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Association Schemes Related to the Quantum Group $U_q(sl(2))$

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This is an abbreviated version of a paper [6] in which we present a relationship between $U_q(sl(2))$, the quantum enveloping algebra of $sl(2)$, and certain distance-regular graphs. The starting point of this paper is the observation that the Terwilliger algebras of the Hamming cubes possess a natural $U(sl(2))$ structure.

Let $\Gamma=(X,R)$ denote a distance-regular graph with diameter D and adjacency matrix $A\in Mat_X(\mathbf{C})$, where $Mat_X(\mathbf{C})$ denotes the $\mathbf{C}$-algebra of matrices in $\mathbf{C}$ whose rows and columns are indexed by X. Fix $x\in X$, and write $E_i^*=E_i^*(x)\in Mat_X(\mathbf{C})$ to denote the diagonal matrix with (y,y)-entry 1 if $\partial(x,y)=i$ and 0 otherwise. The algebra $T=T(x)$ generated by A and $E_0^*, E_1^*, \ldots, E_D^*$ is called the *Terwilliger algebra (with respect to x) of* Γ.

Let $\Gamma=(X,R)$ denote a Hamming D-cube. Fix $x\in X$, and write $T=T(x)$. Set

$$L=\sum_{i=0}^{D-1}E_i^*AE_{i+1}^*, \qquad R=\sum_{i=1}^{D}E_i^*AE_{i-1}^*, \qquad Z=\sum_{i=0}^{D}(D-2i)E_i^*.$$

It is easy to verify that

$$ZL-LZ=2L, \quad ZR-RZ=-2R, \quad LR-RL=Z,$$

the relations of the standard presentation of $U(sl(2))$. Moreover, L, R, and Z generate T. Thus T is a homomorphic image of $U(sl(2))$. (We prove these facts in Theorem 2.3).

The matrices L and R are called the *lowering* and *raising matrices* of T, respectively. These matrices have the following combinatorial interpretation. For the moment, identify each vertex of Γ with its characteristic column vector (and thereby allow T to act on the vertices of Γ). Fix $y\in X$, and let i denote the distance between x and y. Then L maps y to the sum of those vertices which are adjacent to y and at distance $i-1$ from x, and R maps y to the sum of those vertices which are adjacent to y and at distance $i+1$ from x. Thus the lowering and raising matrices are lowering and raising the distance from x while preserving adjacency. Thus the usual generators of $U(sl(2))$ are mapped to combinatorially significant elements of T in the homomorphism described above.

This leads us to investigate the following questions. Are there any other distance-regular graphs with a similar $U(sl(2))$ structure? Are there examples of distance-regular graphs with a similar $U_q(sl(2))$ structure, where $U_q(sl(2))$ is the quantum universal enveloping algebra of $sl(2)$? Can we find all such examples? We answer these questions, showing that only the Hamming cubes have a natural $U(sl(2))$ structure, and only the 2-homogeneous bipartite distance-regular graphs have a natural $U_q(sl(2))$ structure.

In the next section we review some background material. We then return to the $U(sl(2))$ structure on the Hamming cubes, followed by a description of the $U_q(sl(2))$ structure on the 2-homogeneous bipartite distance-regular graphs. We omit many of the proofs, and we omit a discussion of the module theory for $U_q(sl(2))$ and T. Instead, we will focus the combinatorial aspects of these relationships.

1 Association Schemes

In this section, we present some basic material concerning association schemes and their Terwilliger algebras. For more information about association schemes see [1, 2], and for more information about their Terwilliger algebras see [11, 12, 13].

Let X be a finite non-empty set, and let $Mat_X(\mathbf{C})$ denote the $\mathbf{C}$-algebra of matrices with entries in $\mathbf{C}$ whose rows and columns are indexed by X. For all $A \in Mat_X(\mathbf{C})$ and for all a, $b \in X$, we write $A(a,b)$ to denote the (a,b)-entry of A. For any set $G \subseteq Mat_X(\mathbf{C})$, the smallest subalgebra of $Mat_X(\mathbf{C})$ which contains G and the identity matrix of $Mat_X(\mathbf{C})$ is called the *subalgebra of* $Mat_X(\mathbf{C})$ *generated by* G.

