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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1998, 1062: 34-46</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1998-09</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62410">http://hdl.handle.net/2433/62410</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
<td>publisher</td>
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GROWTH AND COEFFICIENT ESTIMATES FOR UNIFORMLY
LOCALLYUNIVALENT FUNCTIONS ON THE UNIT DISK

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ABSTRACT. In this note, we shall give a sharp growth estimate for a uniformly locally
univalent holomorphic function on the unit disk. As applications, we shall investigate
the growth of coefficients of such a function and mention the connection with Hardy
spaces. We also give norm estimates for typical classes of univalent functions.

1. INTRODUCTION

We will call a holomorphic function $f$ on the unit disk $\mathbb{D}$ uniformly locally univalent if
$f$ is univalent on each hyperbolic disk $D(a, \rho) = \{z \in \mathbb{D}; |z - a| < \tanh \rho\}$ with radius $\rho$
and center $a \in \mathbb{D}$ for a positive constant $\rho$. In particular, a holomorphic universal covering
map of a plane domain $D$ is uniformly locally univalent if and only if the boundary of $D$
is uniformly perfect (cf. [12] or [17]). Also it is well-known (cf. [20]) that a holomorphic
function $f$ on the unit disk is uniformly locally univalent if and only if the pre-Schwarzian
derivative (or nonlinearity) $T_f = f''/f'$ of $f$ is hyperbolically bounded, i.e., the norm

$$\|T_f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)|T_f(z)|$$

is finite. This quantity can be regarded as the Bloch norm of the function $\log f'$. Remark
that a holomorphic function $f$ is locally univalent at the point $z$ if and only if $T_f = f''/f'$
is a well-defined holomorphic function near $z$. Roughly speaking, the quantity $T_f$ measures
the deviation of $f$ from orientation-preserving similarities (non-constant linear functions).

Because $T_f$ is invariant under the post-composition by a non-constant linear function,
we may assume that a holomorphic function $f$ on the unit disk is normalized so that
$f(0) = 0$ and $f'(0) = 1$. We denote by $A$ the set of such normalized holomorphic functions
on the unit disk. And we denote by $B$ the set of normalized uniformly locally univalent
functions: $B = \{f \in A; \|T_f\| < \infty\}$. The space $B$ has a structure of non-separable complex
Banach space under the Hornich operation ([19]).

For a non-negative real number $\lambda$ we set

$$B(\lambda) = \{f \in A; \|T_f\| \leq 2\lambda\},$$

here the number 2 is due to some technical reason. The functions in $B(\lambda)$ can be character-
ized as the following.

Date: June 9, 1998.
1991 Mathematics Subject Classification. Primary 30C45, 30C50; Secondary 30C80.
Key words and phrases. pre-Schwarzian derivative, uniformly locally univalent, growth estimate, co-
efficient estimate.
Proposition 1.1. Let a non-negative constant \( \lambda \) be given. A locally univalent function \( f \in A \) belongs to \( B(\lambda) \) if and only if for any pair of points \( z_1, z_2 \) in \( \mathbb{D} \) it holds that

\[
|g(z_1) - g(z_2)| \leq 2\lambda d_{\mathbb{D}}(z_1, z_2),
\]

where \( g(z) = \log f'(z) \) and \( d_{\mathbb{D}}(z_1, z_2) = \tanh^{-1} \left| \frac{z_1 - z_2}{1 - \overline{z}_1 z_2} \right| \) stands for the hyperbolic distance between \( z_1 \) and \( z_2 \) in the unit disk \( \mathbb{D} \).

Proof. First of all, note that we can take a holomorphic branch \( g \) of \( \log f' \) for a locally univalent holomorphic function \( f \) on the unit disk. The "only if " part is shown by integrating the inequality \( |g'(z)| = |T_f(z)| \leq 2\lambda/(1 - |z|^2) \) along the hyperbolic geodesic joining \( z_1 \) and \( z_2 \). The "if " part directly follows from the observation:

\[
\lim_{z' \to z} \frac{|g(z') - g(z)|}{d_{\mathbb{D}}(z', z)} = (1 - |z|^2)|g'(z)|.
\]

The following theorem is significant in connection with univalent function theory.

Theorem A (Becker and Pommerenke [3], [4]). The set \( S \) of normalized univalent holomorphic functions on the unit disk is contained in \( B(3) \) and contains \( B(\frac{1}{2}) \). The result is sharp.

We note that the Schwarzian derivative \( S_f \) of \( f \) can be written as \( S_f = (T_f)' - (T_f)^2/2 \). Thus the space \( B \) has a close connection with (the Bers embedding of) the Teichmüller spaces. Especially, it is expected to be useful when considering the Bers boundary of the Teichmüller spaces since the quantity \( T_f \) is much easier to treat than \( S_f \). In fact, the space \( T_1 := \{ T_f; f \in S \text{ has a quasiconformal extension to the Riemann sphere } \} \) can be regarded as a model of the universal Teichmüller space (cf. [1] and [23]).

