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ON THE ORDER OF STRONGLY MEROMORPHIC STARLIKENESS OF STRONGLY MEROMORPHIC CONVEX FUNCTIONS

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ABSTRACT. In [1], Nunokawa proved that if \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is analytic in \(|z| < 1\) and
\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2} \alpha(\beta) \quad \text{in} \quad |z| < 1.
\]
where \( \alpha(\beta) \) satisfies the condition of Theorem A, then
\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \beta \quad \text{in} \quad |z| < 1.
\]

It is the purpose of the present paper that if \( f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \) is analytic in \( 0 < |z| < 1 \) and
\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| > \frac{\pi}{2} (1 + \delta(\beta)) \quad \text{in} \quad |z| < 1.
\]
where \( \delta(\beta) \) satisfies the condition of the Main Theorem, then we have
\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| > \frac{\pi}{2} (1 + 1 - \beta) \quad \text{in} \quad |z| < 1.
\]

1. Introduction

Let \( \Sigma \) denote the class of the form
\[
f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n
\]
which are analytic and univalent in the punctured disk \( \mathcal{E} = \{ z : 0 < |z| < 1 \} \).

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A function \( f(z) \in \Sigma \) is called to be strongly meromorphic starlike of order \( \beta \) \((-1 < \beta < 1)\) if
\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| > \frac{\pi}{2} (1 + \beta) \quad \text{in} \quad \mathcal{E}.
\]
We denote by \( SMS(\beta) \) the class of all strongly meromorphic starlike functions of order \( \beta \). Similarly, a function \( f(z) \in \Sigma \) is called to be strongly meromorphic convex of order \( \beta \) \((-1 < \beta < 1)\) if
\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| > \frac{\pi}{2} (1 + \beta) \quad \text{in} \quad \mathcal{E}.
\]
We denote by \( SMc(\beta) \) the class of all strongly meromorphic convex function of order \( \beta \).

In [1], Nunokawa obtained the following theorem.

**Theorem A.** If \( f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n} \) is analytic in \( |z| < 1 \) and
\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2} \alpha(\beta) \quad \text{in} \quad |z| < 1
\]
where \( 0 < \beta \leq 1 \),
\[
\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta q(\beta) \sin \frac{\pi}{2} (1 - \beta)}{p(\beta) + \beta q(\beta) \cos \frac{\pi}{2} (1 - \beta)},
\]
\[
p(\beta) = (1 + \beta)^{1+\beta} \quad \text{and} \quad q(\beta) = (1 - \beta)^{\frac{\beta-1}{2}},
\]
then we have
\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \beta \quad \text{in} \quad |z| < 1.
\]

It is the purpose of the present paper to obtain an analogous result for meromorphic starlike and convex functions.

2. Preliminary

**Lemma.** Let \( p(z) \) be analytic in \( \mathcal{U} = \{ z : |z| < 1 \} \), \( p(0) = 1, \ p(z) \neq 0 \) in \( \mathcal{U} \) and suppose that there exists a point \( z_{0} \in \mathcal{U} \) such that
\[
| \arg p(z) | < \frac{\pi \alpha}{2} \quad \text{for} \quad |z| < |z_{0}|
\]
and
\[
| \arg p(z_{0}) | = \frac{\pi \alpha}{2}
\]
where \( 0 < \alpha \). Then we have
\[
\frac{z_{0}p'(z_{0})}{p(z_{0})} = ik\alpha
\]
where

\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi \alpha}{2} \]

and

\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi \alpha}{2} \]

where

\[ p(z_0)^{\frac{1}{\alpha}} = \pm ia, \quad \text{and} \quad a > 0. \]

We owe this lemma to [1].

3. Main result

**Theorem.** If \( f(z) \in SMC(\delta(\beta)) \), then \( f(z) \in SMS(1 - \beta) \), where

\[
\delta(\beta) = \beta - 1 + \frac{2}{\pi} \tan^{-1} \frac{\beta q(\beta) \sin \frac{\pi}{2}(1 - \beta)}{\beta q(\beta) \cos \frac{\pi}{2}(1 - \beta) - p(\beta)},
\]

\[ p(\beta) = (1 + \beta)^{\frac{1 + \beta}{2}}, \quad q(\beta) = (1 - \beta)^{\frac{\beta + 1}{2}}, \]

and \( 0 < \beta < 1 \).