By a *commutative association scheme* (or simply *scheme* hereafter) we mean a pair $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$, where X is a finite non-empty set, and where $A_0, A_1, \ldots, A_D \in Mat_X(\mathbf{C})$ are non-zero (0,1)-matrices satisfying the following conditions: (i) $\sum_{i=0}^{D} A_i = J$ (the all ones matrix), (ii) $A_0 = I$ (the identity matrix), (iii) for all i $(0 \le i \le D)$ there exists an i' $(0 \le i' \le D)$ such that $A_i^t = A_{i'}$, and (iv) for all h, i, and j $(0 \le h, i, j \le D)$ there exists an integer p_{ij}^h such that $A_iA_j = A_jA_i = \sum_{h=0}^{D} p_{ij}^h A_h$. A_i is called the i^{th} *associate matrix* of $\mathcal{X}$. The numbers p_{ij}^h $(0 \le h, i, j \le D)$ are called the *intersection numbers* of $\mathcal{X}$.

Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ be a scheme. From (i)–(iv) we see that $A_0, A_1, \ldots, A_D$ form a linear basis for a commutative subalgebra M of $Mat_X(\mathbf{C})$. We refer to M as the *Bose-Mesner algebra* of $\mathcal{X}$. By [1], M has a basis $E_0, E_1, \ldots, E_D$ satisfying: (i) $\sum_{i=0}^{D} E_i = I$, (ii) $E_0 = |X|^{-1}J$, (iii) for all i $(0 \le i \le D)$ there exists an $\hat{\imath}$ $(0 \le \hat{\imath} \le D)$ such that $E_i^t = \bar{E}_i = E_{\hat{\imath}}$, and (iv) $E_iE_j = \delta_{ij}E_i$ $(0 \le i, j \le D)$. We refer to $E_0, E_1, \ldots, E_D$ as the *primitive idempotents* of $\mathcal{X}$. For all i $(0 \le i \le D)$ $\dim E_iV = \operatorname{rank} E_i = \operatorname{trace} E_i = q_{ii}^0$.

Observe that M is closed under entry-wise multiplication, $\circ$, and that the A_i are the primitive idempotents of M under $\circ$, i.e. $A_i \circ A_j = \delta_{ij} A_i$. For all h, i, and j $(0 \le h, i, j \le D)$ there exists a scalar q_{ij}^h such that $E_i \circ E_j = \sum_{h=0}^{D} q_{ij}^h E_h$. The numbers q_{ij}^h are called the *Krein parameters* of $\mathcal{X}$. The Krein parameters are non-negative real numbers [1, Theorem II.3.8].

Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ denote a scheme. Fix any $x \in X$. For each integer i $(0 \le i \le D)$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $Mat_X(\mathbf{C})$ with (y,y)-entry $E_i^*(y,y) = A_i(x,y)$. Observe that (i) $\sum_{i=0}^{D} E_i^* = I$, (ii) $E_i^{*t} = E_i^*$ $(0 \le i \le D)$, and (iii) $E_i^*E_j^* = \delta_{ij}E_i^*$ $(0 \le i, j \le D)$. E_i^* is called the i^{th} *dual-idempotent of* $\mathcal{X}$ *with respect to* x. For all i $(0 \le i \le D)$ $\dim E_i^*V = \operatorname{rank} E_i^* = \operatorname{trace} E_i^* = p_{ii}^0$. From (i)–(iii) we see that E_0^*, $E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra M^* of $Mat_X(\mathbf{C})$. We refer to M^* as the *dual-Bose-Mesner algebra of* $\mathcal{X}$ *with respect to* x.

For each integer i $(0 \le i \le D)$, let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $Mat_X(\mathbf{C})$ with (y,y)-entry $A_i^*(y,y) = |X|E_i(x,y)$. By [11], $A_0^*, A_1^*, \ldots, A_D^*$ form a basis for M^* and satisfy: (i) $\sum_{i=0}^{D} A_i^* = |X|E_0^*$, (ii) $A_0^* = I$, (iii) $A_i^{*t} = A_i^*$ $(0 \le i \le D)$, and (iv) for all i, j $(0 \le i, j \le D)$ $A_i^*A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$. We refer to $A_0^*, A_1^*, \ldots, A_D^*$ as the *dual-associate matrices* of $\mathcal{X}$.

