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ON SUFFICIENT CONDITIONS FOR
MEROMORPHIC STARLIKE FUNCTIONS

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ABSTRACT. The object of the present paper is to show certain sufficient conditions for starlikeness and close-to-convexity of meromorphic functions in the punctured unit disk.

1. Introduction

Let \( \Sigma \) denote the class of functions of the form

\[
f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (1.1)
\]

which are analytic in the punctured unit disk \( D = \{ z : 0 < |z| < 1 \} \). For \( f \) and \( g \) which are analytic in \( U = \{ z : |z| < 1 \} \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \) or \( f(z) \prec g(z) \), if \( g \) is univalent, \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

For \( 0 < \alpha \leq 1 \), let \( \text{SM}_\alpha \Sigma \) denote the class of functions \( f \in \Sigma \) which are starlike of order \( \alpha \); that is, which satisfy

\[
-\frac{zf'(z)}{f(z)} < \left( \frac{1+z}{1-z} \right)^\alpha \quad (z \in U). \quad (1.2)
\]

We note that the equation (1.2) can be rewritten by the following form;

\[
\left| \arg \left( \frac{-zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U).
\]

Also, we note that if \( \alpha = 1 \), \( \text{SM}_1 \Sigma \) coincides with \( \Sigma^* \), the well known class of meromorphic starlike(univalent) functions with respect to origin.

In [1], Bajpai and Mehrok proved that the functions of the form (1.1) satisfying the condition

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Re \left\{ \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \left( \alpha + \beta \right) \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U)

are univalent and meromorphic starlike, where \alpha and \beta are real numbers. For various other interesting developments involving analytic functions in the open unit disk \( U \), the reader may be referred (for example) to the recent work of Nunokawa[3].

In this paper, we investigate some sufficient conditions for starlikeness and close-to-convexity of functions belonging to \( \Sigma \).

2. Main results

In proving our theorems, we need the following lemma due to Nunokawa [2].

**Lemma 2.1** Let \( p \) be analytic in \( U \), \( p(0) = 1 \) and \( p(z) \neq 0 \) in \( U \). Suppose that there exists a point \( z_0 \in U \) such that

\[
|\arg p(z)| < \frac{\pi}{2}\delta \quad \text{for} \quad |z| < |z_0|
\]

and

\[
|\arg p(z_0)| = \frac{\pi}{2}\delta \quad (0 < \delta \leq 1).
\]

Then we have

\[
\frac{z_0p'(z_0)}{p(z_0)} = i\delta k,
\]

where

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\delta,
\]

\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\delta,
\]

and

\[
\{p(z_0)\}^{-\frac{1}{2}} = \pm ia \quad (a > 0).
\]
Applying Lemma 2.1, we have the following

**Theorem 2.1.** Let $p$ be analytic in $U$ with $p(0) = 1$. If

$$\left| \arg \left( \beta p(z) + \alpha \frac{zp'(z)}{p(z)} \right) \right| < \frac{\pi}{2} \gamma(\alpha, \beta, \delta) \ (\alpha, \beta > 0, 0 < \delta < 1, z \in U),$$

where

$$\gamma(\alpha, \beta, \delta) = \frac{2}{\pi} \tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta}{\beta (1 + \delta) \frac{1 + \delta}{4} (1 - \delta) \frac{1 - \delta}{2} \cos \frac{\pi}{2} \delta} \right\},$$

then

$$|\arg p(z)| < \frac{\pi}{2} \delta.$$

**Proof.** If there exists a point $z_0 \in U$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.1) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

From (2.7), we note that $p(z) \neq 0$ in $U$. In fact, if $p$ has a zero of order $m$ at $z = z_1 \in U$, then $p$ can be written as

$$p(z) = (z - z_1)^m q(z) \ (m \in N = \{1, 2, \cdots\}),$$

where $q$ is analytic in $U$ and $q(z_1) \neq 0$. Hence we have

$$\beta p(z) + \alpha \frac{zp'(z)}{p(z)} = \frac{\alpha mz}{z - z_1} + \alpha \frac{zq'(z)}{q(z)} + \beta (z - z_1)^m q(z).$$

But choosing $z \to z_1$ suitably, the argument of the right hand side of (2.9) can take any value between $0$ and $2\pi$. This contradicts (2.7). Hence we have $p(z) \neq 0 \ (z \in U)$. Then we obtain

$$\beta p(z_0) + \alpha \frac{z_0p'(z_0)}{p(z_0)} = \beta (\pm i a)^\delta + i \alpha \delta k \quad \text{for} \quad k = 1, 2, \cdots, \beta a^\delta \cos \frac{\pi}{2} \delta + i \left\{ \beta a^\delta \sin \frac{\pi}{2} \delta + \alpha \delta k \right\}.$$

Now we suppose that

$$\{p(z_0)\}^\frac{1}{\delta} = i a \ (a > 0).$$
Then we have
\[
\arg \left( \beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) = \tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta k}{\beta \cos \frac{\pi}{2} \delta} \right\},
\]
where
\[
ka^{-\delta} \geq \frac{1}{2} (a^{-1} - 1 - \alpha) \equiv g(a) \ (a > 0).
\]
Hence, by a simple calculation, we see that the function \( g(a) \) takes the minimum value at \( a = \sqrt{\frac{1 + \alpha}{1 - \alpha}} \). Hence we have
\[
\arg \left( \beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) \leq \tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta}{\beta(1 + \delta)(1 - \alpha)^{\frac{1 - \alpha}{2}} \cos \frac{\pi}{2} \delta} \right\}
= -\frac{\pi}{2} \gamma(\alpha, \beta, \delta),
\]
where \( \gamma(\alpha, \beta, \delta) \) is given by (2.8). This evidently contradicts the assumption of Theorem 2.1.

Next, we suppose that
\[
\{p(z_0)\}^{\frac{1}{2}} = -ia \ (a > 0).
\]
Applying the same method as the above, we have
\[
\arg \left( \beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) \geq -\tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta}{\beta(1 + \delta)^{\frac{1 + \delta}{2}} (1 - \alpha)^{\frac{1 - \alpha}{2}} \cos \frac{\pi}{2} \delta} \right\}
= -\frac{\pi}{2} \gamma(\alpha, \beta, \delta),
\]
where \( \gamma(\alpha, \beta, \delta) \) is given by (2.8), which is a contradiction to the assumption of Theorem 2.1. Therefore, we complete the proof of Theorem 2.1.

Taking \( p(z) = -\frac{zf'(z)}{f(z)} \) in Theorem 2.1, we have

**Corollary 2.1.** If \( f \in \Sigma \) satisfies the condition
\[
\left| \arg \left\{ \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - (\alpha + \beta) \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \gamma(\alpha, \beta, \delta) \ (\alpha, \beta > 0, 0 < \delta < 1, z \in U),
\]
where $\gamma(\alpha, \beta, \delta)$ is given by (2.8), then $f \in \mathcal{SMS}(\delta)$.

Next, we prove

**Theorem 2.2.** Let $\alpha \geq 0$ or $\alpha \leq -2\beta (\beta > 0)$. If $p$ satisfies the condition

\[ \beta p(z) + \alpha \frac{zp'(z)}{p(z)} \neq ik \quad (z \in U), \]

where $k$ is a real number with $|k| \geq \sqrt{(\alpha + 2\beta)\alpha}$. Then $\text{Re} \ p(z) > 0 (z \in U)$.

**Proof.** For the case $\alpha = 0$, it is obvious and so we suppose $\alpha \neq 0$. By using the same method of the proof in Theorem 2.1, we can see easily that $p(z) \neq 0$ in $U$. Suppose that there exists a point $z_0 \in U$ such that

\[ \text{Re} \ p(z) > 0 \quad \text{for} \quad |z| < |z_0|, \]

\[ \text{Re} \ p(z_0) = 0 \quad \text{and} \quad p(z_0) = ia \quad (a \neq 0). \]

For the case $\alpha > 0$, from Lemma 2.1 with $\delta = 1$, we have

\[ \beta p(z_0) + \alpha \frac{z_0p'(z_0)}{p(z_0)} = i(\beta a + \alpha k), \]

and

\[ \beta a + \alpha k \geq \frac{1}{2} ((\alpha + 2\beta) a + \frac{\alpha}{a}) \geq \sqrt{(\alpha + 2\beta)\alpha} \quad \text{when} \ a > 0, \]

and

\[ \beta a + \alpha k \leq -\frac{1}{2} ((\alpha + 2\beta)|a| + \frac{\alpha}{|a|}) \leq -\sqrt{(\alpha + 2\beta)\alpha} \quad \text{when} \ a < 0, \]

which contradict (2.10). Therefore we have $\text{Re} \ p(z) > 0$ in $U$. For the case $a \leq -2b$, applying the same method as the above, we easily have the same conclusion. This completes the proof of our theorem.

Letting $p(z) = -\frac{zf''(z)}{f'(z)}$ in Theorem 2.2, we easily have the following

**Corollary 2.2.** Let $\alpha \geq 0$ or $\alpha \leq -2\beta (\beta > 0)$. If $f \in \Sigma$ satisfies the condition

\[ \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - (\alpha + \beta) \frac{zf'(z)}{f(z)} \neq ik \quad (z \in U), \]
where \( k \) is real number with \( |k| \geq \sqrt{(\alpha + 2\beta)\alpha} \), then \( f \in \Sigma^* \).

Making \( \alpha = \beta = 1 \) in Corollary 2.2, we obtain

**Corollary 2.3.** Let \( f \in \Sigma \) and suppose that there exists a real number \( R \) for which

\[
\left| \frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)} - R \right| < \sqrt{(R + 1)^2 + 3} \quad (z \in U).
\]

Then \( f \) is meromorphic starlike in \( U \).

Putting \( p(z) = -z^2 f'(z) \) in Theorem 2.2, we get

**Corollary 2.4.** Let \( \alpha \geq 0 \) or \( \alpha \leq -2\beta (\beta > 0) \). If \( f \in \Sigma \) satisfies the condition

\[
\alpha \left( 2 + \frac{zf''(z)}{f'(z)} \right) - \beta z^2 f'(z) \neq ik \quad (z \in U),
\]

where \( k \) is given by Corollary 2.2. Then \( f \) is meromorphic univalent (or close-to-convex) in \( U \).

Similarly, from Corollary 2.4, we have

**Corollary 2.5.** Let \( f \in \Sigma \) and suppose that there exists a real number \( R \) for which

\[
\left| \frac{zf''(z)}{f'(z)} - z^2 f'(z) - R \right| < \sqrt{(R + 2)^2 + 3} \quad (z \in U).
\]

Then \( f \) is meromorphic univalent (or close-to-convex) in \( U \).

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**References**

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