<table>
<thead>
<tr>
<th>Title</th>
<th>Starlikeness of Libera transformation (II) (Applications of Complex Function Theory to Differential Equations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nunokawa, Mamoru; Ota, Yoshiaki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1998年 1062号 62-68</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1998-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62406">http://hdl.handle.net/2433/62406</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
1. Introduction.

Let $A$ denote the class of function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in $E = \{z : |z| < 1\}$.

A function $f(z) \in A$ is called to be starlike if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in} \quad E.$$

Similarly, $f(z) \in A$ is called to be convex if and only if

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in} \quad E.$$

The following interesting results are due to Libera [2].

**Theorem A.** If $f(z)$ is starlike in $E$, then so does the function $F(z)$, defined by

$$F(z) = \frac{2}{z} \int_{0}^{z} f(t) dt.$$

**Theorem B.** If $f(z)$ is convex in $E$, then so does the function $F(z)$, defined by

$$F(z) = \frac{2}{z} \int_{0}^{z} f(t) dt.$$

On the other hand, S. Singh and R. Singh [5, Theorem 1] and [4, Theorem 1] proved the following Theorem C and D and Nunokawa [3] proved Theorem E.

**Theorem C.** If $f(z) \in A$ and $\Re f'(z) > 0$ in $E$, then the function

$$F(z) = \frac{c+1}{z^c} \int_{0}^{z} t^{c-1} f(t) dt,$$

is starlike in $E$ for all $c$, $-1 < c \leq 0$. 
Theorem D. If \( f(z) \in A \) and \( |zf''(z)/f'(z)| < 3/2 \) in \( E \), then the function

\[
F(z) = \frac{2}{z} \int_{0}^{z} f(t) dt.
\]
is convex.

**Theorem E.** Let \( f(z) \in A \) and suppose that

\[
\text{Re} f'(z) > \frac{1}{12} \left\{ \log \frac{4}{e} \right\} \left( \tan^{2} \frac{\pi}{2} - 3 \right)
\]
in \( E \).

where

\[
-0.01759 < \frac{1}{12} \left\{ \log \frac{4}{e} \right\} \left( \tan^{2} \frac{\pi}{2} - 3 \right) < -0.01751.
\]

Then \( F(z) \) is starlike in \( E \), where

\[
F(z) = \frac{2}{z} \int_{0}^{z} f(t) dt.
\]

2. Preliminary.

**Lemma.** Let \( f(z) \in A \) and \( f(z) \neq 0 \) in \( 0 < |z| < 1 \) and suppose that

\[
|\arg f'(z)| < \frac{\pi}{2} \alpha
\]
in \( E \)

where

\[
\alpha = \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta}{2} \quad \text{and} \quad \beta > 0.
\]

Then we have

\[
|\arg g'(z)| < \frac{\pi}{2} \alpha
\]
in \( E \)

and

\[
|\arg \frac{g(z)}{z}| < \frac{\pi}{2} \gamma
\]
in \( E \),

where

\[
0 < \gamma, \quad \beta = \gamma + \frac{2}{\pi} \tan^{-1} \gamma
\]

and

\[
g(z) = \frac{2}{z} \int_{0}^{z} f(t) dt.
\]
Proof. From the hypothesis, it follows that

\[ f(z) = g(z) + zg'(z), \]

\[ f'(z) = 2g'(z) + zg''(z) \]

and so

\[ \arg f'(z) = \arg g'(z) + \arg(2 + \frac{zg''(z)}{g'(z)}). \]

Now, if there exists a point \( z_0 \in E \) such that

\[ |\arg g'(z)| < \frac{\pi}{2} \beta \quad \text{for} \quad |z| < |z_0| \]

and

\[ |\arg g(z_0)| = \frac{\pi}{2} \beta \]

Then, from [3, Lemma], we have

\[ \frac{z_0g''(z_0)}{g'(z_0)} = i\beta k \]

where

\[ k \geq \frac{1}{2}(a + \frac{1}{a}) \quad \text{when} \quad \arg g'(z_0) = \frac{\pi}{2} \beta \]

and

\[ k \leq -\frac{1}{2}(a + \frac{1}{a}) \quad \text{when} \quad \arg g'(z_0) = -\frac{\pi}{2} \beta \]

where

\[ p(z_0)^{\frac{1}{\beta}} = \pm ia, \quad a > 0. \]

If \( \arg g'(z_0) = \pi \beta/2 \), then we have

\[ \arg f'(z_0) = \arg g'(z_0) + \arg(2 + \frac{zg''(z_0)}{g'(z_0)}) \]

\[ = \frac{\pi}{2} \beta + \arg(2 + i\beta k) \]

\[ \geq \frac{\pi}{2} \left( \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta}{2} \right). \]
This contradicts the hypothesis. On the other hand, if \( \arg g'(z_0) = -\pi \beta /2 \), then we have

\[
\arg f'(z_0) = -\frac{\pi}{2} \beta + \arg (2 + i\beta k)
\]
\[
\leq -\frac{\pi}{2} - \tan^{-1} \frac{\beta}{2}
\]
\[
= -\frac{\pi}{2} (\beta + \frac{2}{\pi} \tan^{-1} \frac{\beta}{2}).
\]

This also contradicts the hypothesis. Therefore we must have

\[
| \arg g(z) | < \frac{\pi}{2} \beta \quad \text{in} \quad E.
\]

Putting

\[
p(z) = \frac{g(z)}{z}, \quad p(0) = 1,
\]

then we have

\[
g'(z) = p(z) + zp'(z)
\]

and

\[
\arg g'(z) = \arg p(z) + \arg (1 + \frac{zp'(z)}{p(z)}).
\]