Let $T = T(x)$ denote the subalgebra of $Mat_X(\mathbf{C})$ generated by M and M^*. The algebra T is called the *Terwilliger (or subconstituent) algebra of* Γ *with respect to* x.

Definition 1.1 Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\dots,D})$ denote a scheme. We say that $\mathcal{X}$ is *P-polynomial* (with respect to a given ordering $A_0 = I, A_1, \dots, A_D$ of the associate matrices) whenever $D \geq 1$, and for all integers h, i, j $(0 \leq h, i, j \leq D)$, $p^h_{ij} = 0$ if one of h, i, j is larger than the sum of the other two, and $p^h_{ij} \neq 0$ if one of h, i, j equals the sum of the other two.

Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\dots,D})$ denote a P-polynomial scheme, and write $A = A_1$. The Bose-Mesner algebra of a P-polynomial scheme is generated by A. Let $\Gamma = (X, R)$ denote the graph with adjacency matrix A, and write ∂ to denote the shortest-path distance function on Γ. Then for all $x, y \in X$, $A_i(x, y) = 1$ if $\partial(x, y) = i$ and 0 otherwise $(0 \leq i \leq D)$. The axioms of a scheme imply that for all h $(0 \leq h \leq D)$ and all $x, y \in X$ with $\partial(x, y) = h$, the number $|\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|$ is independent of x and y for all i, j $(0 \leq i, j \leq D)$. Such a graph is said to be *distance-regular.* (See, for example, [1, pp. 188-193] or [2, pp. 58-59]). Throughout this paper we will use the notation of a scheme for a distance-regular graph, referring to the above construction of the associate matrices from such a graph. We will write $\Gamma_i(x) = \{y \in X \mid A_i(x, y) \neq 0\}$, the set of vertices at distance i from x in the graph Γ.

Suppose $\mathcal{X} = (X, \{A_i\}_{i=0,1,\dots,D})$ is a P-polynomial scheme. We set $c_i = p^i_{1i-1}$ $(1 \leq i \leq D)$, $a_i = p^i_{1i}$ $(0 \leq i \leq D)$, and $b_i = p^i_{1i+1}$ $(0 \leq i \leq D-1)$. We define $c_0 = b_D = 0$. Recall that $c_i + a_i + b_i = b_0$ $(0 \leq i \leq D)$ [2, p. 126].

Define

$$L = \sum_{i=0}^{D-1} E^*_i A E^*_{i+1}, \qquad F = \sum_{i=0}^{D} E^*_i A E^*_i, \qquad R = \sum_{i=1}^{D} E^*_i A E^*_{i-1}.$$

Observe that $A = R + L + F$.

Lemma 1.2 *[3] Let* $\mathcal{X} = (X, \{A_i\}_{i=0,1,\dots,D})$ *denote a P-polynomial scheme with* $D \geq 2$. *Fix* $x \in X$, *and write* $T = T(x)$. *Then the following are equivalent.*

(i) $a_i = 0$ $(0 \leq i \leq D)$.

(ii) $F = 0$.

(iii) *There exists, up to isomorphism, a unique simple* T*-module with endpoint 1, it is thin, and it has diameter* $D - 2$.

A P-polynomial scheme satisfying (i)–(iii) of Lemma 1.2 is said to be *bipartite.*

Definition 1.3 Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\dots,D})$ denote a scheme. We say that $\mathcal{X}$ is *Q-polynomial* (with respect to a given ordering $E_0 = |X|^{-1}J, E_1, \dots, E_D$ of the primitive idempotents) whenever $D \geq 1$, and for all integers h, i, j $(0 \leq h, i, j \leq D)$, $q^h_{ij} = 0$ if one of h, i, j is larger than the sum of the other two, and $q^h_{ij} \neq 0$ if one of h, i, j equals the sum of the other two.

Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\dots,D})$ denote a Q-polynomial scheme, and write $A^* = A_1^*$. The dual-Bose-Mesner algebra of a Q-polynomial scheme is generated by A^* [11, Lemma 3.11].

Suppose $\mathcal{X} = (X, \{A_i\}_{i=0,1,\dots,D})$ is a Q-polynomial scheme. We set $c_i^* = q_{1i-1}^i$ $(1 \le i \le D)$, $a_i^* = q_{1i}^i$ $(0 \le i \le D)$, and $b_i^* = q_{1i+1}^i$ $(0 \le i \le D-1)$. We define $c_0^* = b_D^* = 0$. Recall $c_i^* + a_i^* + b_i^* = b_0^*$ $(0 \le i \le D)$ [1, Proposition 3.7].

