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ON STARLIKENESS AND CONVEXITY OF FUNCTIONS AND THE SCHWARZIAN DERIVATIVE

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ABSTRACT. The purpose of this paper is to generalize Miller and Mocanu's result [2].

1. Introduction

Let $A$ denote the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{ z : z \in \mathbb{C}, \text{ and } |z| < 1 \}$. Also, let $S$ denote the class of all functions in $A$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to the class $S$ is said to be in the class $S^*$ if and only if

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}$$

and is said to be in the class $C$ if and only if

$$1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}.$$

We denote by $\{f, z\}$ the Schwarzian derivative, which is characterized by the equality

$$\{f, z\} = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.$$

In [1], Nunokawa et al. obtained the following result:

**Theorem A.** Let $f(z) \in A$ and suppose that

$$\text{Re} \left[ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} + z^2 \{f, z\} \right) \right] \geq -\frac{1}{2} \quad \text{in} \quad \mathcal{U}.$$

Then $f(z) \in S^*$.

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Remark. Theorem A is an extension of Miller and Mocanu [2], where the right hand side of (2) is improved from 0 to $-\frac{1}{2}$.

Further Miller and Mocanu [2] obtained the following results:

**Theorem B.** Let $f(z) \in A$ satisfy

$$\text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^2 + z^2 \{f, z\} \right] > 0 \quad \text{in } \mathcal{U}.$$  

Then $f(z) \in C$.

**Theorem C.** Let $f(z) \in A$ satisfy

$$\text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) e^{z^2 \{f, z\}} \right] > 0 \quad \text{in } \mathcal{U}.$$  

Then $f(z) \in C$.

Let us investigate improvements of these results in the next section.

### 2. Main Results

The following result will be required in our investigation:

**Lemma.** [3] Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and suppose that there exists a point $z_0 \in \mathcal{U}$ such that $\text{Re} \{p(z)\} > 0$ for $|z| < |z_0|$ and $\text{Re} \{p(z_0)\} = 0$ ($p(z_0) \neq 0$). Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where $k$ is a real number and

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \geq 1 \quad \text{when } p(z_0) = ia, \ a > 0,$$

and

$$k \leq \frac{1}{2} \left( a + \frac{1}{a} \right) \leq -1 \quad \text{when } p(z_0) = ia, \ a < 0.$$

Now we state our main result.

**Theorem 1.** Let $f(z) \in A$ and satisfy one of the following inequalities:

$$(3) \quad \text{Re} \left[ \left( \frac{zf''(z)}{f'(z)} \right)^{4m-1} \left( 1 + \frac{zf''(z)}{f'(z)} + z^2 \{f, z\} \right) \right]$$

$$< \frac{1}{2} \left| \frac{zf'(z)}{f(z)} \right|^{4m-2} \left( 3 \left| \frac{zf'(z)}{f(z)} \right|^2 + 1 \right) \quad \text{in } \mathcal{U},$$
\[
\text{Re} \left[ \left( \frac{zf'(Z)}{f(z)} \right)^{4m-3} \left( 1 + \frac{zf''(z)}{f'(z)} + z^2 \{f, z\} \right) \right] > -\frac{1}{2} \left| \frac{zf'(Z)}{f(z)} \right|^{4m-4} \left( 3 \left| \frac{zf'(z)}{f(z)} \right|^2 + 1 \right) \quad \text{in} \quad \mathcal{U},
\]

where \(m\) is a positive integer. Then \(f(z) \in S^*\).

