数理解析研究所講究録
1062 巻 1998 年 57-61

ON THE ORDER OF STRONGLY MEROMORPHIC STARLIKENESS OF STRONGLY MEROMORPHIC CONVEX FUNCTIONS

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ABSTRACT. In [1], Nunokawa proved that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $|z| < 1$ and

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha(\beta) \quad \text{in} \quad |z| < 1.$$

where $\alpha(\beta)$ satisfies the condition of Theorem A, then

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta \quad \text{in} \quad |z| < 1.$$

It is the purpose of the present paper that if $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ is analytic in $0 < |z| < 1$ and

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| > \frac{\pi}{2}(1 + \delta(\beta)) \quad \text{in} \quad |z| < 1.$$

where $\delta(\beta)$ satisfies the condition of the Main Theorem, then we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| > \frac{\pi}{2}(1 + 1 - \beta) \quad \text{in} \quad |z| < 1.$$

1. Introduction

Let Σ denote the class of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

which are analytic and univalent in the punctured disk $\mathcal{E} = \{z : 0 < |z| < 1\}$.

1991 *Mathematics Subject Classification.* 30C45.

A function $f(z) \in \Sigma$ is called to be strongly meromorphic starlike of order β $(-1 < \beta < 1)$ if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| > \frac{\pi}{2}(1+\beta) \quad \text{in} \quad \mathcal{E}.$$

We denote by $SMS(\beta)$ the class of all strongly meromorphic starlike functions of order β. Similarly, a function $f(z) \in \Sigma$ is called to be strongly meromorphic convex of order β $(-1 < \beta < 1)$ if

$$\left|arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| > \frac{\pi}{2}(1+\beta) \quad \text{in} \quad \mathcal{E}.$$

We denote by $SMC(\beta)$ the class of all strongly meromorphic convex function of order β.

In [1], Nunokawa obtained the following theorem.

Theorem A. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *is analytic in* $|z| < 1$ *and*

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha(\beta) \quad \textit{in} \quad |z| < 1$$

where $0 < \beta \leq 1$,

$$\alpha(\beta) = \beta + \frac{2}{\pi}\tan^{-1}\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{p(\beta)+\beta q(\beta)\cos\frac{\pi}{2}(1-\beta)},$$

$$p(\beta) = (1+\beta)^{\frac{1+\beta}{2}} \quad \textit{and} \quad q(\beta) = (1-\beta)^{\frac{\beta-1}{2}},$$

then we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta \quad \textit{in} \quad |z| < 1.$$

It is the purpose of the present paper to obtain an analogous result for meromorphic starlike and convex functions.

2. Preliminary

Lemma. *Let* $p(z)$ *be analytic in* $\mathcal{U} = \{z : |z| < 1\}$, $p(0) = 1$, $p(z) \neq 0$ *in* $\mathcal{U}$ *and suppose that there exists a point* $z_0 \in \mathcal{U}$ *such that*

$$|\arg p(z)| < \frac{\pi\alpha}{2} \quad \textit{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\alpha}{2}$$

where $0 < \alpha$. *Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{when} \qquad \arg p(z_0) = \frac{\pi\alpha}{2}$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{when} \qquad \arg p(z_0) = -\frac{\pi\alpha}{2}$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia, \qquad \textit{and} \qquad a > 0.$$

We owe this lemma to [1].

3. Main result

Theorem. *If* $f(z) \in SMC(\delta(\beta))$*, then* $f(z) \in SMS(1-\beta)$*, where*

$$\delta(\beta) = \beta - 1 + \frac{2}{\pi}\tan^{-1}\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{\beta q(\beta)\cos\frac{\pi}{2}(1-\beta) - p(\beta)},$$

$$p(\beta) = (1+\beta)^{\frac{1+\beta}{2}}, \qquad q(\beta) = (1-\beta)^{\frac{\beta-1}{2}},$$

and $0 < \beta < 1$.