Define

$$L^* = \sum_{i=0}^{D-1} E_i A^* E_{i+1}, \qquad F^* = \sum_{i=0}^{D} E_i A^* E_i, \qquad R^* = \sum_{i=1}^{D} E_i A^* E_{i-1}.$$

Observe that $A^* = R^* + L^* + F^*$.

2 A $U(sl(2))$ structure on the Hamming cubes

In this section we describe a natural $sl(2)$ structure on the Hamming cubes. The results in this section are observations of Terwilliger. We present them here to motivate our current work. Recall the following presentation of $U(sl(2))$.

Definition 2.1 The *universal enveloping algebra of* $sl(2)$ is the associative algebra $U(sl(2))$ generated by X^-, X^+, and Z with relations

$$ZX^- - X^-Z = 2X^-, \tag{1}$$
$$ZX^+ - X^+Z = -2X^+, \tag{2}$$
$$X^-X^+ - X^+X^- = Z. \tag{3}$$

Also recall the following construction of the Hamming cubes.

Definition 2.2 The *Hamming D-cube* is the graph with vertex set $X = \{0, 1\}^D$ (the D-tuples with (0,1)-entries) such that two vertices are adjacent if and only if they differ in precisely one coordinate.

The Hamming D-cube has been characterized as the unique distance-regular graph with intersection numbers $c_i = i$, $b_i = D - i$, and $a_i = 0$ $(0 \le i \le D)$ [8, 7].

It follows from Definition 2.2 that for all integers i $(1 \le i \le D)$ and for all vertices x, y, $z \in X$ with $\partial(y, z) = 2$, $\partial(x, y) = \partial(x, z) = i$,

$$|\Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_{i-1}(x)| = |\Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_{i+1}(x)| = 1. \tag{4}$$

With this observation we are ready to prove the first result.

Lemma 2.3 *Let $\mathcal{X}$ denote the Hamming D-cube, $D \ge 2$. Fix $x \in X$, and write $T = T(x)$. Write*

$$X^- = L, \qquad X^+ = R, \qquad Z = \sum_{i=0}^{D} (D - 2i) E_i^*.$$

(i) *X^-, X^+, and Z satisfy the defining relations of $U(sl(2))$ given in Definition 2.1.*

(ii) *X^-, X^+, and Z generate T.*

Proof. (i): We verify (1) with the following computations.

$$ZX^- = \left(\sum_{j=0}^{D}(D-2j)E_j^*\right)\left(\sum_{i=0}^{D-1}E_i^*AE_{i+1}^*\right) = \sum_{i=0}^{D-1}(D-2i)E_i^*AE_{i+1}^*,$$

$$X^-Z = \left(\sum_{i=0}^{D-1}E_i^*AE_{i+1}^*\right)\left(\sum_{j=0}^{D}(D-2j)E_j^*\right) = \sum_{i=0}^{D-1}(D-2i-2)E_i^*AE_{i+1}^*.$$

Now (1) follows. The relation (2) is verified similarly.

We now show that (3) holds. Since $\sum_{i=0}^{D}E_i^* = I$, it is enough to show that for all i $(0 \le i \le D)$

$$(LR - RL)E_i^* = (D-2i)E_i^*. \qquad (5)$$

Fix i $(0 \le i \le D)$, and pick y, $z \in X$ with $\partial(x,y) = \partial(x,z) = i$. Let r, s, t denote the (y,z)-entries of LRE_i^*, RLE_i^*, and E_i^*, respectively. First suppose $\partial(y,z) > 2$. Then $r = s = t = 0$. Suppose $\partial(y,z) = 2$. Then by (4) $r = 1$, $s = 1$, and $t = 0$. Suppose $\partial(y,z) = 1$. Then $r = s = t = 0$ since $a_i = 0$. Finally suppose $y = z$. Then $r = b_i = D - i$, $s = c_i = i$, and $t = 1$. In all cases $r = s + (D-2i)t$, so (5) holds.