Here, as a result in this direction, we mention the following.

Corollary. For a constant \( k \in [0, 1) \), let \( S_k \) be the subset of \( S \) consisting of those functions which can be extended to \( k \)-quasiconformal self-mappings of the Riemann sphere \( \hat{\mathbb{C}} \). Then, we have

\[
B(k/2) \subset S_k.
\]

This implication is easily obtained by the \( \lambda \)-lemma (see, for example, [13, p. 121]). This already (implicitly) appeared in the paper [3] by Becker.

2. Growth estimate for the class \( B(\lambda) \)

In the class \( B(\lambda) \) for \( 0 \leq \lambda < \infty \) the function

\[
F_\lambda(z) = \int_0^z \left( \frac{1 + t}{1 - t} \right)^\lambda dt
\]

is extremal as we shall see later. We remark that \( F_\lambda \in A \) can be defined for any complex number \( \lambda \) and satisfies \( T_{F_\lambda} = 2\lambda(1 - z^2)^{-1} \), thus \( \|T_{F_\lambda}\| = 2|\lambda| \). \( F_\lambda \) may provide an example of a function with small pre-Schwarzian norm which does not belong to typical classes of univalent functions when \( \lambda \) is sufficiently small and \( \lambda \notin \mathbb{R} \).

In practice, it is important to know the univalence of \( F_\lambda \).
Lemma 2.1. For a non-negative number $\lambda$, the function $F_\lambda$ is univalent in the unit disk if and only if $0 \leq \lambda \leq 1$.

Proof. First, we compute the Schwarzian derivative $S_{F_\lambda}$ of $F_\lambda$. Then, we have
\[
\sup_{z \in \mathbb{D}}(1 - |z|^2)^2 |S_{F_\lambda}(z)| = \sup_{z \in \mathbb{D}}(1 - |z|^2)^2 \frac{2\lambda|2z - \lambda|}{|1 - z^2|^2} = 2\lambda(\lambda + 2).
\]
In particular, if $1 < \lambda$, then $2\lambda(\lambda + 2) > 6$, thus the Nehari-Kraus theorem implies that $F_\lambda$ is not univalent.

On the other hand, if $0 \leq \lambda \leq 1$, we have $\Re F'_\lambda(z) > 0$ in the unit disk, hence the Noshiro-Warschawski theorem ensures the univalence of $F_\lambda$ in this case. $\square$

The following result is elementary and might be known. But we shall include the proof because of its importance for our aim.

Theorem 2.2 (Distortion Theorem). Let $\lambda$ be a non-negative real number. For an $f \in B(\lambda)$ it holds that
\[
F'_\lambda(-|z|) = \left(\frac{1 - |z|}{1 + |z|}\right)^\lambda \leq |f'(z)| \leq \left(\frac{1 + |z|}{1 - |z|}\right)^\lambda = F'_\lambda(|z|), \quad \text{and}
\]
\[
|f(z)| \leq F_\lambda(|z|).
\]
in the unit disk. Furthermore, if $f$ is univalent then
\[
-F_\lambda(-|z|) \leq |f(z)| \leq F_\lambda(|z|).
\]
If the equality occurs in any of the above inequalities at some point $z_0 \neq 0$, then $f$ must be a rotation of $F_\lambda$, i.e., $f(z) = \overline{\mu}F_\lambda(\mu z)$ for a unimodular constant $\mu$.

Proof. Applying Proposition 1.1 in the case of $z_1 = z$ and $z_2 = 0$, we see
\[
|\log f'(z)| \leq \lambda \log \frac{1 + |z|}{1 - |z|},
\]
Taking the real part of $\log f'$, we obtain (2.1). And the integration of (2.1) yields (2.2). The inequality (2.3) can be shown by the same method as in the proof of the Koebe distortion theorem. The equality cases are obvious. (Note that the inequality (2.3) is sharp only for $\lambda \leq 1$ by Lemma 2.1.) $\square$

Since $\int_0^1 (\frac{1 + t}{1 - t})^\lambda \, dt < \infty$ for $\lambda < 1$ and $\int_0^r (\frac{1 + t}{1 - t})^\lambda \, dt \leq \frac{2^\lambda}{\lambda - 1}(1 - r)^{1 - \lambda}$ for $\lambda > 1$, we have the following

Corollary 2.3. For $\lambda > 1$ any $f \in B(\lambda)$ satisfies the growth condition
\[
f(z) = O(1 - |z|)^{1 - \lambda}
\]
as $|z| \to 1$. Furthermore, if $f$ is univalent, then $f(\mathbb{D})$ contains the disk $\{ |z| < -F_\lambda(-1) \}$. This constant $-F_\lambda(-1)$ is best possible for $0 \leq \lambda \leq 1$.

