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THE SCHWARZIAN DERIVATIVE AND STARLIKENESS

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ABSTRACT. In this paper, we obtain sufficient conditions for starlikeness by applying the Schwarzian derivative.

1. Introduction

Let $\mathcal{A}$ be the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$.

2. Preliminaries

In [1, p.304], Miller and Mocanu obtained the following result:

Theorem A. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} + z^2\{f,z\}\right)\right] > 0 \qquad in \qquad \mathcal{U}$$

or

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}(1 + z^2\{f,z\})\right] > 0 \qquad in \qquad \mathcal{U}$$

where $\{f,z\}$ is the Schwarzian derivative

$$\{f,z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Then it follows that

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad in \qquad \mathcal{U}$$

or $f(z)$ is starlike in $\mathcal{U}$.

We will improve Theorem A.

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3. Main result

Theorem 1. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\text{(1)} \qquad \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}+z^2\{f,z\}\right)\right] \geq -\frac{1}{2} \qquad \textit{in} \qquad \mathcal{U}.$$

Then $f(z)$ is starlike in $\mathcal{U}$.

Proof. Let us put

$$p(z)=\frac{zf'(z)}{f(z)}, \qquad p(0)=1.$$

Then it follows that

$$1+\frac{zf''(z)}{f'(z)}=p(z)+\frac{zp'(z)}{p(z)},$$

and

$$z^2\{f,z\}=\frac{zp'(z)}{p(z)}+\frac{z^2p''(z)}{p(z)}-\frac{3}{2}\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{1}{2}(1-p(z)^2).$$

Applying the same method as the proof of [2, Lemma 1], we have

$$f'(z) \neq 0 \qquad \text{in} \qquad \mathcal{U},$$

because if $f'(z)$ has zero in $\mathcal{U}$, then it contradicts (1). Therefore we have

$$p(z) \neq 0 \qquad \text{in} \qquad \mathcal{U}.$$

On the other hand, if there exists a point $z_0 \in \mathcal{U}$ such that

$$\mathrm{Re}p(z) > 0 \qquad \text{for} \qquad |z| < |z_0|$$

and

$$\mathrm{Re}p(z_0) = 0 \qquad (p(z_0) \neq 0).$$

Then, from [3], we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik$$

where

$$k \geq \frac{1}{2}\left(a+\frac{1}{a}\right) \qquad \text{when} \qquad p(z_0)=ia, \qquad a>0$$

and

$$k \leq \frac{1}{2}\left(a+\frac{1}{a}\right) \qquad \text{when} \qquad p(z_0)=ia, \qquad a<0.$$

This shows that

$$p'(z_0) \neq 0.$$

Therefore we have

$$\frac{z_0^2 p''(z_0)}{p(z_0)} = \frac{z_0 p''(z_0)}{p'(z_0)} \frac{z_0 p'(z_0)}{p(z_0)} = ik\left(1 + \frac{z^2 p''(z_0)}{p'(z_0)} - 1\right)$$

and it follows

$$\text{(2)} \quad \begin{aligned} &\operatorname{Re}\left[\frac{z_0 f'(z_0)}{f(z_0)}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} + z_0^2\{f, z_0\}\right)\right] \\ &= \operatorname{Re}\left[ia\left(ia + ik + ik + ik\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)} - 1\right) + \frac{1}{2}\left(3k^2 + a^2 + 1\right)\right)\right] \\ &= -a^2 - ak - ak\operatorname{Re}\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \end{aligned}$$

From [1, Theorem 4, (ii)] or [4, p.3], we have

$$\text{(3)} \qquad 1 + \operatorname{Re}\frac{z_0 p''(z_0)}{p'(z_0)} \geq 0.$$

On the other hand, we have

$$\text{(4)} \qquad -ak \leq -\frac{1}{2}(1 + a^2) < -\frac{1}{2}$$

where $p(z_0) = ia$ and

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik.$$

From (2), (3) and (4), we have

$$\operatorname{Re}\left[\frac{z_0 f'(z_0)}{f(z_0)}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} + z_0^2\{f, z_0\}\right)\right] < -\frac{1}{2}.$$

This contradicts (1). Therefore we have

$$\operatorname{Re} p(z) = \operatorname{Re}\frac{z f'(z)}{f(z)} > 0 \quad \text{in} \quad \mathcal{U}.$$

This completes the proof and improves Theorem A. Applying the same method as the proof of Theorem 1, we have the following theorem.

Theorem 2. *Let $f(z) \in A$ and suppose that*

$$\operatorname{Re}\left[\frac{z f'(z)}{f(z)} z^2\{f, z\}\right] \geq 0 \quad \text{in} \quad \mathcal{U}.$$

Then $f(z)$ is starlike in $\mathcal{U}$.

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