(ii): Observe that $Z^j = \sum_{i=0}^{D}(D-2i)^jE_i^*$ since the E_i^* are idempotents. (We take for the Z^0 expression $I = \sum_{i=0}^{D}E_i^*$). Viewing these expressions for Z^j $(0 \le i \le D)$ as equations in the unknowns E_i^* $(0 \le i \le D)$ gives a system with a Vandermonde coefficient matrix. Thus we may express each E_i^* as a linear combination of nonnegative powers of Z. Observe that $A = L + R$ since $\mathcal{X}$ is bipartite, so A, E_0^*, E_1^*, ..., E_D^* are contained in the subalgebra of $Mat_X(\mathbb{C})$ generated by L, R, and Z. It follows that T is generated by L, R, and Z since T is generated by A, E_0^*, E_1^*, ..., E_D^*. ∎

The Hamming D-cube is Q-polynomial with $q_{ij}^h = p_{ij}^h$ for all h, i, j, $(0 \le h, i, j \le D)$. This duality between the P- and Q-polynomial structure extends to the Terwilliger algebra. This gives a natural $sl(2)$ structure on the Q-polynomial structure Hamming cubes. Let us state without proof the dual version of Lemma 2.3.

Lemma 2.4 *Let $\mathcal{X}$ denote the Hamming D-cube, $D \ge 2$. Fix $x \in X$, and write $T = T(x)$. Write*

$$X^- = L^*, \qquad X^+ = R^*, \qquad Z = \sum_{i=0}^{D}(D-2i)E_i.$$

(i) *X^-, X^+, and Z satisfy the defining relations of $U(sl(2))$ given in Definition 2.1.*

(ii) *X^-, X^+, and Z generate T.*

Having found examples of distance-regular graphs which have a natural $U(sl(2))$ structure, we consider the possibility of finding other examples. In this section we show that the Hamming cubes are the only examples the natural structure described in Lemmas 2.3 and 2.4. Our main result is the following theorem, which we state without proof.

Theorem 2.5 *Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ denote a scheme with $D \ge 2$. Fix $x \in X$, and write $T = T(x)$. The following are equivalent.*

(i) *$\mathcal{X}$ is the Hamming D-cube.*

(ii) *$\mathcal{X}$ is P-polynomial and T is generated by elements X^-, X^+, and Z of the form*

$$X^- = \sum_{i=0}^{D} x_i^- E_i^* A E_{i+1}^*, \qquad X^+ = \sum_{i=0}^{D-1} x_i^+ E_i^* A E_{i-1}^*, \qquad Z = \sum_{i=1}^{D} z_i E_i^*$$

which satisfy the relations of $U(sl(2))$ of Definition 2.1.

(iii) *$\mathcal{X}$ is Q-polynomial and T is generated by elements X^-, X^+, and Z of the form*

$$X^- = \sum_{i=0}^{D-1} x_i^- E_i A^* E_{i+1}, \qquad X^+ = \sum_{i=1}^{D} x_i^+ E_i A^* E_{i-1}, \qquad Z = \sum_{i=0}^{D} z_i E_i$$

which satisfy the relations of $U(sl(2))$ of Definition 2.1.

Suppose (i) – (iii) hold. Then in both (ii) and (iii)

$$z_i = D - 2i \qquad (0 \le i \le D), \tag{6}$$

$$x_i^- x_{i+1}^+ = 1 \qquad (0 \le i \le D-1). \tag{7}$$

3 A $U_q(sl(2))$ structure on the 2-homogeneous bipartite distance-regular graphs

In this section show that the 2-homogeneous bipartite distance-regular graphs have a natural $U_q(sl(2))$ structure similar to the $U(sl(2))$ structure on the Hamming cubes. Recall the following presentation of the $U_q(sl(2))$.

Definition 3.1 For $q \in \mathbf{C}\backslash\{0, 1, -1\}$, the *quantum universal enveloping algebra* of $sl(2)$ is the associative algebra $U_q(sl(2))$ generated by X^-, X^+, Y, and Y^{-1} with relations

$$YY^{-1} = 1 = Y^{-1}Y,$$

$$YX^-Y^{-1} = q^2X^-, \tag{8}$$

$$YX^+Y^{-1} = q^{-2}X^+, \tag{9}$$

$$X^-X^+ - X^+X^- = (Y - Y^{-1})/(q - q^{-1}). \tag{10}$$

The algebra $U(sl(2))$ can be viewed as the classical limit $q \to 1$ of $U_q(sl(2))$ (see [9, Section VI.2] for further details). In the module theory of $U_q(sl(2))$, the following q-analogues of the integers appear: For any integer n, set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We will write $[n]$ for $[n]_q$ when q is clear from the context. We write $[n]! = [n][n-1]\cdots[1]$ for each positive integer n. The following family of distance-regular graphs can be viewed as q-analogues of the Hamming graphs. (They are not related to the bilinear forms graphs which are sometimes called the "q-analogues of the Hamming graph" [2, p. 280]).