On the other hand, for $\lambda < 1$, a function $f \in B(\lambda)$ is always bounded with a uniform bound $F_\lambda(1)$.
We note again that for $\lambda \leq 1/2$ the function $f \in \mathcal{B}(\lambda)$ must be univalent. We also note that, for $0 \leq \lambda \leq 1$, we have $-F_{\lambda}(-1) \leq -F_{1}(-1) = 2\log 2 - 1 = 0.38629 \cdots$, therefore the result above is better than the Koebe one-quarter theorem.


$$
\frac{F_{\lambda}(z)}{z} = \int_{0}^{1} \left( \frac{1 + tz}{1 - tz} \right)^{\lambda} dt
= \sum_{k=0}^{\infty} \binom{\lambda}{k} z^{k} \int_{0}^{1} t^{k}(1 - tz)^{-\lambda} dt
= \sum_{k=0}^{\infty} \frac{\Gamma(\lambda + k)}{(k + 1)!\Gamma(\lambda)} z^{k} F(-\lambda, k + 1; k + 2; -z),
$$

where $F(a, b; c; z)$ denotes the Gauss hypergeometric function. Also, the values $F_{\lambda}(1)$ and $-F_{\lambda}(-1)$ can be expressed in terms of the Gauss hypergeometric function. For example, by [14] p.491,

$$
-F_{\lambda}(-1) = \int_{0}^{1} \left( \frac{1 - t}{1 + t} \right)^{\lambda} dt = \frac{1}{\lambda + 1} F(1, \lambda; \lambda + 2; -1)
= \frac{1}{2^\lambda(\lambda + 1)} F(\lambda, \lambda + 1; \lambda + 2; 1/2)
= \sum_{k=0}^{\infty} \frac{\Gamma(\lambda + k)}{k!(\lambda + k + 1)!\Gamma(\lambda)2^\lambda k},
$$

which may also be rewritten in terms of the difference of two Digamma functions ([14], p.489, Eq.12):

$$
-F_{\lambda}(-1) = \lambda \left[ \psi \left( \frac{\lambda + 1}{2} \right) - \psi \left( \frac{\lambda}{2} \right) \right] - 1 \quad \left( \psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \right).
$$

Similarly, we have $F_{\lambda}(1) = \lambda[\psi(-\lambda/2) - \psi((1 + \lambda)/2)] - 1$. It may be useful to note the following elementary estimate:

$$
\frac{1}{(\lambda + 1)2^\lambda} < -F_{\lambda}(-1) < \frac{1}{\lambda + 1}.
$$

In the above theorem, the case $\lambda = 1$ is critical. In this case, by Theorem 2.2, we can see that for $f \in \mathcal{B}(1)$

$$
|f(z)| \leq F_{1}(|z|) = 2\log \left( \frac{1}{1 - |z|} \right) - |z|.
$$

In particular, a function in $\mathcal{B}(1)$ need not be bounded (for instance, $F_{1}$). The next proposition gives a boundedness criterion for functions in $\mathcal{B}(1)$.

**Proposition 2.4.** If a holomorphic function $f$ on the unit disk satisfies that

$$
\beta(f) := \lim_{|z| \to 1-0} \left\{ (1 - |z|^2)|T_f(z)| - 2 \right\} \log \left( \frac{1}{1 - |z|^2} \right) < -2
$$
then $f$ is bounded. Here, the constant $-2$ in the right hand side is sharp.

**Proof.** By assumption, there exists a $\beta < -2$ such that the left-hand side in (2.5) is less than $\beta$. Thus, for some $0 < r_0 < 1$, $(1 - |z|^2)|T_f(z)| - 2 \leq \frac{\beta}{\log \frac{1}{1-r_0^2}}$, i.e.,

\[
|T_f(z)| \leq \frac{2}{1 - |z|^2} + \frac{\beta}{(1 - |z|^2) \log \frac{1}{1 - |z|^2}},
\]

for any $z \in \mathbb{C}$ with $r_0 < |z| < 1$. Here, we may choose $r_0$ sufficiently close to 1 so that $1 - r_0^2 < e^{-1}$ and that $\beta_1 := (1 + r_0)\beta/2 < -2$.

Integrating the inequality (2.6), we see that, for $|z| > r_0$,

\[
|\log f'(z)| \leq \log \frac{1+|z|}{1-|z|} + \int_{r_0}^{|z|} \frac{\beta dt}{(1-t^2) \log \frac{1}{1-t^2}} + C_1 \leq \log \frac{1+|z|}{1-|z|} + \int_{r_0}^{|z|} \frac{\beta_1 dt}{2(1-t) \log \frac{1}{2(1-t)}} + C_2,
\]

where $C_1$ and $C_2$ are constants depending only on $f$ and $r_0$. In particular, we have

\[
|f'(z)| \leq e^{C_2} \frac{1+|z|}{1-|z|} \left( \log \frac{1}{2(1-|z|)} \right)^{\beta_1/2}.
\]

Since $\beta_1/2 < -1$ the function $\frac{1+t}{1-t}(\log \frac{1}{2(1-t)})^{\beta_1/2}$ is integrable on the interval $[r_0, 1)$. Thus $f$ is bounded.