If there exists a point \( z_0 \in E \) such that

\[
| \arg p(z) | < \frac{\pi}{2} \gamma \quad \text{for} \quad |z| < |z_0|
\]

and

\[
| \arg p(z_0) | = \frac{\pi}{2} \gamma,
\]

then we have

\[
\frac{z_0 p'(z_0)}{p(z_0)} = i\gamma k
\]

where

\[
k \geq \frac{1}{2} (a + \frac{1}{a}) \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2} \gamma
\]

and

\[
k \leq -\frac{1}{2} (a + \frac{1}{a}) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2} \gamma
\]
where 
\[ \sqrt{p(z_0)} = \pm ia, \quad a > 0. \]

Then, applying the same method as the above, we have
\[ |\arg g'(z_0)| < \frac{\pi}{2}(\gamma + \frac{2}{\pi}\tan^{-1}\gamma). \]

This contradicts the hypothesis. Therefore we must have
\[ |\arg \frac{g(z)}{z}| < \frac{\pi}{2}\gamma \quad \text{in} \quad E. \]

3. Main result.

Theorem. Let \( f(z) \in A, f(z) \neq 0 \) in \( 0 < |z| < 1 \),

\[ |\arg f'(z)| < \frac{\pi}{2}\left(\frac{5}{3} - \gamma_0\right) \quad \text{in} \quad E \]

where \( \gamma_0 \) is the smallest positive root of the equation
\[ \frac{5}{6} = 2\gamma + \frac{2}{\pi}\tan^{-1}\gamma + \frac{2}{\pi}\tan^{-1}\frac{1}{2}(\gamma + \frac{2}{\pi}\tan^{-1}\gamma) \]

and \( 0.266 < \gamma_0 < 0.267 \).

Let us put

\[ g(z) = \frac{2}{z} \int_0^z f(t)dt. \]

Then \( g(z) \) is starlike in \( E \).

Proof. From (1) and Lemma, we easily have
\[ |\arg \frac{g(z)}{z}| < \frac{\pi}{2}\gamma_0 \quad \text{in} \quad E. \]

From (2), we have
\[ 2f(z) = g(z) + zg'(z) \]

and
\[ 2f'(z) = 2g'(z) + zg''(z) \]

Let us put
\[ \frac{zg'(z)}{g(z)} = \frac{1 + w(z)}{1 - w(z)}, \quad w(0) = 0 \]

then \( w(z) \) is analytic in \( E \) and \( w(z) \neq 1 \) in \( E \).
Then it follows that

\[ 2f'(z) = 2g'(z) + zg''(z) \]

\[ = zg'(z) \left[ \frac{1 + w(z)}{1 - w(z)} \right]^2 + \frac{2zw'(z)}{(1 - w(z))^2} + \frac{1 + w(z)}{1 - w(z)} \].

If there exists a point \( z_0 \in E \) such that

\[ \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1, \]

then from Jack's lemma [1, Lemma], we have

\[ \frac{z_0w'(z_0)}{w(z_0)} = k \geq 1 \]

Putting \( w(z_0) = e^{i\theta}, 0 \leq \theta < 2\pi \), we have

\[ 2f'(z_0) = 2g'(z_0) + z_0g''(z_0) \]

\[ = g(z_0) \left[ \frac{(1 + e^{i\theta})^2}{1 - e^{i\theta}} + \frac{2ke^{i\theta}}{(1 - e^{i\theta})^2} + \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right] \]

\[ = g(z_0) \left[ -\frac{\sin^2 \theta}{(1 - \cos \theta)^2} - \frac{k}{1 - \cos \theta} + \frac{i \sin \theta}{1 - \cos \theta} \right]. \]

Then we have

\[ \arg f'(z_0) \geq \arg \left( -\frac{\sin^2 \theta + k(1 - \cos \theta)}{(1 - \cos \theta)^2} - \frac{i \sin(1 - \cos \theta)}{(1 - \cos \theta)^2} \right) - |\arg g(z_0)| \]

\[ \geq \arg \left( -\frac{\sin^2 \theta + 1 - \cos \theta}{(1 - \cos \theta)^2} - \frac{i \sin \theta(1 - \cos \theta)}{(1 - \cos \theta)^2} \right) - \frac{\pi}{2} \gamma_0 \]

\[ = \pi - \tan^{-1} \frac{|\sin \theta|(1 - \cos \theta)}{\sin^2 \theta + 1 - \cos \theta} - \frac{\pi}{2} \gamma_0 \]

\[ = \pi - \tan^{-1} \frac{\sin \theta}{2 + \cos \theta} - \frac{\pi}{2} \gamma_0 \]

\[ \geq \pi - \tan^{-1} \frac{1}{\sqrt{3}} - \frac{\pi}{2} \gamma_0 \]

\[ = \frac{5}{6} \pi - \frac{\pi}{2} \gamma_0 \]

\[ = \frac{\pi}{2} \left( \frac{5}{3} - \gamma_0 \right). \]

This contradicts (1). Therefore we must have

\[ |w(z)| < 1 \quad \text{in} \quad E. \]

This show that

\[ \Re \frac{zg'(z)}{g(z)} > 0 \quad \text{in} \quad E. \]

or \( g(z) \) is starlike in \( E \).
References


Mamoru Nunokawa
Yoshiaki Ota
Department of Mathematics
University of Gunma
Aramaki, Maebashi, Gunma 371, Japan