Definition 3.2 Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ denote a bipartite distance-regular graph with diameter $D \ge 3$ and $b_0 \ge 3$. $\mathcal{X}$ is said to be *2-homogeneous* whenever for all integers i $(1 \le i \le D)$, the number $|\Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_{i-1}(x)|$ is independent of the choice of x, y, $z \in X$ with $\partial(y, z) = 2$, $\partial(x, y) = \partial(x, z) = i$. In this case we write γ_i to denote the number $|\Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_{i-1}(x)|$.

Observe that the Hamming cubes are 2-homogeneous bipartite distance-regular graphs. We will exclude them from further consideration below. The following theorem shows how to interpret the 2-homogeneous bipartite distance-regular graphs as q-analogues of the Hamming cubes.

Theorem 3.3 *([5, Theorem 35]) Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ denote a distance-regular graph with diameter $D \geq 3$ and $b_0 \geq 3$. Assume $\mathcal{X}$ is not a Hamming cube. Then the following are equivalent.*

(i) *$\mathcal{X}$ is a 2-homogeneous bipartite distance-regular graph.*

(ii) *There exists a complex scalar $q \notin \{0, 1, -1\}$ such that*

$$c_i = \frac{q^{i-1}(q^D + q^2)}{(q^D + q^{2i})}[i]_q, \qquad b_i = \frac{q^{i-1}(q^D + q^2)}{(q^D + q^{2i})}[D-i]_q \qquad (0 \leq i \leq D). \tag{11}$$

Suppose (i) *and* (ii) *hold. Then q is real and*

$$\gamma_i = \frac{(q^D + q^2)}{(q^D + q^4)} \frac{(q^D + q^{2i+2})}{(q^D + q^{2i})} \qquad (1 \leq i \leq D-1). \tag{12}$$

The parameter q of Theorem 3.3 is determined by the graph structure as follows.

Lemma 3.4 *([5, Corollary 36]) Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ denote a 2-homogeneous bipartite distance-regular graph with diameter $D \geq 3$ and $b_0 \geq 3$. Assume $\mathcal{X}$ is not a Hamming cube. Let Φ denote the set of all $q \in \mathbf{C}$ satisfying the parameterization of Theorem 3.3.*

(i) *Suppose D is even. Then $\Phi = \{q \in \mathbf{C} \mid (q + q^{-1})^2 = c_2^2(b_0 - 2)/((c_2 - 1)b_2)\}$. In particular, $\Phi = \{a,\ a^{-1},\ -a,\ -a^{-1}\}$ for some real number $a > 1$.*

(ii) *Suppose D is odd. Then $\Phi = \{q \in \mathbf{C} \mid q + q^{-1} = c_2\gamma_r^{-1}\}$, where $r = (D-1)/2$. In particular, $\Phi = \{a,\ a^{-1}\}$ for some real number $a > 1$.*

We are ready to describe a $U_q(sl(2))$ structure on the 2-homogeneous bipartite distance-regular graphs.

Lemma 3.5 *Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ denote a 2-homogeneous bipartite distance-regular graph with diameter $D \geq 3$ and $b_0 \geq 3$. Assume $\mathcal{X}$ is not a Hamming cube, and let q be as in Theorem 3.3. Fix $x \in X$, and write $T = T(x)$. Write*

$$X^- = \sum_{j=1}^{D-1} \frac{q^D + q^{2j}}{q^j(q^D + q^2)} E_j^* A E_{j-1}^*, \qquad X^+ = \sum_{j=0}^{D} \frac{q^D + q^{2j}}{q^j(q^D + q^2)} E_j^* A E_{j+1}^*,$$

$$Y = \sum_{j=0}^{d} q^{D-2j} E_j^*, \qquad Y^{-1} = \sum_{j=0}^{d} q^{2j-D} E_j^*.$$