The sharpness follows from the example below. \hfill \square

**Example 2.1.** Let a constant $\beta < 0$ be given. Choose a constant $c > 0$ so that $c\beta + 2 \geq 0$. Now we consider the function $f \in A$ determined by

\[
f'(z) = \frac{K}{1-z} \left( 1 + c \log \frac{2}{1-z} \right)^\beta,
\]

where $K = (1 + c \log 2)^{-\beta}$. Then this function satisfies that $\|T_f\| = 2$. And moreover, $f$ is bounded in the unit disk if and only if $\beta < -1$.

In fact, first observe that

\[
T_f(z) = \frac{1}{1-z} + \frac{c\beta}{(1-z)(1+c \log \frac{2}{1-z})} = \frac{1}{1-z} \left[ 1 + \frac{\beta}{c + \log \frac{2}{1-z}} \right].
\]

By the fact that $\text{Re} \frac{2}{1-z} > 1$, one can conclude that $\text{Re} w > \frac{1}{c} \geq -\beta/2$ and $|\text{Im} w| < \pi/2$, where $w = \frac{1}{c} + \log \frac{2}{1-z}$. Noting that $|1 + \beta/w|^2 = 1 + \beta(2\text{Re} w + \beta)/|w|^2 \leq 1$, one can see that $|T_f(z)| \leq \frac{1}{|1-z|} \leq \frac{1}{1-|z|}$. In particular, it holds that $(1 - |z|^2)|T_f(z)| \leq 1 + |z| < 2$. On the other hand, it is easy to see that $\lim_{x\to 1-0}(1 - x^2)|T_f(x)| = 2$, thus $\|T_f\| = 2$.

Next, we shall show that $\beta(f) = 2\beta$. Since $|1 + \beta/w| = |1 + \beta(2\text{Re} w + \beta)/|w|^2|^{1/2} \sim 1 + \beta(\text{Re} w + \beta/2)/|w|^2 \sim 1 + \beta/\text{Re} w \sim 1 - \beta/\log |1-z|$ as $z \to 1$ and since the function
if $0$ of function of if sharp properties different functions. pair we those have various holomorphic Proposition estimate this course, owing consider integral function function is sufficiently the any we noting of the absolute absolute constant, any having coefficient the fact, that the condition that the exponent to the constant.

Let $f(z) = z + a_1 z + a_2 z^2 + \cdots \in B(\lambda)$. Then, by definition, $|T_f(0)| \leq 2\lambda$, which implies $|a_2| \leq \lambda$. Of course, this is sharp because the equality holds for the function $F_\lambda$. But, a function in $B(\lambda)$ essentially different from $F_\lambda$ may attain this maximum. For instance, consider the function $f(z) = (e^{2\lambda z} - 1)/2\lambda$.

If the origin is a critical point of the function $(1 - |z|^2)|T_f(z)|$ then $(T_f)'(0) = 6a_3 - (2a_2)^2 = 0$ though this condition need not be sufficient for $|a_2| = \lambda$.

As for the growth of coefficients of a holomorphic function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ in the unit disk, it is convenient to consider the integral mean of exponent $p \in \mathbb{R}$:

$$I_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$ 

In fact, we have the following elementary
Lemma 3.2. If $I_1(r, f) = O(1 - r)^{-\alpha}$ as $r \to 1$ for a constant $\alpha \geq 0$, then we have $a_n = O(n^\alpha)$ as $n \to \infty$.

Proof. Suppose that $I_1(r, f) \leq M(1 - r)^{-\alpha}$ for $0 \leq r < 1$. Then, for $n > 1$ and $r = 1 - 1/n$, it follows from Cauchy's integral formula that

$$|a_n| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})(re^{i\theta})^{-n}d\theta \right| \leq r^{-n} I_1(r, f) \leq Mr^{-n}(1 - r)^{-\alpha}$$

$$= M \left( 1 - \frac{1}{n} \right)^{-n} n^\alpha < \frac{eMn}{n-1} n^\alpha.$$  

Thus $|a_n| < 2eMn^\alpha$.

