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INTEGRAL MEANS OF THE FRACTIONAL DERIVATIVE OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. By using the definition of fractional derivative (cf., [2]), we investigate the sharp integral means inequalities of the fractional derivatives of univalent functions with negative coefficients and extend the sharp results of H. Silverman [5, Theorem 2.2].

1. Introduction and Definitions

Let $A$ denote the class of $f(z)$ normalized by

(1.1) \[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \]

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $S$ denote the class of all functions in $A$ which are univalent in $U$. Then a function $f(z)$ belonging to the class $S$ is said to be in the class $K$ if and only if

(1.2) \[ \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in U). \]

We denote by $T$ the subclass of $S$ whose functions may be represented by

(1.3) \[ f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \]

Silverman [4] showed that $f$ of the form (1.3) is in $T$ if and only if $\sum_{k=2}^{\infty} ka_k \leq 1$, and that the extreme points of $T$ are

(1.4) \[ f_1(z) = z \quad \text{and} \quad f_m(z) = z - z^m/m, \quad m = 2, 3, \ldots. \]

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Further a function $f$ of the form (1.3) is in $C = T \cap K$ if and only if $\sum_{k=2}^{\infty} k^2 a_k \leq 1$, and that the extreme points of $C$ are $g_1(z) = z$ and $g_2(z) = z - zm/m^2$ ($m = 2, 3, \cdots$).

For analytic functions $g(z)$ and $h(z)$ with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $w(z)$ so that $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathcal{U}$) and $g(z) = h(w(z))$, we denote this subordination by $g(z) \prec h(z)$.

Many essentially equivalent definition of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [3], [6, p 45] and [7]). We find it to be convenient to recall here the following definition which were used recently by Owa [2] (and by Srivastava and Owa [7]).

**Definition 1.** The fractional derivative of order $\lambda$ is defined, for a function $f(z)$, by

\begin{equation}
D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_{0}^{z} \frac{f(z)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leq \lambda < 1),
\end{equation}

where $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring for $\log(z-\zeta)$ to be real for $z > \zeta$.

**Definition 2.** Under the hypotheses of Definition 1, the fractional derivative of order $n + \lambda$ is defined by

\begin{equation}
D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^\lambda f(z) \quad (0 \leq \lambda < 1; n \in \mathbb{N}_0 := \{0, 1, 2, \cdots \}).
\end{equation}

In [5] it is proven that

\begin{equation}
\int_{0}^{2\pi} |f(re^{i\theta})|^{\beta} d\theta \leq \int_{0}^{2\pi} |f_2(re^{i\theta})|^{\beta} d\theta
\end{equation}

for all $f \in \mathcal{T}$, $\beta > 0$ and $0 < r < 1$. In this paper, by using the fractional derivative, we prove that

\begin{equation}
\int_{0}^{2\pi} |D_z^\lambda f(re^{i\theta})|^{\beta} d\theta \leq \int_{0}^{2\pi} |D_z^\lambda f_2(re^{i\theta})|^{\beta} d\theta
\end{equation}

for all $f \in \mathcal{T}$, $\beta > 0$, $0 < r < 1$ and $0 \leq \lambda < 1$. We also obtain the integral means inequality for $D_z^{n+\lambda} f(z)$ ($n = 1, 2$) if $f \in \mathcal{C}$ (or $T$).

2. Main Results

The following result will be required in our investigation.
Lemma. (Littlewood [1]) If $f$ and $g$ are analytic in $\mathcal{U}$ with $g < f$, then, for $\beta > 0$ and $0 < r < 1$,

\begin{equation}
\int_{0}^{2\pi} |g(re^{i\theta})|^\beta d\theta \leq \int_{0}^{2\pi} |f(re^{i\theta})|^\beta d\theta.
\end{equation}

Applying the above lemma, we prove

Theorem 1. Let $\beta > 0$ and $f_2(z)$ is defined by (1.4). If $f \in \mathcal{T}$, then for $z = re^{i\theta}$ and $0 < r < 1$,

(i) $\int_{0}^{2\pi} |D_z^\lambda f(z)|^\beta d\theta \leq \int_{0}^{2\pi} |D_z^\lambda f_2(z)|^\beta d\theta$ \hspace{1cm} $(0 \leq \lambda < 1)$

(ii) $\int_{0}^{2\pi} |D_z^{2+\lambda} f(z)|^\beta d\theta \leq \int_{0}^{2\beta\pi} |D_z^{2+\lambda} f_2(z)|^\beta d\theta$ \hspace{1cm} $(0 < \lambda < 1)$.