Proof. Let us put

$$p(z) = -\frac{zf'(z)}{f(z)}, \qquad (p(0) = 1)$$

then we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} - p(z).$$

From the assumption of the theorem, we have

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| > \frac{\pi}{2}(1 + \delta(\beta)) \qquad \text{in} \qquad \mathcal{U}.$$

Therefore, we have $f'(z) \neq 0$ in $\mathcal{U}$, because if $f'(z)$ has a zero at $z = z_0$ in $\mathcal{U}$, then $(1 + \frac{zf''(z)}{f'(z)})$ can be infinite and $\arg(1 + \frac{zf''(z)}{f'(z)})$ can take any value of θ, for $0 \leq \theta \leq 2\pi$ when z approaches to z_0 from certain direction. This shows that

$$p(z) \neq 0 \qquad \text{in} \qquad \mathcal{U}.$$

If there exists a point $z_0 \in \mathcal{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\beta \qquad \text{for} \qquad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\beta$$

then from the lemma, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta$$

where k is a real

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \text{when} \qquad \arg p(z_0) = \frac{\pi\beta}{2}$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \text{when} \qquad \arg p(z_0) = -\frac{\pi\beta}{2}$$

where

$$p(z_0)^{\frac{1}{\beta}} = \pm ia, \qquad \text{and} \qquad a > 0.$$

When $\arg p(z_0) = \frac{\pi\beta}{2}$, then from Lemma and applying the same method as the proof of [1, p.236], we have

$$\begin{aligned}
\arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= \arg p(z_0) + \arg\left(\frac{z_0 p'(z_0)}{p(z_0)^2} - 1\right) \\
&= \frac{\pi\beta}{2} + \arg\left(e^{i(1-\beta)\frac{\pi}{2}}\beta k \frac{1}{a^\beta} - 1\right) \\
&\leq \frac{\pi\beta}{2} + \arg\left(e^{i(1-\beta)\frac{\pi}{2}}\frac{\beta}{2}(a^{1-\beta} + a^{-1-\beta}) - 1\right) \\
&\leq \frac{\pi\beta}{2} + \arg\left\{e^{i(1-\beta)\frac{\pi}{2}}\frac{\beta}{2}\left(\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1-\beta}{2}} + \left(\frac{1+\beta}{1-\beta}\right)^{-\frac{1+\beta}{2}}\right) - 1\right\} \\
&= \frac{\pi}{2}\beta + \tan^{-1}\frac{(\frac{\beta}{1-\beta})(\frac{1-\beta}{1+\beta})^{\frac{1+\beta}{2}}\sin\frac{\pi}{2}(1-\beta)}{(\frac{\beta}{1-\beta})(\frac{1-\beta}{1+\beta})^{\frac{1+\beta}{2}}\cos\frac{\pi}{2}(1-\beta) - 1} \\
&= \frac{\pi}{2}\beta + \tan^{-1}\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{\beta q(\beta)\cos\frac{\pi}{2}(1-\beta) - p(\beta)} \\
&= \frac{\pi}{2}(1 + \delta(\beta))
\end{aligned}$$

and if $\arg p(z_0) = -\frac{\pi\beta}{2}$, applying the same method as the above, we have

$$\arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) \geq -\frac{\pi}{2}(1 + \delta(\beta)).$$

These contradict the assumption of the theorem, therefore we have

$$\left|\arg\left(-\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi\beta}{2} \qquad \text{in} \qquad \mathcal{U}$$

or

$$\left|\arg\frac{zf'(z)}{f(z)}\right| > \frac{\pi}{2}(2 - \beta) \qquad \text{in} \qquad \mathcal{U}.$$

This shows that

$$f(z) \in SMS(1 - \beta).$$

This completes the proof.

Remark. *It is trivial that* $-1 < \delta(\beta) < 0$ *for* $0 < \beta < 1$.

References

1. M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. **69** (1993), 234–237.

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