In particular, for a function $f(z) = z + a_2z^2 + \cdots$ in $B(\lambda)$, by Theorem 2.2, we have $I_1(r, f') = O(1 - r)^{-\lambda}$, thus an estimate $|a_n| = O(n^{\lambda - 1 + \epsilon})$ as $n \to \infty$. But we can improve the exponent of this order. For $\lambda > 0$, we set

$$\alpha(\lambda) = \frac{\sqrt{1 + 4\lambda^2} - 1}{2}.$$  

Noting $\alpha(\lambda) = 2\lambda^2/(\sqrt{1 + 4\lambda^2} + 1)$, then we have

$$\frac{\lambda^2}{\lambda + 1} < \alpha(\lambda) < \min \left\{ \lambda^2, \frac{2\lambda^2}{2\lambda + 1} \right\} \leq \min \{\lambda^2, \lambda\}.$$  

We also note that

$$\alpha(\lambda) = \lambda - \frac{1}{2} + \frac{1}{8\lambda} + O \left( \frac{1}{\lambda^3} \right) \quad (\lambda \to \infty).$$  

For this number, we have the next result.

Theorem 3.3. Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be in $B(\lambda)$. Then, for any $\varepsilon > 0$ and a real number $p$, we have $I_p(r, f') = O(1 - r)^{-\alpha(|p|\lambda) + \epsilon}$, in particular, $a_n = O(n^{\alpha(\lambda) - 1 + \epsilon})$.

This immediately follows from the next result.

Theorem C ([10, Lemma 5.3]). Let $h$ be a holomorphic function in the unit disk such that

$$f(z) = \frac{h'(z)}{h(z)} \leq c \quad (r_0 \leq |z| < 1)$$

for constants $c > 0$ and $r_0 < 1$. Then, $I_p(r, h) = O(1 - r)^{-\beta}$, where $\beta = (\sqrt{1 + 4p^2c^2} - 1)/2$ and $p \in \mathbb{R}$.

We note that this is a consequence of the Fuchsian differential inequality:

$$I''_p(r, h) \leq \frac{p^2}{2\pi} \int_0^{2\pi} |h(z)|^p \left| \frac{h'(z)}{h(z)} \right|^2 d\theta \leq \frac{p^2c^2}{(1 - r)^2} I_p(r, h).$$

Moreover if $f$ is univalent, we may have a better growth estimate for the coefficients. First we remind the reader of the following result due to Littlewood, Paley, Clunie, Pommerenke and Baernstein II (see [2], [13, Theorem 8.8] and [8, Theorem 3.7]).
Theorem D. Suppose that \( f(z) = z + a_2z^2 + \cdots \in S \) satisfies \( f(z) = O(1 - |z|)^{-\alpha} \). If \( 0.491 < \alpha \leq 2 \), then \( \int_0^{2\pi} |f'(re^{i\theta})|d\theta = O(1 - r)^{-\alpha} \) and \( a_n = O(n^{\alpha - 1}) \). If \( \alpha = 0 \), in other words, if \( f \) is bounded, then \( \int_0^{2\pi} |f'(re^{i\theta})|d\theta = O(1 - r)^{-0.491} \) and \( a_n = O(n^{0.491 - 1}) \).

In view of Corollary 2.3 we have the following result as a corollary.

Theorem 3.4. Let \( f(z) = z + a_2z^2 + \cdots \in S \). If \( f \in B(\lambda) \) with \( 1.491 < \lambda \leq 3 \), then it holds that \( a_n = O(n^{\lambda - 2}) \) as \( n \to \infty \). This order estimate is best possible.

In order to see the sharpness, we may consider the function \( f(z) = (1 - z)^{1-\lambda} = 1 + a_1z + a_2z^2 + \cdots \) for \( 1 < \lambda \). We note that \( f \) is univalent in the unit disk if \( 1 < \lambda \leq 3 \). For this function, we can see that \( ||T_f|| = 2\lambda \) and \( a_n = \Gamma(\lambda + n - 1)/n!\Gamma(\lambda - 1) \sim n^{\lambda - 2} \) as \( n \to \infty \) by Stirling’s formula.

On the other hand, in the case that \( f \) is univalent with \( ||T_f|| < 3 \), the situation seems rather complicated. Given a holomorphic function \( f(z) = z + a_2z^2 + \cdots \) in the unit disk, let \( \gamma(f) \) denote the infimum of exponents \( \gamma \) such that \( a_n = O(n^{\gamma - 1}) \) as \( n \to \infty \), i.e.,

\[
\gamma(f) = \lim_{n \to \infty} \frac{\log n|a_n|}{\log n}.
\]

And, for a subset \( X \) of \( A \), we denote by \( \gamma(X) \) the supremum of \( \{\gamma(f); f \in X\} \). As for \( \gamma(S_b) \), where \( S_b \) denotes the class of normalized bounded univalent functions in the unit disk, it has been shown ([5] and [9]) that \( 0.24 < \gamma(S_b) < 0.4886 \), and conjectured by Carleson and Jones that \( \gamma(S_b) = 0.25 \). We also remark that the growth of coefficients seems to involve an irregularity of the boundary of image under \( f \) when \( f \) is bounded univalent (see [13, Chapter 10]) and, recently, Makarov and Pommerenke observed a remarkable phenomenon of phase transition of the functional \( \gamma(f) \) with respect to the Minkowski dimension of the boundary curve [9].