**Proof.** Let us put
\[
p(z) = \frac{zf'(z)}{f(z)},
\]
then we easily have
\[
1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}
\]
and from (1), by a simple calculation, we have
\[
z^2 \{f, z\} = z^2 \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{zf''(z)}{f'(z)} \right)^2
\]
\[
= \frac{zp'(z)}{p(z)} + \frac{z^2 p'(z)}{p(z)} - \frac{3}{2} \left( \frac{zp'(z)}{p(z)} \right)^2 + \frac{1}{2} \{1 - p(z)^2\}.
\]
To prove \(\text{Re} \{zf'(z)/f(z)\} > 0\) in \(\mathcal{U}\), we show \(\text{Re} \{p(z)\} > 0\) in \(\mathcal{U}\). If there exists a point \(z_0 \in \mathcal{U}\) such that
\[
\text{Re} \{p(z)\} > 0 \quad \text{for} \quad |z| < |z_0|
\]
and
\[
\text{Re} \{p(z_0)\} = 0 \quad (p(z_0) \neq 0),
\]
then from Lemma we have
\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik,
\]
and (3), (4) and (5) imply
\[
(z_0 f'(z_0) f(z_0)) \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} + z_0^2 \{f, z_0\} \right)
\]
\[
= (ia)^l \left[ i a + ik + ik + \frac{z_0^2 p'(z_0)}{p(z_0)} - \frac{3}{2} (ik)^2 + \frac{1}{2} \{1 - (ia)^2\} \right]
\]
\[
= (ia)^l \left[ i \left\{ a + k + k \left( 1 + \frac{z_0 p'(z_0)}{p(z_0)} \right) \right\} + \frac{3}{2} k^2 + \frac{1}{2} (1 + a^2) \right],
\]
where \(l\) is a positive integer. Let \(J\) be the right hand side of (6). For the case \(l = 2n - 1\),
\[
J = (ia)^{2n-1} \left[ i \left\{ a + k + k \left( 1 + \frac{z_0 p'(z_0)}{p(z_0)} \right) \right\} + \frac{3}{2} k^2 + \frac{1}{2} (1 + a^2) \right],
\]
Therefore we have
\[ \text{Re}\{J\} = (-1)^n a^{2n-1} \left[ a + k + k \left( 1 + \text{Re} \left\{ \frac{z_0p''(z_0)}{p'(z_0)} \right\} \right) \right]. \]

Considering the geometrical property, we notice that the tangential vector of the curve \( p(z) = p(z_0) \) moves to positive direction near the point \( p(z_0) \). In short, \( p(z) \) is convex in the neighborhood of the point \( p(z_0) \), or
\[ 1 + \text{Re} \left\{ \frac{z_0p'(z_0)}{p'(z_0)} \right\} \geq 0. \]

(i) Case \( n = 2m \):
\[
\text{Re}\{J\} = (-1)^{2m} a^{4m-1} \left[ a + k + k \left( 1 + \text{Re} \left\{ \frac{z_0p''(z_0)}{p'(z_0)} \right\} \right) \right] \geq a^{4m-1} (a + k) = -a^{4m-2} \left( -a^2 - ak \right) \geq -a^{4m-2} \left\{ -a^2 - \frac{1}{2} (a^2 + 1) \right\} = \frac{1}{2} \left| \frac{z_0f'(z_0)}{f(z_0)} \right|^{4m-2} \left( 3 \left| \frac{z_0f'(z_0)}{f(z_0)} \right|^2 + 1 \right).
\]
(ii) Case \( n = 2m - 1 \):
\[
\text{Re}\{J\} = (-1)^{2m-1} a^{4m-3} \left[ a + k + k \left( 1 + \text{Re} \left\{ \frac{z_0p''(z_0)}{p'(z_0)} \right\} \right) \right] \leq -a^{4m-3} (a + k) = a^{4m-4} \left( -a^2 - ak \right) \leq a^{4m-4} \left\{ -a^2 - \frac{1}{2} (a^2 + 1) \right\} = a^{4m-4} \left( -\frac{3}{2} a^2 - \frac{1}{2} \right) = -\frac{1}{2} \left| \frac{z_0f'(z_0)}{f(z_0)} \right|^{4m-4} \left( 3 \left| \frac{z_0f'(z_0)}{f(z_0)} \right|^2 + 1 \right).
\]

These contradict (3) and (4), respectively. Hence we must have
\[ \text{Re}\{p(z)\} > 0 \quad \text{in} \quad \mathcal{U} \]
or
\[ \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}, \]
which means \( f(z) \in S^* \). This completes our proof.

Setting \( \alpha = 1 \) in Theorem 1, we obtain
Corollary 1. Let $f(z) \in A$ and suppose that
\[
\text{Re} \left[ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} + z^2 \{f, z\} \right) \right] > -\frac{1}{2} \left( 1 + 3 \left| \frac{zf'(z)}{f(z)} \right|^2 \right)
\] in $\mathcal{U}$.

Then $f(z) \in S^*$.  

Corollary 1 is better than Theorem A.

Theorem 2. Let $f(z) \in A$ and suppose that
\[
(7) \quad \text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{2n} + z^2 \{f, z\} \right] + (-1)^{n+1} \left| \frac{zf''(z)}{f'(z)} \right|^{2n} > 0 \quad \text{in $\mathcal{U}$}
\]

for positive integer $n$. Then $f(z) \in C$.