(i) *X^-, X^+, and Y satisfy the defining relations of $U_q(sl(2))$ given in Definition 3.1.*

(ii) *X^-, X^+, and Y generate T.*

Proof. (i): Write $e_i = q^{-i}(q^D + q^{2i})/(q^D + q^2)$. We verify (8) with the following computation.

$$
\begin{aligned}
YX^-Y^{-1} &= \left(\sum_{j=0}^{D} q^{D-2j}E_j^*\right)\left(\sum_{i=0}^{D-1} e_iE_i^*AE_{i+1}^*\right)\left(\sum_{k=0}^{D} q^{2k-D}E_k^*\right)\\
&= \sum_{i=0}^{D-1} e_iq^{D-2i}q^{2i+2-D}E_i^*AE_{i+1}^* = q^2X^-.
\end{aligned}
$$

The relation (9) is verified similarly.

We now verify (10). Since $I = \sum_{i=0}^{D} E_i^*$, it is enough to show that for all i $(0 \leq i \leq D)$

$$e_ie_{i+1}LRE_i^* - e_{i-1}e_iRLE_i^* = [i]_qE_i^*. \qquad (13)$$

Fix i $(0 \leq i \leq D)$, and pick $y,\ z \in X$ such that $\partial(x, y) = \partial(x, z) = i$. Let r, s, and t denote the (y, z)-entries of LRE_i^*, RLE_i^*, and E_i^*, respectively. First suppose $\partial(y, z) > 2$. Then $r = s = t = 0$. Suppose $\partial(y, z) = 2$. Then by the definition of 2-homogeneous $r = c_2 - \gamma_i$, $s = \gamma_i$, and $t = 0$. It follows from (11) and (12) that $e_ie_{i+1}r - e_{i-1}e_it = 0$. Suppose $\partial(y, z) = 1$. Then $r = s = t = 0$ since $a_i = 0$. Finally suppose $y = z$. Then $r = b_i$, $s = c_i$, and $t = 1$. It follows from (11) that $e_ie_{i+1}r - e_{i-1}e_is = [i]_qt$. Now (13) follows.

(ii): Recall that q is real and not ± 1 by Lemma 3.4, so the coefficients q^{D-2i} in the expression for Y are distinct. Observe that $Y^j = \sum_{i=0}^{D} q^{j(D-2i)}E_i^*$ since the E_i^* are idempotents. Viewing these expressions for Y^j $(0 \leq j \leq D)$ as equations in the unknowns E_i^* gives a system with a Vandermonde coefficient matrix. Thus we may express each E_i^* $(0 \leq i \leq D)$ as a linear combination of powers of Y. Now observe that $R = \left(\sum_{i=0}^{D} e_i^{-1}E_i^*\right)X^+$ and $L = \left(\sum_{i=0}^{D} e_i^{-1}E_i^*\right)X^-$, and recall that $A = R + L$ since $\mathcal{X}$ is bipartite. It follows that A, E_0^*, E_1^*, ..., E_D^* are contained in the subalgebra of $Mat_X(\mathbf{C})$ generated by X^-, X^+, and Y. It follows that X^-, X^+, and Y generate T since A, E_0^*, E_1^*, ..., E_D^* generate T. ∎

Let us rewrite (13):

$$LRE_j^* = \frac{q^{D+2} + q^{2i}}{q^D + q^{2i+2}}RLE_j^* + \frac{(q^D + q^2)^2(q^D - q^{2i})}{q^D(q^2 - 1)(q^D + q^{2i+2})}E_j^* \qquad (0 \leq j \leq D). \qquad (14)$$

It can be shown using [5, Theorem 13] that the linear dependence of LRE_i^*, RLE_i^*, and E_i^* for all i $(1 \leq i \leq D-1)$ is equivalent to the 2-homogeneous condition for the bipartite distance-regular graphs.

The 2-homogeneous bipartite distance-regular graphs are Q-polynomial with $q_{ij}^h = p_{ij}^h$ for all h, i, j, $(0 \leq h,\ i,\ j \leq D)$ (see [5, Theorem 42, Corollary 43] and [1, Theorem III.5.1]). As was the case for the Hamming cubes, the Q-polynomial structure of the 2-homogeneous bipartite distance-regular graphs also has a natural $U_q(sl(2))$ structure.