Now we turn to our case. Theorem 3.3 and the above example \((1 - z)^{1-\lambda}\) (or, \(-\log(1 - z)\) when \( \lambda = 1 \)) yield

\[
\lambda - 1 \leq \gamma(B(\lambda)) \leq \alpha(\lambda) = \frac{\sqrt{1 + 4\lambda^2} - 1}{2}.
\]

By standard calculations, we can see that the extremal function \( F_\lambda \) also satisfies \( \gamma(F_\lambda) = \lambda - 1 \).

For \( 0 < \lambda \leq 1/2 \), we note that \( \alpha(\lambda) \leq \lambda^2 - 2\lambda^4/3 \leq 5/24 = 0.2083 \cdots \), because \( \sqrt{1 + x} < 1 + x/2 - x^2/(6 + 4\sqrt{2}) < 1 + x/2 - x^2/12 \) for \( 0 < x \leq 1 \). Remark again that \( B(1/2) \subset S_b \).

Remark. Actually, by Theorem C, for any \( f \in A \), we have the estimate

\[
\gamma(f) \leq \frac{1}{2} \left( \sqrt{1 + 4 \left( \lim_{|z| \to 1} (1 - |z|)|T_f(z)| \right)^\frac{2}{\alpha}} - 1 \right).
\]

Next we consider the relationship between the class \( B(\lambda) \) and Hardy spaces. The following are fundamental results in the univalent function theory.
Theorem E (cf. [13]). Let $\beta$ be a constant with $0 \leq \beta \leq 2$. If a univalent function $f \in S$ satisfies that $f(z) = O(1 - |z|)^{-\beta}$ as $|z| \to 1$, then the following holds.

For $0 < p < 1/\beta$, we have $f \in H^p$. For $1/\beta < p$, we have $M_p(r, f) = O(1 - r)^{1/p - \beta}$ ($r \to 1$).

Where $M_p(r, f)$ denotes $L^p$-integral mean of $f$, i.e., $M_p(r, f) = \frac{1}{r} \frac{1}{\log \frac{1}{1-r}} \frac{1}{\log \frac{1}{1-r}} f'(re^{i\theta}) | \theta | d\theta$.

Theorem F (Pommerenke [11]). Let $f$ be a univalent holomorphic function on the unit disk. Then, $f \in BMOA$ if and only if $f$ is Bloch, i.e., $\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty$.

Combining these theorems with Theorem 2.2, we have the following results.

Theorem 3.5. Let $f \in S$ and set $||T_f|| = 2\lambda$.

If $\lambda < 1$ then $f \in H^\infty$.

If $\lambda > 1$ then $f \in H^p$ for any $0 < p < 1/(\lambda - 1)$.

If $\lambda = 1$ then $f \in BMOA$.

Note that $H^\infty \subset BMOA \subset \bigcap_{0 < p < \infty} H^p$.

Remark. Most of the above results can be extended to the case of $p$-valent, more generally, mean $p$-valent functions with $p < \infty$ (see Hayman [8]).

We shall mention a connection with integral means for univalent functions. For a univalent function $f \in S$ and a real number $p$, we set (cf. [13, Chapter 8])

$$\beta_f(p) = \lim_{r \to 1} \frac{\log \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta}{\log \frac{1}{1-r}} = \lim_{r \to 1} \frac{\log I_p(r, f')}{\log \frac{1}{1-r}}.$$

The Brennan conjecture asserts that $\beta_f(-2) \leq 1$ for every univalent holomorphic function $f$.

For $f \in B(\lambda)$, as a corollary of Theorem 3.3, we have the next

Theorem 3.6. For $f \in B(\lambda)$ and $p \in \mathbb{R}$ the inequality

$$\beta_f(p) \leq \alpha(|p| \lambda) = \frac{\sqrt{1 + 4p^2\lambda^2} - 1}{2}$$

holds. In particular, the Brennan conjecture is true for any univalent function $f$ with $||T_f|| \leq \sqrt{2}$.

4. Norm estimate for various classes of univalent functions

In this section, we provide several norm estimates for well-known classes of univalent functions. These enable us to obtain growth and coefficient estimates for those classes, which agree with known results in many cases.

The following is due to S. Yamashita. (The case of strongly starlike functions was first shown by [18].)

Theorem G (Yamashita [22]). Let $0 \leq \alpha < 1$ and $f \in S$.

If $f$ is starlike of order $\alpha$, i.e., $\text{Re}(zf'(z)/f(z)) > \alpha$, then $||T_f|| \leq 6 - 4\alpha$.

