<table>
<thead>
<tr>
<th>Title</th>
<th>UNIVALENCE OF CERTAIN INTEGRAL OPERATORS (Applications of Complex Function Theory to Differential Equations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Pescar, Virgli; Owa, Shigeyoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1998), 1062: 84-88</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1998-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62402">http://hdl.handle.net/2433/62402</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

京都大学
UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

Virgil PESCAR  Shigeyoshi OWA

30.01.1998

Abstract
In this work some integral operators are studied and the authors determine conditions for the univalence of these integral operators.

1  INTRODUCTION

Let $A$ be the class of the functions $f$ which are analytic in the unit disc
$U = \{ z \in \mathbb{C}; |z| < 1 \}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $S$ the class of the function $f \in A$ which are univalent in $U$.

Many authors studied the problem of integral operators which preserve the class $S$. In this sense an important result is due to J. Pfaltzgraff [4].

**THEOREM A** [4]. If $f$ is univalent in $U$, $\alpha$ a complex number and $|\alpha| \leq \frac{1}{4}$, then the function

$$ G_\alpha(z) = \int_0^z [f'(\xi)]^\alpha d\xi $$

is univalent in $U$.

**THEOREM B**[3]. If the function $g \in S$ and $\alpha$ is a complex number, $|\alpha| \leq \frac{1}{4n}$, then the function defined by

$$ G_{\alpha,n}(z) = \int_0^z [g'(u^n)]^\alpha du $$

is univalent in $U$ for all positive integer $n$.

2  PRELIMINARY RESULTS

We will need the following theorems in this work.

**THEOREM C** [2]. Let $\alpha$ be a complex number, $\text{Re} \alpha > 0$ and $f \in A$. If
for all $z \in U$, then for any complex number $\beta, \text{Re} \beta \geq \text{Re} \alpha$ the function
\[ F_\beta(z) = \left[ \beta \int_0^\infty u^{\beta-1} f'(u) du \right]^\frac{1}{\beta} \] is in the class $S$.

**THEOREM D** [1]. If the function $g$ is regular in $U$ and $|g(z)| < 1$ in $U$, then for all $\xi \in U$ and $z \in U$ the following inequalities hold
\[ \left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)} \right| \leq \frac{|\xi - z|}{1 - \overline{z}\xi} \] (5)
and
\[ |g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}, \] (6)
the equalities hold only in the case $g(z) = \epsilon \frac{z}{1 + \overline{z}u}$, where $|\epsilon| = 1$ and $|u| < 1$.

**REMARK** [1]. For $z = 0$, from inequality (5)
\[ \left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\epsilon)} \right| \leq |\xi| \] (7)
and, hence
\[ |g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\epsilon|}. \] (8)
Considering $g(0) = a$ and $\xi = z$,
\[ |g(z)| \leq \frac{|z| + |a|}{1 + |a||z|}. \] (9)
for all $z \in U$.

**LEMMA SCHWARZ** [1]. If the function $g$ is regular in $U$, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold
\[ |g(z)| \leq |z| \] (10)
for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (10) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

## 3 MAIN RESULTS

**THEOREM 1.** Let $\alpha, \gamma$ be complex numbers, $\text{Re} \alpha = a > 0$ and $g \in A$. If
\[ \left| \frac{g''(z)}{g'(z)} \right| \leq \frac{1}{n} \] (11)
for all $z \in U$ and
$$|\gamma| \leq \frac{n + 2a}{2} \left( \frac{n + 2a}{n} \right)^{\frac{n}{2}}$$  \hspace{1cm} (12)$$

then for any complex number $\beta$, $\text{Re}\beta \geq a$, the function

$$G_{\beta, \gamma, n}(z) = \left\{ \beta \int_0^z u^{\beta-1} \left[ g'(u^n) \right]^{\gamma} du \right\}^{\frac{1}{\gamma}}$$  \hspace{1cm} (13)

is in the class $S$ for all $n \in N^{\ast} - \{1\}$.

**Proof.** Let us consider the function

$$f(z) = \int_0^z \left[ g'(u^n) \right]^{\gamma} du.$$  \hspace{1cm} (14)

The function

$$p(z) = \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)},$$  \hspace{1cm} (15)

where the constant $|\gamma|$ satisfies the inequality (12), is regular in $U$.

