Re \alpha = a > 0 \) and \( g \in A \). If 
\[ \left| \frac{g''(z)}{g'(z)} \right| \leq \frac{1}{n} \]
for all \( z \in U \) and
\[ |\gamma| \leq \frac{n + 2a}{2} \left( \frac{n + 2a}{n} \right)^{\frac{a}{2}} \]  \hspace{1cm} (12)

then for any complex number \( \beta \), \( \Re \beta \geq a \), the function

\[ G_{\beta, \gamma, n}(z) = \left\{ \beta \int_{0}^{z} u^{\beta-1} \left[ g'(u^n) \right]^\gamma du \right\}^\frac{1}{\gamma} \]  \hspace{1cm} (13)

is in the class \( S \) for all \( n \in N^* \setminus \{1\} \).

**Proof.** Let us consider the function

\[ f(z) = \int_{0}^{z} \left[ g'(u^n) \right]^\gamma du. \]  \hspace{1cm} (14)

The function

\[ p(z) = \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)}, \]  \hspace{1cm} (15)

where the constant \( |\gamma| \) satisfies the inequality (12), is regular in \( U \).

From (15) and (14) we obtain

\[ p(z) = \frac{\gamma}{|\gamma|} \left[ \frac{nx^{n-1}g''(z^n)}{g'(z^n)} \right]. \]  \hspace{1cm} (16)

Using (16) and (11) we obtain

\[ |p(z)| < 1 \]  \hspace{1cm} (17)

for all \( z \in U \). For \( z = 0 \) we have \( p(0) = 0 \).

From (16) and Schwarz-Lemma it results that

\[ \frac{1}{|\gamma|} \left| \frac{f''(z)}{f'(z)} \right| \leq |z|^{n-1} \leq |z| \]  \hspace{1cm} (18)

for all \( z \in U \), and hence

\[ \left( \frac{1 - |z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \left( \frac{1 - |z|^{2a}}{a} \right) |z|^n. \]  \hspace{1cm} (19)

Let us consider \( Q:[0,1] \rightarrow R , \quad Q(x) = \frac{(1-x^{2a})}{a} x^n \), \( x = |x| \). We have

\[ Q(x) \leq \frac{2}{n + 2a} \left( \frac{n}{n + 2a} \right)^{\frac{a}{2}} \]  \hspace{1cm} (20)

for all \( x \in [0,1] \). From (20), (19) and (12) we obtain

\[ \left( \frac{1 - |z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \]  \hspace{1cm} (21)

for all \( z \in U \). Then, from (21) and Theorem C it follows that the function \( G_{\beta, \gamma, n} \), is in the class \( S \).
**Theorem 2.** Let $\alpha, \gamma$ be complex numbers, $\Re \alpha = b > 0$ and the function $g \in A$, 

$$g(z) = z + a_2 z^2 + \ldots$$

If

$$\left| \frac{g''(z)}{g'(z)} \right| < 1$$

for all $z \in U$ and the constant $|\gamma|$ satisfies the condition

$$|\gamma| \leq \frac{1}{\max_{|z| \leq 1} \left[ \frac{1 - |z|^{2b}}{b} |x| \left| \frac{|z| + 2|a_2|}{1 + 2|a_2| |z|} \right| \right]}$$

then for any complex number $\beta$, $\Re \beta \geq b$ the function

$$G_{\beta, \gamma}(z) = \left\{ \beta \int_{0}^{z} u^{\beta-1} [g'(u)]^{\gamma} du \right\}^{\frac{1}{\beta}}$$

is in the class $S$.

**Proof.** Let us consider the function

$$f(z) = \int_{0}^{z} [g'(u)]^{\gamma} du.$$  \hspace{1cm} (25)

The function

$$h(z) = \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)},$$

where the constant $|\gamma|$ satisfies the inequality (23), is regular in $U$.

From (26) and (25) we have

$$h(z) = \frac{\gamma}{|\gamma|} \frac{g''(z)}{g'(z)}.$$ \hspace{1cm} (27)

Using (27) and (22) we obtain

$$|h(z)| < 1,$$ \hspace{1cm} (28)

for all $z \in U$ and $|h(0)| = 2|a_2|$.

The above Remark applied to the function $h$ gives

$$\frac{1}{|\gamma|} \left| \frac{f''(z)}{f'(z)} \right| \leq \frac{|z| + 2|a_2|}{1 + 2|a_2||z|}$$ \hspace{1cm} (29)

for all $z \in U$.

From (29) we obtain

$$\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \left[ \frac{1 - |z|^{2b}}{b} \right] \frac{|z| + 2|a_2|}{1 + 2|a_2||z|}$$ \hspace{1cm} (30)

for all $z \in U$. Hence, we have

$$\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \max_{|z| \leq 1} \left[ \frac{1 - |z|^{2b}}{b} \right] \frac{|z| + 2|a_2|}{1 + 2|a_2||z|}.$$ \hspace{1cm} (31)
From (31) and (23) we obtain
\[
\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1
\]  
for all \( z \in U \). From Theorem C, it follows that the function \( G_{\beta,\gamma} \) defined by (24) is in the class \( S \).

References