If $f$ is convex of order $\alpha$, i.e., $\text{Re}(1 + zf''(z)/f'(z)) > \alpha$, then $||T_f|| \leq 4(1 - \alpha)$. 
If \( f \) is strongly starlike of order \( \alpha \), i.e., \( \arg(zf'(z)/f(z)) < \pi \alpha/2 \), then \( \|T_f\| \leq M(\alpha) + 2\alpha \), where \( M(\alpha) \) is a specified constant depending only on \( \alpha \) satisfying \( 2\alpha < M(\alpha) < 2\alpha(1 + \alpha) \).

All of the bounds are sharp.

Remark. For the equality cases and more detailed and greatly general results, consult the paper [22] by S. Yamashita. For information about the constant \( M(\alpha) \) see [18] or [22].

Now we state general and useful principles for estimation of the norm of \( T_f \). A holomorphic function \( f \) on the unit disk is said to be subordinate to another \( g \) if \( f \) can be written as \( f = g \circ \omega \), where \( \omega \) is a holomorphic self-mapping of the unit disk with \( \omega(0) = 0 \). Remark that the Schwarz lemma implies that \( |\omega(z)| \leq |z| \) and also Pick's version of the Schwarz lemma does that

\[
(4.1) \quad \frac{|\omega'(z)|}{1-|\omega(z)|^2} \leq \frac{1}{1-|z|^2}
\]

for any point \( z \in \mathbb{D} \).

We also note that if \( g \in S \), then \( f \) is subordinate to \( g \) if and only if \( f(0) = 0 \) and \( f(\mathbb{D}) \subset g(\mathbb{D}) \).

The following always generates a sharp result for fixed \( g \). The idea is due to Littlewood.

Theorem 4.1 (Subordination Principle I). Let \( g \in B \) be given. For \( f \in A \), if \( f' \) is subordinate to \( g' \) then we have \( \|T_f\| \leq \|T_g\| \). In particular, \( f \) is uniformly locally univalent on the unit disk.

Proof. By assumption, there exists a holomorphic function \( \omega : \mathbb{D} \rightarrow \mathbb{D} \) with \( \omega(0) = 0 \) such that \( f' = g' \circ \omega \). Therefore, \( T_f = T_g \circ \omega \cdot \omega' \). Thus (4.1) implies the following:

\[
(1-|z|^2)|T_f(z)| = (1-|z|^2)|T_g(\omega)||\omega'| \leq (1-|\omega|^2)|T_g(\omega)| \leq \|T_g\|
\]

which leads to the conclusion.

As a typical application of the Subordination Principle, we exhibit the following.

Theorem 4.2. If \( f \in A \) satisfies that \( \text{Re} f' > 0 \) on the unit disk, then \( \|T_f\| \leq 2 \). The bound is sharp.

Remark. The Noshiro-Warschawski theorem says that such an \( f \) must be univalent.

Proof. The condition \( \text{Re} f' > 0 \) is equivalent to the statement that \( f' \) is subordinate to the function \( F_1(z) = \frac{1+z}{1-z} \). Thus we have \( \|T_f\| \leq \|T_{F_1}\| = 2 \).

We note that \( f' \) is a Gelfer function if \( \text{Re} f' > 0 \), where a holomorphic function \( g \) on the unit disk with \( g(0) = 1 \) is called Gelfer when \( g(z) + g(w) \neq 0 \) for all \( z, w \in \mathbb{D} \).

Therefore the next result can be viewed as a natural generalization of the above theorem.

Theorem 4.3. Suppose that \( f' \) is a Gelfer function for an \( f \in A \). Then we have \( \|T_f\| \leq 2 \). This bound is sharp.
Proof. For a Gelfer function $g(z) = f'(z)$ it is known to hold that
\[ \left| \frac{g'(z)}{g(z)} \right| \leq \frac{2}{1 - |z|^2} \] (see [21]). Hence, the result immediately follows. \qed

The next is a variant of the subordination principle.

**Theorem 4.4 (Subordination Principle II).** Let $g \in \mathcal{B}$ be given. For $f \in \mathcal{A}$, if $zf'(z)/f(z)$ is subordinate to $g'$ then we have
\[(4.2) \quad \|T_f\| \leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left( \left| \frac{g'(z) - 1}{z} \right| + |T_g(z)| \right)\]
\[(4.3) \quad \leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{g'(z) - 1}{z} \right| + \|T_g\| .\]

Proof. By assumption, there exists a holomorphic function $\omega : \mathbb{D} \rightarrow \mathbb{D}$ with $\omega(0) = 0$ such that $zf'(z)/f(z) = g'(\omega(z))$. By taking logarithmic derivative, we have the following formula.
\[ T_f = \frac{f'}{f} - \frac{1}{z} + \frac{g''(\omega)}{g'(\omega)} \omega' = \frac{\omega g'(\omega) - 1}{\omega} + T_g(\omega) \omega'. \]
From this, we can easily have the desired estimate. \qed

The following is a simple application of this principle.

