ON STARLIKENESS AND CONVEXITY OF FUNCTIONS AND THE SCHWARZIAN DERIVATIVE (Applications of Complex Function Theory to Differential Equations)

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ON STARLIKENESS AND CONVEXITY OF FUNCTIONS AND THE SCHWARZIAN DERIVATIVE

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ABSTRACT. The purpose of this paper is to generalize Miller and Mocanu's result [2].

1. Introduction

Let \( \mathcal{A} \) denote the class of functions \( f(z) \) defined by

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]

which are analytic in the open unit disk \( \mathcal{U} = \{ z : z \in \mathbb{C}, \text{and} |z| < 1 \} \). Also, let \( \mathcal{S} \) denote the class of all functions in \( \mathcal{A} \) which are univalent in \( \mathcal{U} \). A function \( f(z) \) belonging to the class \( \mathcal{S} \) is said to be in the class \( \mathcal{S}^* \) if and only if

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}
\]

and is said to be in the class \( \mathcal{C} \) if and only if

\[
1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}.
\]

We denote by \( \{f, z\} \) the Schwarzian derivative, which is characterized by the equality

\[
\{f, z\} = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]

In [1], Nunokawa et al. obtained the following result:

**Theorem A.** Let \( f(z) \in \mathcal{A} \) and suppose that

\[
\text{Re} \left[ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} + z^2 \{f, z\} \right) \right] \geq -\frac{1}{2} \quad \text{in} \quad \mathcal{U}.
\]

Then \( f(z) \in \mathcal{S}^* \).
Remark. Theorem A is an extension of Miller and Mocanu [2], where the right hand side of (2) is improved from 0 to $-\frac{1}{2}$.

Further Miller and Mocanu [2] obtained the following results:

Theorem B. Let $f(z) \in A$ satisfy

$$\text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^2 + z^2 \{f, z\} \right] > 0 \quad \text{in } \mathcal{U}.$$

Then $f(z) \in C$.

Theorem C. Let $f(z) \in A$ satisfy

$$\text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) e^{z^2 \{f, z\}} \right] > 0 \quad \text{in } \mathcal{U}.$$

Then $f(z) \in C$.

Let us investigate improvements of these results in the next section.

2. Main Results

The following result will be required in our investigation:

Lemma. [3] Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and suppose that there exists a point $z_0 \in \mathcal{U}$ such that $\text{Re} \{p(z)\} > 0$ for $|z| < |z_0|$ and $\text{Re} \{p(z_0)\} = 0$ ($p(z_0) \neq 0$). Then we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik,$$

where $k$ is a real number and

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \geq 1 \quad \text{when } \ p(z_0) = ia, \ a > 0,$$

and

$$k \leq \frac{1}{2} \left( a + \frac{1}{a} \right) \leq -1 \quad \text{when } \ p(z_0) = ia, \ a < 0.$$

Now we state our main result.

Theorem 1. Let $f(z) \in A$ and satisfy one of the following inequalities:

$$\text{Re} \left[ \left( \frac{zf''(z)}{f'(z)} \right)^{4m-1} \left( 1 + \frac{zf''(z)}{f'(z)} + z^2 \{f, z\} \right) \right]$$

$$\leq \frac{1}{2} \left| \frac{zf'(z)}{f(z)} \right|^{4m-2} \left( 3 \left| \frac{zf'(z)}{f(z)} \right|^2 + 1 \right) \quad \text{in } \mathcal{U},$$
$\text{Re}\left[\left(\frac{zf'(Z)}{f(z)}\right)^{4m-3}\left(1 + \frac{zf''(z)}{f'(z)} + z^2\{f, z\}\right)\right]$

$> - \frac{1}{2} \left|\frac{zf'(Z)}{f(z)}\right|^{4m-4} \left(3 \left|\frac{zf'(z)}{f(z)}\right|^2 + 1\right)$ in $\mathcal{U}$,

where $m$ is a positive integer. Then $f(z) \in S^*$.