Proof. We prove (i). The proof of (ii) is similar and will be omitted. If $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ ($a_k \geq 0$), then

$$D_z^\lambda f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \sum_{k=2}^{\infty} \Phi(k) k a_k z^{k-1}\right),$$

where

\begin{equation}
\Phi(k) = \frac{\Gamma(k) \Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \hspace{1cm} (k \geq 2).
\end{equation}

Note that $\Phi(k)$ is a non-increasing function of $k$,

\begin{equation}
0 < \Phi(k) \leq \Phi(2) = \frac{1}{2-\lambda}.
\end{equation}

Since

$$D_z^\lambda f_2(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \frac{1}{2-\lambda} z\right),$$

we must show that

$$\int_{0}^{2\pi} \left|1 - \sum_{k=2}^{\infty} \Phi(k) k a_k z^{k-1}\right|^\beta d\theta \leq \int_{0}^{2\pi} \left|1 - \frac{1}{2-\lambda} z\right|^\beta d\theta.$$

By Lemma, it suffices to prove that

$$1 - \sum_{k=2}^{\infty} \Phi(k) k a_k z^{k-1} < 1 - \frac{1}{2-\lambda} z.$$
Setting
\[ (2.4) \quad 1 - \sum_{k=2}^{\infty} \Phi(k)ka_{k}z^{k-1} = 1 - \frac{w(z)}{2 - \lambda}. \]

From (2.3) and (2.4), we obtain
\[ |w(z)| \leq \left| \sum_{k=2}^{\infty} (2 - \lambda)\Phi(2)k^{\wedge}akz^{k-1} \right| \leq |z| \sum_{k=2}^{\infty} k\alpha_k \leq |z|. \]

This completes the proof of (i).

**Remark.** If \( \lambda = 0 \) in (i) of Theorem 1, then it would immediately yield the result of Silverman [5, Theorem 2.2].

For the fractional derivative of order \( 1 + \lambda \), we have

**Theorem 2.** If \( f \in C \) and \( \beta > 0 \), then for \( z = re^{i\theta} \) and \( 0 < r < 1 \),

(i) \( \int_{0}^{2\pi} |D_{z}^{1+\lambda}f_{2}(z)|^{\beta} d\theta \leq \int_{0}^{2\pi} |D_{z}^{1+\lambda}f_{2}(z)|^{\beta} d\theta \) \( (0 \leq \lambda < 1) \)

(ii) \( \int_{0}^{2\pi} |D_{z}^{1+\lambda}f(z)|^{\beta} d\theta \leq \int_{0}^{2\pi} |D_{z}^{2+\lambda}g_{2}(z)|^{\beta} d\theta \) \( (0 \leq \lambda \leq 2/3) \).

**Proof.** (i) From the definition (1.6), we have
\[ (2.5) \quad D_{z}^{1+\lambda}f(z) = \frac{z^{-\lambda}}{\Gamma(1 - \lambda)} \left( 1 - \sum_{k=2}^{\infty} \Psi(k)k(k-1)a_{k}z^{k-1} \right), \]

where
\[ \Psi(k) = \frac{\Gamma(k-1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)} \] \( (k \geq 2) \).

Note that \( 0 < \Psi(k) \leq \Psi(2) = 1/(1 - \lambda) \).

Since
\[ D_{z}^{1+\lambda}f_{2}(z) = \frac{z^{-\lambda}}{\Gamma(1 - \lambda)} \left( 1 - \frac{1}{1 - \lambda}z \right), \]
it suffices to show that
\[ 1 - \sum_{k=2}^{\infty} \Psi(k)k(k-1)a_{k}z^{k-1} \prec 1 - \frac{1}{1 - \lambda}z. \]
Setting

$$1 - \sum_{k=2}^{\infty} \Psi(k)(k-1)a_k z^{k-1} = 1 - \frac{w(z)}{1 - \lambda},$$

$$|w(z)| \leq \sum_{k=2}^{\infty} k(k-1)a_k z^{k-1} \leq |z| \sum_{k=2}^{\infty} k^2 a_k \leq |z|.$$ 

By Lemma, the proof of (i) is completed.