**Theorem 4.5.** If $f \in \mathcal{A}$ satisfies that $|zf'(z)/f(z) - 1| < 1$, then we have an estimate $\|T_f\| \leq 2.25$. The equality holds if and only if $f$ is a rotation of the function $ze^z$.

**Remark.** In this case, $f$ satisfies $\text{Re} f'(z)/f(z) > 0$ thus $f$ is starlike, in particular, univalent in the unit disk.

Proof. We have only to apply the estimate (4.2) with $g(z) = z + z^2/2$. Then, we have $\|T_f\| \leq \sup (2 + |z| - |z|^2) = 9/4$, where the supremum is attained only by $|z| = 1/2$. Thus, if $\|T_f\| = 9/4$, then $|\omega|$ must be the constant 1, whence $f$ is a rotation of $ze^z$. Conversely, it is clear that the function $f(z) = ze^{\mu z}$ with $|\mu| = 1$ satisfies $\|T_f\| = 9/4$. \qed

Finally, we consider uniformly convex functions:
\[ \text{UCV} = \left\{ f \in \mathcal{S}; \text{Re} \left( 1 + (z - \zeta) f''(z) \right) f'(z) \geq 0, \forall z, \forall \zeta \in \mathbb{D} \right\} . \]
For the geometric meaning of this class, see [7]. Rønning gave a simple characterization for this class.

**Theorem H (Rønning [16]).** A function $f \in \mathcal{A}$ is uniformly convex if and only if $zf_T(z) \in W$ for any $z \in \mathbb{D}$, where $W$ is the domain $\{ w = u + iv; v^2 < 2u + 1 \}$. 


We note that a conformal map $g : \mathbb{D} \to W$ with $g(0) = 0$ is given by
\[
g(z) = \frac{2}{\pi^2} \left( \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 = \frac{8z}{\pi^2} \left( 1 + \frac{z}{3} + \frac{z^2}{5} + \frac{z^3}{7} + \cdots \right)^2.
\]
Therefore, $f \in \mathcal{A}$ is uniformly convex if and only if $zT_f(z)$ is subordinate to the function $g$, i.e., there exists a holomorphic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $zT_f(z) = g(\omega(z))$. Since $g$ has positive Taylor coefficients, we see that $|zT_f(z)| \leq g(|\omega(z)|) \leq g(|z|)$.

Hence, we have
\[
\|T_f(z)\| \leq \sup_{0<x<1} (1-x^2) \frac{g(x)}{x} = \sup_{0<t<\infty} h(t),
\]
where
\[
h(t) = \frac{8t^2 \cosh t}{\pi^2 \sinh^2 t}
\]
and $\frac{1+\sqrt{x}}{1-\sqrt{x}} = e^t$. By the logarithmic differentiation, we have
\[
\frac{h'(t)}{h(t)} = \frac{2 \sinh 2t - t(\cosh 2t + 3)}{t \sinh 2t} = \frac{N(t)}{t \sinh 2t}.
\]
Since $N''(t) = \frac{4 \tanh 2t}{\cosh 2t}$ has the unique zero $t_0$ in $(0, \infty)$, the function $N'(t) = 3(\cosh 2t - 1) - 2t \sinh 2t$ attains its maximum at $t_0$. Since $N'(0) = 0$ and $N'(t) \to -\infty$ as $t \to \infty$, the function $N(t)$ has the unique zero $t_1 > t_0$ in $(0, \infty)$. By exactly same reason, the function $N(t)$ has the unique zero $t_2 > t_1$ in $(0, \infty)$. Thus, $h(t)$ assumes its maximum at the point $t = t_2$. By a numerical calculation, we have $t_2 = 1.6061152988 \cdots$, and $h(t_2) = 0.94774221287 \cdots$. Therefore, we summarize as follows.

**Theorem 4.6.** If $f \in \mathcal{A}$ is uniformly convex, then we have
\[
\|T_f\| \leq h(t_2) = 0.94774 \cdots,
\]
where the equality occurs only when $f$ is a rotation of the function $F \in \mathcal{A}$ determined by $T_F(z) = g(z)/z$.

**Remark.** By the corollary of Theorem A, we see that a uniformly convex function can be extended to a quasiconformal self-homeomorphism of the Riemann sphere with maximal dilatation at most $K_0 = 37.2718 \cdots$.

**Acknowledgements.** A part of this work was carried out during authors' visit to Fukuoka University, February 1998. They sincerely thank Fukuoka University, especially, Professor Megumi Saigo for the invitation of them.

The first named author was partially supported by BSRI-98-1401 and KOSEF. The second named author was partially supported by the Ministry of Education, Science, Culture and Sports, Grant-in-Aid for Encouragement of Young Scientists, 9740056, 1997.

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