**Proof.** Let us put

$$p(z) = \frac{zf''(z)}{f'(z)},$$

then we easily have

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}$$

and from (1), by a simple calculation, we have

$$z^2\{f, z\} = z^2 \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2} \left(\frac{zf''(z)}{f'(z)}\right)^2$$

$$= \frac{zp'(z)}{p(z)} + \frac{z^2p''(z)}{p(z)} - \frac{3}{2} \left(\frac{zp'(z)}{p(z)}\right)^2 + \frac{1}{2}\{1 - p(z)^2\}.$$  

To prove $\text{Re}\left\{zf'(Z)/f(z)\right\} > 0$ in $\mathcal{U}$, we show $\text{Re}\{p(z)\} > 0$ in $\mathcal{U}$. If there exists a point $z_0 \in \mathcal{U}$ such that

$$\text{Re}\{p(z)\} > 0 \quad \text{for} \quad |z| < |z_0|$$

and

$$\text{Re}\{p(z_0)\} = 0 \quad (p(z_0) \neq 0),$$

then from Lemma we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik,$$

and (3), (4) and (5) imply

$$\left(\frac{z_0f'(z_0)}{f(z_0)}\right)^l \left(1 + \frac{z_0f''(z_0)}{f'(z_0)} + z_0^2\{f, z_0\}\right)$$

$$= (ia)^l \left[ia + ik + ik + \frac{z_0^2p''(z_0)}{p(z_0)} - \frac{3}{2}(ik)^2 + \frac{1}{2}\{1 - (ia)^2\}\right]$$

$$= (ia)^l \left[i\left\{a + k + k \left(1 + \frac{z_0p''(z_0)}{p'(z_0)}\right)\right\} + \frac{3}{2}k^2 + \frac{1}{2}(1 + a^2)\right],$$

where $l$ is a positive integer. Let $J$ be the right hand side of (6). For the case $l = 2n - 1$,

$$J = (ia)^{2n-1} \left[i\left\{a + k + k \left(1 + \frac{z_0p''(z_0)}{p'(z_0)}\right)\right\} + \frac{3}{2}k^2 + \frac{1}{2}(1 + a^2)\right],$$
Therefore we have
\[
\text{Re} \{ J \} = (-1)^n a^{2n-1} \left[ a + k + k \left( 1 + \text{Re} \left\{ \frac{z_0 p''(z_0)}{p'(z_0)} \right\} \right) \right].
\]

Considering the geometrical property, we notice that the tangential vector of the curve \( p(z) = p(z_0) \) moves to positive direction near the point \( p(z_0) \). In short, \( p(z) \) is convex in the neighborhood of the point \( p(z_0) \), or
\[
1 + \text{Re} \left\{ \frac{z_0 p''(z_0)}{p'(z_0)} \right\} \geq 0.
\]

(i) Case \( n = 2m \):
\[
\text{Re} \{ J \} = (-1)^{2m} a^{4m-1} \left[ a + k + k \left( 1 + \text{Re} \left\{ \frac{z_0 p''(z_0)}{p'(z_0)} \right\} \right) \right]
\geq a^{4m-1} (a + k)
= -a^{4m-2} (a^2 - ak)
\geq -a^{4m-2} \left\{ -a^2 - \frac{1}{2} (a^2 + 1) \right\}
= -a^{4m-2} \left( -\frac{3}{2} a^2 - \frac{1}{2} \right)
= \frac{1}{2} \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|^{4m-2} \left( 3 \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|^2 + 1 \right).
\]

(ii) Case \( n = 2m - 1 \):
\[
\text{Re} \{ J \} = (-1)^{2m-1} a^{4m-3} \left[ a + k + k \left( 1 + \text{Re} \left\{ \frac{z_0 p''(z_0)}{p'(z_0)} \right\} \right) \right]
\leq -a^{4m-3} (a + k)
= a^{4m-4} (-a^2 - ak)
\leq a^{4m-4} \left\{ -a^2 - \frac{1}{2} (a^2 + 1) \right\}
= a^{4m-4} \left( -\frac{3}{2} a^2 - \frac{1}{2} \right)
= -\frac{1}{2} \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|^{4m-4} \left( 3 \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|^2 + 1 \right).
\]

These contradict (3) and (4), respectively. Hence we must have
\[
\text{Re} \{ p(z) \} > 0 \text{ in } \mathcal{U}
\]
or
\[
\text{Re} \left\{ \frac{zf''(z)}{f(z)} \right\} > 0 \text{ in } \mathcal{U},
\]
which means \( f(z) \in S^* \). This completes our proof.

Setting \( \alpha = 1 \) in Theorem 1, we obtain
Corollary 1. Let \( f(z) \in A \) and suppose that

\[
\text{Re} \left[ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} + z^2 \{f, z\} \right) \right] > -\frac{1}{2} \left( 1 + 3 \left| \frac{zf'(z)}{f(z)} \right|^2 \right) \quad \text{in } \mathcal{U}.
\]

Then \( f(z) \in S^* \).

Corollary 1 is better than Theorem A.