**Proof.** Let us put

\[ p(z) = -\frac{zf'(z)}{f(z)}, \quad (p(0) = 1) \]

then we have

\[ 1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} - p(z). \]

From the assumption of the theorem, we have

\[ \left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| > \frac{\pi}{2}(1 + \delta(\beta)) \quad \text{in} \quad \mathcal{U}. \]

Therefore, we have \( f'(z) \neq 0 \) in \( \mathcal{U} \), because if \( f'(z) \) has a zero at \( z = z_0 \) in \( \mathcal{U} \), then \( (1 + z \frac{f''(z)}{f'(z)}) \) can be infinite and \( \arg(1 + \frac{zf''(z)}{f'(z)}) \) can take any value of \( \theta \), for \( 0 \leq \theta \leq 2\pi \) when \( z \) approaches to \( z_0 \) from certain direction. This shows that

\[ p(z) \neq 0 \quad \text{in} \quad \mathcal{U}. \]

If there exists a point \( z_0 \in \mathcal{U} \) such that

\[ |\arg p(z)| < \frac{\pi}{2} \beta \quad \text{for} \quad |z| < |z_0| \]

and

\[ |\arg p(z_0)| = \frac{\pi}{2} \beta \]
then from the lemma, we have

\[
\frac{z_0 p'(z_0)}{p(z_0)} = i k \beta
\]

where \( k \) is a real

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi \beta}{2}
\]

and

\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi \beta}{2}
\]

where

\[
p(z_0)^{\frac{1}{\beta}} = \pm ia, \quad \text{and} \quad a > 0.
\]

When \( \arg p(z_0) = \frac{\pi \beta}{2} \), then from Lemma and applying the same method as the proof of [1, p.236], we have

\[
\arg \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) = \arg p(z_0) + \arg \left( \frac{z_0 p'(z_0)}{p(z_0)^2} - 1 \right)
\]

\[
= \frac{\pi \beta}{2} + \arg \left( e^{i(1-\beta) \frac{\pi}{2} \beta k \frac{1}{a^\beta}} - 1 \right)
\]

\[
\leq \frac{\pi \beta}{2} + \arg \left( e^{i(1-\beta) \frac{\pi}{2} \beta \left( a^{1-\beta} + a^{-1-\beta} \right)} - 1 \right)
\]

\[
\leq \frac{\pi \beta}{2} + \arg \left\{ e^{i(1-\beta) \frac{\pi}{2} \beta \left( \frac{1 + \beta}{1 - \beta} \right)^{1-\beta} + \left( \frac{1 + \beta}{1 - \beta} \right)^{-1+\beta}} - 1 \right\}
\]

\[
= \frac{\pi \beta}{2} + \tan^{-1} \left( \frac{\beta q(\beta) \sin \frac{\pi}{2}(1-\beta)}{\beta q(\beta) \cos \frac{\pi}{2}(1-\beta) - p(\beta)} \right)
\]

\[
= \frac{\pi}{2} (1 + \delta(\beta))
\]

and if \( \arg p(z_0) = -\frac{\pi \beta}{2} \), applying the same method as the above, we have

\[
\arg \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \geq -\frac{\pi}{2} (1 + \delta(\beta)).
\]

These contradict the assumption of the theorem, therefore we have

\[
\left| \arg \left( \frac{-zf'(z)}{f(z)} \right) \right| < \frac{\pi \beta}{2} \quad \text{in} \quad \mathcal{U}
\]

or

\[
\left| \arg \frac{zf'(z)}{f(z)} \right| > \frac{\pi}{2} (2 - \beta) \quad \text{in} \quad \mathcal{U}.
\]

This shows that

\[ f(z) \in SMS(1 - \beta). \]

This completes the proof.
Remark. *It is trivial that* \(-1 < \delta(\beta) < 0\) *for* \(0 < \beta < 1\).

**REFERENCES**


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