(ii) Making use of (1.6) and (2.5), we obtain

$$D_z^{1+\lambda} f(z) = \frac{z^{-\lambda}}{\Gamma(1-\lambda)} \left( 1 - \sum_{k=2}^{\infty} \Theta(k) k^2 a_k z^{k-1} \right),$$

where

$$\Theta(k) = \frac{\Gamma(k) \Gamma(1-\lambda)}{k \Gamma(k-\lambda)} \quad (k \geq 2).$$

We note that $0 < \Theta(k) \leq \Theta(2) = 1/2(1-\lambda)$ for $0 \leq \lambda \leq 2/3$. Thus the proof of (ii) is much akin to that of (i), and we omit the details involved.

Denote by $T^*(\alpha)$ and $C(\alpha)$, $0 \leq \alpha < 1$, the subclasses of $T$ that are, respectively, starlike of order $\alpha$ and convex of order $\alpha$. In [4], Silverman showed that $f \in T^*(\alpha)$ if and only if $\sum_{k=2}^{\infty} ((k-\alpha)/(1-\alpha)) a_k \leq 1$ and $f \in C(\alpha)$ if and only if $\sum_{k=2}^{\infty} ((k-\alpha)/(1-\alpha)) a_k \leq 1$. In addition, the extreme points of $T^*(\alpha)$ and $C(\alpha)$ are $h_m(z) = z - ((1-\alpha)/(m-\alpha)) z^m$ and $k_m(z) = z - ((1-\alpha)/m(m-\alpha)) z^m$ for $m \geq 2$.

For the cases of $T^*(\alpha)$ and $C(\alpha)$, the proof is much akin to that of Theorem 1 and Theorem 2, and we omit the details involved.

**Theorem 3.** (i) If $f \in T^*(\alpha)$ and $\beta > 0$, then for $0 < r < 1$,

$$\int_0^{2\pi} \left| D_z^\lambda f(re^{i\theta}) \right|^\beta d\theta \leq \int_0^{2\pi} \left| D_z^\lambda h_2(re^{i\theta}) \right|^\beta d\theta \quad (0 \leq \lambda < 1)$$

and

$$\int_0^{2\pi} \left| D_z^{2+\lambda} f(re^{i\theta}) \right|^\beta d\theta \leq \int_0^{2\pi} \left| D_z^{2+\lambda} h_2(re^{i\theta}) \right|^\beta d\theta \quad (0 < \lambda < 1).$$

(ii) If $f \in C(\alpha)$ and $\beta > 0$, then for $0 < r < 1$,

$$\int_0^{2\pi} \left| D_z^\lambda f(re^{i\theta}) \right|^\beta d\theta \leq \int_0^{2\pi} \left| D_z^\lambda k_2(re^{i\theta}) \right|^\beta d\theta \quad (0 \leq \lambda < 1),$$

and

$$\int_0^{2\pi} \left| D_z^{2+\lambda} f(re^{i\theta}) \right|^\beta d\theta \leq \int_0^{2\pi} \left| D_z^{2+\lambda} k_2(re^{i\theta}) \right|^\beta d\theta \quad (0 < \lambda < 1).$$
\[
\int_{0}^{2\pi} |D_{z}^{1+\lambda} f(re^{i\theta})|^{\beta} d\theta \leq \int_{0}^{2\pi} |D_{z}^{1+\lambda} h_2(re^{i\theta})|^{\beta} d\theta \quad (0 \leq \lambda < 1),
\]

\[
\int_{0}^{2\pi} |D_{z}^{1+\lambda} f(re^{i\theta})|^{\beta} d\theta \leq \int_{0}^{2\pi} |D_{z}^{1+\lambda} k_2(re^{i\theta})|^{\beta} d\theta \quad (0 \leq \lambda \leq 2/3)
\]

and

\[
\int_{0}^{2\pi} |D_{z}^{2+\lambda} f(re^{i\theta})|^{\beta} d\theta \leq \int_{0}^{2\pi} |D_{z}^{2+\lambda} k_2(re^{i\theta})|^{\beta} d\theta \quad (0 < \lambda < 1).
\]

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