Theorem 2. Let \( f(z) \in A \) and suppose that

\[
(7) \quad \text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{2n} + z^2 \{f, z\} \right] + (-1)^{n+1} \left| 1 + \frac{zf''(z)}{f'(z)} \right|^{2n} > 0 \quad \text{in } \mathcal{U}
\]

for positive integer \( n \). Then \( f(z) \in C \).

Proof. Let us put

\[
q(z) = 1 + \frac{zf''(z)}{f'(z)}.
\]

Note that \( q(0) = 1 \). Then from (1), we easily have

\[
(8) \quad z^2 \{f, z\} = zq'(z) - \frac{1}{2}q(z)^2 + \frac{1}{2}.
\]

To prove \( 1 + \text{Re} \{zf''(z)/f'(z)\} > 0 \) in \( \mathcal{U} \), we show \( \text{Re} \{q(z)\} > 0 \) in \( \mathcal{U} \). If there exists a point \( z_0 \in \mathcal{U} \) such that

\[
\text{Re} \{q(z)\} > 0 \quad \text{for } |z| < |z_0|
\]

and

\[
\text{Re} \{q(z_0)\} = 0 \quad (q(z_0) \neq 0),
\]

then from Lemma a real number \( k (k \neq 0) \) exists such that

\[
\frac{z_0q'(z_0)}{q(z_0)} = ik.
\]

From (7) and (8), we have

\[
\text{Re} \left[ \left( 1 + \frac{z_0f''(z_0)}{f'(z_0)} \right)^{2n} + z_0^2 \{f, z_0\} \right] + (-1)^{n+1} \left| 1 + \frac{z_0f''(z_0)}{f'(z_0)} \right|^{2n}
\]

\[
= \text{Re} \left\{ (q(z_0))^{2n} + z_0q'(z_0) - \frac{1}{2}q(z_0)^2 + \frac{1}{2} \right\} + (-1)^{n+1} |q(z_0)|^{2n}
\]

\[
= \text{Re} \left\{ (ia)^{2n} - ak - \frac{1}{2} (ia)^2 + \frac{1}{2} \right\} + (-1)^{n+1} |ia|^{2n}
\]

\[
\leq (-1)^n a^{2n} - \frac{1}{2} \left( a^2 + 1 \right) + \frac{1}{2} \left( a^2 + 1 \right) + (-1)^{n+1} |a|^{2n}
\]

\[
= 0.
\]
This is in contradiction to (7). Hence we must have

\[ \text{Re}\{q(z)\} > 0 \quad \text{in} \quad \mathcal{U} \]

or

\[ 1 + \text{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}. \]

Therefore \( f(z) \in C \) and our result is established.

Taking \( n = 1 \) in Theorem 2, we have

**Corollary 2.** Let \( f(z) \in \mathcal{A} \) and suppose that

\[ \text{Re}\left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^2 + z^2 \{f, z\} \right] + \left| 1 + \frac{zf''(z)}{f'(z)} \right|^2 > 0 \quad \text{in} \quad \mathcal{U}. \]

Then \( f(z) \in C \).

Corollary 2 is better than Theorem B.

**Theorem 3.** Let \( f(z) \in \mathcal{A} \) and suppose that

\[ \text{Re}\left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{2n-1} e^{z^2\{f, z\}} \right] \neq 0 \quad \text{in} \quad \mathcal{U}. \]

Then \( f(z) \in C \).

**Proof.** Let us take the same function \( q(z) \) as in the proof of Theorem 2. Then from the assumption of theorem and (8), we find

\[
\begin{align*}
\text{Re}\left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{2n-1} e^{z^2\{f, z\}} \right] &= \text{Re}\left[ (q(z))^2 e^{zq'(z)} - \frac{1}{2} q(z)^2 + \frac{1}{2} \right] \\
&= \text{Re}\left[ (i\alpha)^{2n-1} e^{-ak + \frac{1}{2} a^2 + \frac{1}{2}} \right] \\
&= \text{Re}\left[ i (-1)^{n+1} d^{2n-1} e^{-ak + \frac{1}{2} a^2 + \frac{1}{2}} \right] \\
&= 0.
\end{align*}
\]

This is a contradiction to the assumption. Hence we must have

\[ \text{Re}\{q(z)\} > 0 \quad \text{in} \quad \mathcal{U} \]

or

\[ 1 + \text{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}, \]

which yields our result.

Putting \( n = 1 \) in Theorem 3, we have
Corollary 3. Let \( f(z) \in A \) and suppose that

\[
{\text{Re}} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) e^{z^2} \right] \neq 0 \quad \text{in} \quad U.
\]

Then \( f(z) \in C \).

Corollary 3 is a revision of Theorem C.

REFERENCES


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