Proof. Let us put
\[
q(z) = 1 + \frac{zf''(z)}{f'(z)}.
\]

Note that $q(0) = 1$. Then from (1), we easily have
\[
(8) \quad z^2 \{f, z\} = zq'(z) - \frac{1}{2}q(z)^2 + \frac{1}{2}.
\]

To prove $1 + \text{Re} \left\{ zf''(z)/f'(z) \right\} > 0$ in $\mathcal{U}$, we show $\text{Re} \{q(z)\} > 0$ in $\mathcal{U}$. If there exists a point $z_0 \in \mathcal{U}$ such that
\[
\text{Re} \{q(z)\} > 0 \quad \text{for} \quad |z| < |z_0|
\]
and
\[
\text{Re} \{q(z_0)\} = 0 \quad (q(z_0) \neq 0),
\]
then from Lemma a real number $k (k \neq 0)$ exists such that
\[
\frac{z_0q'(z_0)}{q(z_0)} = ik.
\]

From (7) and (8), we have
\[
\text{Re} \left[ \left( 1 + \frac{zf''(z_0)}{f'(z_0)} \right)^{2n} + z_0^2 \{f, z_0\} \right] + (-1)^{n+1} \left| \frac{zf''(z_0)}{f'(z_0)} \right|^{2n}
\]
\[
= \text{Re} \left\{ (q(z_0))^{2n} + z_0q'(z_0) - \frac{1}{2}q(z_0)^2 + \frac{1}{2} \right\} + (-1)^{n+1} \left| q(z_0) \right|^{2n}
\]
\[
= \text{Re} \left\{ (ia)^{2n} - ak - \frac{1}{2} (ia)^2 + \frac{1}{2} \right\} + (-1)^{n+1} \left| ia \right|^{2n}
\]
\[
\leq (-1)^n a^{2n} - \frac{1}{2} (a^2 + 1) + \frac{1}{2} (a^2 + 1) + (-1)^{n+1} |a|^{2n}
\]
\[
= 0.
\]
This is in contradiction to (7). Hence we must have
\[ \text{Re}\{q(z)\} > 0 \quad \text{in} \quad \mathcal{U} \]
or
\[ 1 + \text{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}. \]
Therefore \( f(z) \in C \) and our result is established.

Taking \( n = 1 \) in Theorem 2, we have

**Corollary 2.** Let \( f(z) \in A \) and suppose that
\[
\text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^2 + z^2 \{f, z\} \right] + \left| 1 + \frac{zf''(z)}{f'(z)} \right|^2 > 0 \quad \text{in} \quad \mathcal{U}.
\]

Then \( f(z) \in C \).

Corollary 2 is better than Theorem B.

**Theorem 3.** Let \( f(z) \in A \) and suppose that
\[
\text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{2n-1} e^{z^2\{f, z\}} \right] \neq 0 \quad \text{in} \quad \mathcal{U}.
\]

Then \( f(z) \in C \).

**Proof.** Let us take the same function \( q(z) \) as in the proof of Theorem 2. Then from the assumption of theorem and (8), we find
\[
\text{Re} \left[ \left( 1 + \frac{zf''(z_0)}{f'(z_0)} \right)^{2n-1} e^{z_0^2\{f, z_0\}} \right] = \text{Re} \left[ (q(z_0))^{2n-1} e^{z_0q'(z_0)-\frac{1}{2}q(z_0)^2+\frac{1}{2}} \right] = \text{Re} \left[ (ia)^{2n-1} e^{-ak+\frac{1}{2}a^2+\frac{1}{2}} \right] = 0.
\]
This is a contradiction to the assumption. Hence we must have
\[ \text{Re}\{q(z)\} > 0 \quad \text{in} \quad \mathcal{U} \]
or
\[ 1 + \text{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\} > 0 \quad \text{in} \quad \mathcal{U}, \]
which yields our result.

Putting \( n = 1 \) in Theorem 3, we have
Corollary 3. Let \( f(z) \in A \) and suppose that

\[
\text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) e^{z^2} (f,z) \right] \neq 0 \quad \text{in} \quad U.
\]

Then \( f(z) \in C \).

Corollary 3 is a revision of Theorem C.

REFERENCES


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