From (15) and (14) we obtain

$$p(z) = \frac{\gamma}{|\gamma|} \left[ \frac{nz^{n-1}g''(z^n)}{g'(z^n)} \right].$$  \hspace{1cm} (16)

Using (16) and (11) we obtain

$$|p(z)| < 1$$  \hspace{1cm} (17)

for all $z \in U$. For $z = 0$ we have $p(0) = 0$.

From (16) and Schwarz-Lemma it results that

$$\left| \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)} \right| \leq |z|^{n-1} \leq |z|$$  \hspace{1cm} (18)

for all $z \in U$, and hence

$$\left( \frac{1 - |z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \left( \frac{1 - |z|^{2a}}{a} \right) |z|^n.$$  \hspace{1cm} (19)

Let us consider $Q:[0,1] \rightarrow R$, $Q(x) = \frac{(1-x^{2a})}{x^n}$, $x = |z|$. We have

$$Q(x) \leq \frac{2}{n + 2a} \left( \frac{n}{n + 2a} \right)^{\frac{n}{2a}}$$  \hspace{1cm} (20)

for all $x \in [0,1]$. From (20), (19) and (12) we obtain

$$\left( \frac{1 - |z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1$$  \hspace{1cm} (21)

for all $z \in U$. Then, from (21) and Theorem $C$ it follows that the function $G_{\beta, \gamma, n}$, is in the class $S$.  

THEOREM 2. Let $\alpha, \gamma$ be complex numbers, $\text{Re}\alpha = b > 0$ and the function $g \in A$

$g(z) = z + a_2 z^2 + \ldots$. If

\[
\left| \frac{g''(z)}{g'(z)} \right| < 1
\]  

(22)

for all $z \in U$ and the constant $|\gamma|$ satisfies the condition

\[
|\gamma| \leq \frac{1}{\max_{|z| \leq 1} \left[ \frac{1-|z|^{2b}}{b} |z| \frac{|z|+2a_2}{1+2|a_2||z|} \right]}
\]  

(23)

then for any complex number $\beta$, $\text{Re} \beta \geq b$ the function

\[
G_{\beta, \gamma}(z) = \left\{ \beta \int_{0}^{z} u^{\beta-1} \left[ g'(u) \right]^\gamma du \right\}^{\frac{1}{\gamma}}
\]  

(24)

is in the class $S$.

Proof. Let us consider the function

\[
f(z) = \int_{0}^{z} \left[ g'(u) \right]^\gamma du.
\]  

(25)

The function

\[
h(z) = \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)},
\]  

(26)

where the constant $|\gamma|$ satisfies the inequality (23), is regular in $U$.

From (26) and (25) we have

\[
h(z) = \frac{\gamma}{|\gamma|} \frac{g''(z)}{g'(z)}.
\]  

(27)

Using (27) and (22) we obtain

\[
|h(z)| < 1,
\]  

(28)

for all $z \in U$ and $|h(0)| = 2|a_2|$.

The above Remark applied to the function $h$ gives

\[
\frac{1}{|\gamma|} \left| \frac{f''(x)}{f'(x)} \right| \leq \frac{|z| + 2a_2}{1 + 2a_2||z|}
\]  

(29)

for all $z \in U$.

From (29) we obtain

\[
\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(x)} \right| \leq |\gamma| \frac{1 - |z|^{2b}}{b} \frac{|z| + 2a_2}{1 + 2a_2||z|}
\]  

(30)

for all $z \in U$. Hence, we have

\[
\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(x)} \right| \leq |\gamma| \max_{|z| \leq 1} \left[ \frac{1 - |z|^{2b}}{b} \frac{|z| + 2a_2}{1 + 2a_2||z|} \right].
\]  

(31)
From (31) and (23) we obtain
\[
\frac{1-|z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1
\]
for all $z \in U$. From Theorem C, it follows that the function $G_{\beta,\gamma}$ defined by (24) is in the class S.

References


"Transilvania" University of Brașov
Faculty of Science
Department of Mathematics
2200 Brașov
ROMANIA

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan