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<th>Title</th>
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</thead>
<tbody>
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ON SUFFICIENT CONDITIONS FOR
MEROMORPHIC STARLIKE FUNCTIONS

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ABSTRACT. The object of the present paper is to show certain sufficient conditions
for starlikeness and close-to-convexity of meromorphic functions in the punctured
unit disk.

1. Introduction

Let $\Sigma$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the punctured unit disk $D = \{ z : 0 < |z| < 1 \}$. For $f$ and $g$
which are analytic in $U = \{ z : |z| < 1 \}$, we say that $f$ is subordinate to $g$, written
$f \prec g$ or $f(z) \prec g(z)$, if $g$ is univalent, $f(0)=g(0)$ and $f(U) \subset g(U)$.

For $0 < \alpha \leq 1$, let $\mathcal{S}M^{\alpha}$ denote the class of functions $f \in \Sigma$ which are
starlike of order $\alpha$; that is, which satisfy

$$-\frac{zf'(z)}{f(z)} \prec \left( \frac{1+z}{1-z} \right)^{\alpha} \quad (z \in U). \quad (1.2)$$

We note that the equation (1.2) can be rewritten by the following form :

$$\left| \arg \left( -\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U).$$

Also, we note that if $\alpha = 1$, $\mathcal{S}M^{\alpha}$ coincides with $\Sigma^*$, the well known class of
meromorphic starlike(univalent) functions with respect to origin.

In [1], Bajpai and Mehrokr proved that the functions of the form (1.1) satisfying
the condition

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are univalent and meromorphic starlike, where \( \alpha \) and \( \beta \) are real numbers. For various other interesting developments involving analytic functions in the open unit disk \( U \), the reader may be referred (for example) to the recent work of Nunokawa \[3\].

In this paper, we investigate some sufficient conditions for starlikeness and close-to-convexity of functions belonging to \( \Sigma \).

2. Main results

In proving our theorems, we need the following lemma due to Nunokawa \[2\].

**Lemma 2.1**  Let \( p \) be analytic in \( U \), \( p(0) = 1 \) and \( p(z) \neq 0 \) in \( U \). Suppose that there exists a point \( z_0 \in U \) such that

\[
|\arg p(z)| < \frac{\pi}{2} \delta \quad \text{for} \quad |z| < |z_0|
\]

(2.1)

and

\[
|\arg p(z_0)| = \frac{\pi}{2} \delta \quad (0 < \delta \leq 1).
\]

(2.2)

Then we have

\[
\frac{z_0 p'(z_0)}{p(z_0)} = i \delta k,
\]

(2.3)

where

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2} \delta,
\]

(2.4)

\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2} \delta,
\]

(2.5)

and

\[
\{p(z_0)\}^{\frac{1}{2}} = \pm ia \quad (a > 0).
\]

(2.6)
Applying Lemma 2.1, we have the following

**Theorem 2.1.** Let $p$ be analytic in $U$ with $p(0) = 1$. If

$$\left| \arg \left( \beta p(z) + \alpha \frac{zp'(z)}{p(z)} \right) \right| < \frac{\pi}{2} \gamma(\alpha, \beta, \delta) \ (\alpha, \beta > 0, 0 < \delta < 1, z \in U), \quad (2.7)$$

where

$$\gamma(\alpha, \beta, \delta) = \frac{2}{\pi} \tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta}{\beta (1 + \delta) \frac{1 + \delta}{2} \cos \frac{\pi}{2} \delta} \right\}, \quad (2.8)$$

then

$$\left| \arg p(z) \right| < \frac{\pi}{2} \delta.$$

**Proof.** If there exists a point $z_0 \in U$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.1) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

From (2.7), we note that $p(z) \neq 0$ in $U$. In fact, if $p$ has a zero of order $m$ at $z = z_1 \in U$, then $p$ can be written as

$$p(z) = (z - z_1)^m q(z) \ (m \in N = \{1, 2, \cdots \}),$$

where $q$ is analytic in $U$ and $q(z_1) \neq 0$. Hence we have

$$\beta p(z) + \alpha \frac{zp'(z)}{p(z)} = \frac{\alpha mz}{z - z_1} + \alpha \frac{zq'(z)}{q(z)} + \beta (z - z_1)^m q(z). \quad (2.9)$$

But choosing $z \to z_1$ suitably, the argument of the right hand side of (2.9) can take any value between 0 and $2\pi$. This contradicts (2.7). Hence we have $p(z) \neq 0 \ (z \in U)$. Then we obtain

$$\beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} = \beta (\pm ia)^\delta + i \alpha \delta k$$

$$= \beta a^\delta \cos \frac{\pi}{2} \delta + i \left\{ \beta a^\delta \sin \frac{\pi}{2} \delta + \alpha \delta k \right\}.$$

Now we suppose that

$$\{p(z_0)\}^\frac{1}{\delta} = ia \ (a > 0).$$
Then we have

\[
\arg \left( \beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) = \tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta k}{\beta \cos \frac{\pi}{2} \delta} \right\},
\]

where

\[
ka^\delta \geq \frac{1}{2} \left( a^{1-\alpha} + a^{-1-\alpha} \right) \equiv g(a) \ (a > 0).
\]

Hence, by a simple calculation, we see that the function \( g(a) \) takes the minimum value at \( a = \sqrt{\frac{1+\alpha}{1-\alpha}} \). Hence we have

\[
\arg \left( \beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) \leq \tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta}{\beta (1+\delta)^{\frac{1+\delta}{2}} (1-\alpha)^{\frac{1-\alpha}{2}} \cos \frac{\pi}{2} \delta} \right\}
\]

\[
= \frac{\pi}{2} \gamma(\alpha, \beta, \delta),
\]

where \( \gamma(\alpha, \beta, \delta) \) is given by (2.8). This evidently contradicts the assumption of Theorem 2.1.

Next, we suppose that

\[
\{ p(z_0) \}^\frac{1}{2} = -ia \ (a > 0).
\]

Applying the same method as the above, we have

\[
\arg \left( \beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) \geq -\tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta}{\beta (1+\delta)^{\frac{1+\delta}{2}} (1-\alpha)^{\frac{1-\alpha}{2}} \cos \frac{\pi}{2} \delta} \right\}
\]

\[
= -\frac{\pi}{2} \gamma(\alpha, \beta, \delta),
\]

where \( \gamma(\alpha, \beta, \delta) \) is given by (2.8), which is a contradiction to the assumption of Theorem 2.1. Therefore, we complete the proof of Theorem 2.1.

Taking \( p(z) = -\frac{zf'(z)}{f(z)} \) in Theorem 2.1, we have

**Corollary 2.1.** If \( f \in \Sigma \) satisfies the condition

\[
\left| \arg \left\{ \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - (\alpha + \beta) \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \gamma(\alpha, \beta, \delta) \ (\alpha, \beta > 0, 0 < \delta < 1, z \in U),
\]
where $\gamma(\alpha, \beta, \delta)$ is given by (2.8), then $f \in \text{SMS}(\delta)$.

Next, we prove

**Theorem 2.2.** Let $\alpha \geq 0$ or $\alpha \leq -2\beta (\beta > 0)$. If $p$ satisfies the condition

\[(2.10) \quad \beta p(z) + \alpha \frac{zp'(z)}{p(z)} \neq ik \quad (z \in U),\]

where $k$ is a real number with $|k| \geq \sqrt{(\alpha + 2\beta)\alpha}$. Then $\Re p(z) > 0 (z \in U)$.

**Proof.** For the case $\alpha = 0$, it is obvious and so we suppose $\alpha \neq 0$. By using the same method of the proof in Theorem 2.1, we can see easily that $p(z) \neq 0$ in $U$. Suppose that there exists a point $z_0 \in U$ such that

\[\Re p(z) > 0 \quad \text{for} \quad |z| < |z_0|,\]

\[\Re p(z_0) = 0 \quad \text{and} \quad p(z_0) = ia \quad (a \neq 0).\]

For the case $\alpha > 0$, from Lemma 2.1 with $\delta = 1$, we have

\[\beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} = i(\beta a + \alpha k),\]

and

\[\beta a + \alpha k \geq \frac{1}{2} ((\alpha + 2\beta) a + \frac{\alpha}{a}) \geq \sqrt{(\alpha + 2\beta)\alpha} \text{ when } a > 0,\]

and

\[\beta a + \alpha k \leq -\frac{1}{2} ((\alpha + 2\beta)|a| + \frac{\alpha}{|a|}) \leq -\sqrt{(\alpha + 2\beta)\alpha} \text{ when } a < 0,\]

which contradict (2.10). Therefore we have $\Re p(z) > 0$ in $U$. For the case $a \leq -2b$, applying the same method as the above, we easily have the same conclusion. This completes the proof of our theorem.

Letting $p(z) = \frac{-zf'(z)}{f(z)}$ in Theorem 2.2, we easily have the following

**Corollary 2.2.** Let $\alpha \geq 0$ or $\alpha \leq -2\beta (\beta > 0)$. If $f \in \Sigma$ satisfies the condition

\[\alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) - (\alpha + \beta) \frac{zf'(z)}{f(z)} \neq ik \quad (z \in U),\]
where $k$ is real number with $|k| \geq \sqrt{(\alpha + 2\beta)\alpha}$, then $f \in \Sigma^*$. 

Making $\alpha = \beta = 1$ in Corollary 2.2, we obtain

**Corollary 2.3.** Let $f \in \Sigma$ and suppose that there exists a real number $R$ for which

$$\left| \frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)} - R \right| < \sqrt{(R + 1)^2 + 3} \quad (z \in U).$$

Then $f$ is meromorphic starlike in $U$.

Putting $p(z) = -z^2f'(z)$ in Theorem 2.2, we get

**Corollary 2.4.** Let $\alpha \geq 0$ or $\alpha \leq -2\beta (\beta > 0)$. If $f \in \Sigma$ satisfies the condition

$$\alpha \left(2 + \frac{zf''(z)}{f'(z)}\right) - \beta z^2f'(z) \neq ik \quad (z \in U),$$

where $k$ is given by Corollary 2.2. Then $f$ is meromorphic univalent (or close-to-convex) in $U$.

Similarly, from Corollary 2.4, we have

**Corollary 2.5.** Let $f \in \Sigma$ and suppose that there exists a real number $R$ for which

$$\left| \frac{zf''(z)}{f'(z)} - z^2f'(z) - R \right| < \sqrt{(R + 2)^2 + 3} \quad (z \in U).$$

Then $f$ is meromorphic univalent (or close-to-convex) in $U$.

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**References**

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