THE SCHWARZIAN DERIVATIVE AND STARLIKENESS

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ABSTRACT. In this paper, we obtain sufficient conditions for starlikeness by applying the Schwarzian derivative.

1. Introduction

Let $A$ be the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the unit disk $\mathcal{U} = \{ z : |z| < 1 \}$.

2. Preliminaries

In [1, p.304], Miller and Mocanu obtained the following result:

**Theorem A.** Let $f(z) \in A$ and suppose that

$$\Re \left[ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} + z^2 \{f, z\} \right) \right] > 0 \quad \text{in} \quad \mathcal{U}$$

or

$$\Re \left[ \frac{zf'(z)}{f(z)} \left( 1 + z^2 \{f, z\} \right) \right] > 0 \quad \text{in} \quad \mathcal{U}$$

where $\{f, z\}$ is the Schwarzian derivative

$$\{f, z\} = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.$$

Then it follows that

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}$$

or $f(z)$ is starlike in $\mathcal{U}$.

We will improve Theorem A.
3. Main result

Theorem 1. Let \( f(z) \in A \) and suppose that

\[
(1) \quad \text{Re} \left[ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} + z^2 \{f, z\} \right) \right] \geq -\frac{1}{2} \quad \text{in} \quad \mathcal{U}.
\]

Then \( f(z) \) is starlike in \( \mathcal{U} \).

Proof. Let us put

\[
p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1.
\]

Then it follows that

\[
1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)},
\]

and

\[
z^2 \{f, z\} = \frac{zp'(z)}{p(z)} + z^2 \frac{p''(z)}{p(z)} - \frac{3}{2} \left( \frac{zp'(z)}{p(z)} \right)^2 + \frac{1}{2} (1 - p(z)^2).
\]

Applying the same method as the proof of [2, Lemma 1], we have

\[
f'(z) \neq 0 \quad \text{in} \quad \mathcal{U},
\]

because if \( f'(z) \) has zero in \( \mathcal{U} \), then it contradicts (1). Therefore we have

\[
p(z) \neq 0 \quad \text{in} \quad \mathcal{U}.
\]

On the other hand, if there exists a point \( z_0 \in \mathcal{U} \) such that

\[
\text{Re} p(z) > 0 \quad \text{for} \quad |z| < |z_0|
\]

and

\[
\text{Re} p(z_0) = 0 \quad (p(z_0) \neq 0).
\]

Then, from [3], we have

\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik
\]

where

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad p(z_0) = ia, \quad a > 0
\]

and

\[
k \leq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad p(z_0) = ia, \quad a < 0.
\]

This shows that

\[
p'(z_0) \neq 0.
\]
Therefore we have
\[
\frac{z_0^2 p'(z_0)}{p(z_0)} = \frac{z_0 p''(z_0)}{p'(z_0)} \frac{z_0 p'(z_0)}{p(z_0)} = ik \left(1 + \frac{z_0^2 p''(z_0)}{p'(z_0)} - 1\right)
\]
and it follows

(2) \[ \text{Re} \left[ \frac{z_0 f'(z_0)}{f(z_0)} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} + z_0^2 \{f, z_0\}\right) \right] \]
\[= \text{Re} \left[ ia \left(ia + ik + ik \left(1 + \frac{z_0 p''(z_0)}{p'(z_0)} - 1\right) + \frac{1}{2} (3k^2 + a^2 + 1)\right) \right] \]
\[= -a^2 - ak - ak \text{Re} \left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \]

From [1, Theorem 4, (ii)] or [4, p.3], we have

(3) \[ 1 + \text{Re} \frac{z_0 p''(z_0)}{p'(z_0)} \geq 0. \]

On the other hand, we have

(4) \[ -ak \leq -\frac{1}{2} (1 + a^2) < -\frac{1}{2} \]

where \(p(z_0) = ia\) and
\[ \frac{z_0 p'(z_0)}{p(z_0)} = ik. \]

From (2), (3) and (4), we have

\[ \text{Re} \left[ \frac{z_0 f'(z_0)}{f(z_0)} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} + z_0^2 \{f, z_0\}\right) \right] < -\frac{1}{2}. \]

This contradicts (1). Therefore we have

\[ \Re p(z) = \text{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in} \quad \mathcal{U}. \]

This completes the proof and improves Theorem A. Applying the same method as the proof of Theorem 1, we have the following theorem.

**Theorem 2.** Let \(f(z) \in A\) and suppose that

\[ \text{Re} \left[ \frac{zf'(z)}{f(z)} z^2 \{f, z\} \right] \geq 0 \quad \text{in} \quad \mathcal{U}. \]

Then \(f(z)\) is starlike in \(\mathcal{U}\).
REFERENCES


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