Lemma 3.6 *Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ denote a 2-homogeneous bipartite distance-regular graph with diameter $D \geq 3$ and $b_0 \geq 3$. Assume $\mathcal{X}$ is not a Hamming cube, and let q be as in Theorem 3.3. Fix $x \in X$, and write $T = T(x)$. Write*

$$X^- = \sum_{j=1}^{D-1} \frac{q^D + q^{2j}}{q^j(q^D + q^2)}E_jA^*E_{j-1}, \qquad X^+ = \sum_{j=0}^{D} \frac{q^D + q^{2j}}{q^j(q^D + q^2)}E_jA^*E_{j+1},$$

$$Y = \sum_{j=0}^{d} q^{D-2j} E_j, \qquad Y^{-1} = \sum_{j=0}^{d} q^{2j-D} E_j.$$

(i) *X^-, X^+, and Y satisfy the defining relations of $U_q(sl(2))$ given in Definition 3.1.*

(ii) *X^-, X^+, and Y generate T.*

There is a stronger characterization than Theorem 3.3 of the intersection numbers of the 2-homogeneous bipartite P-polynomial schemes. This gives us some very concrete examples of schemes with a natural $U_q(sl(2))$ structure.

Theorem 3.7 *[10, Theorem 1.2] Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ denote a bipartite distance-regular graph with diameter $D \geq 3$ and $b_0 \geq 3$. Suppose $\mathcal{X}$ is not the Hamming D-cube. Then $\mathcal{X}$ is 2-homogeneous if and only if its intersection array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is one of the following.*

(i) $\{k, k-1, 1; 1, k-1, k\}$, $k \geq 3$.

(ii) $\{4\gamma, 4\gamma-1, 2\gamma, 1; 1, 2\gamma, 4\gamma-1, 4\gamma\}$ *for γ a positive integer.*

(iii) $\{k, k-1, k-\mu, \mu, 1; 1, \mu, k-\mu, k-1, k\}$, *with $k = \gamma(\gamma^2 + 3\gamma + 1)$, $\mu = \gamma(\gamma + 1)$ for $\gamma \geq 2$, an integer.*

The array (i) is uniquely realized by the complement of the $2 \times (k+1)$-grid. The graphs with array (ii) are the Hadamard graphs of order 4γ. The array (iii) is uniquely realized by the antipodal 2-cover of the Higman-Sims graph when $\gamma = 2$, and no examples with $\gamma \geq 3$ are known.

One might hope for more examples with the $U_q(sl(2))$ structure described in Theorems 3.5 and 3.6. Indeed, for only a limited set of q does $U_q(sl(2))$ have an interpretation on a 2-homogeneous bipartite distance-regular graph. Moreover, the low diameter of these examples means that they only give rise to simple $U_q(sl(2))$-modules of low dimension (at most 6). One might even wish for examples which help to understand some of the more subtle simple $U_q(sl(2))$-modules which arise in the root of unity case.

Now that we have some examples, we consider the possibility of finding other examples of schemes with a $U_q(sl(2))$ structure similar to those described in Lemmas 3.5 and 3.6. In this section we show that the 2-homogeneous bipartite distance-regular graphs are the only examples. We state this in the following theorem, but omit the proof here.

Theorem 3.8 *Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ denote a scheme with $D \geq 3$ and $b_0 > 3$. Fix $x \in X$ and write $T = T(x)$. Then the following are equivalent.*

(i) *$\mathcal{X}$ is a 2-homogeneous bipartite distance-regular graph.*

(ii) *$\mathcal{X}$ is P-polynomial and T is generated by elements X^-, X^+, Y, and Y^{-1} of the form*

$$X^- = \sum_{i=0}^{D-1} x_i^- E_i^* A E_{i+1}^*, \quad X^+ = \sum_{i=1}^{D} x_i^+ E_i^* A E_{i-1}^*, \quad Y = \sum_{i=0}^{D} y_i E_i^*, \quad Y^{-1} = \sum_{i=0}^{D} y_i^{-1} E_i^*,$$

where $y_i \neq 0$ $(0 \leq i \leq D)$, which satisfy the defining relations of $U_q(sl(2))$ given in Definition 3.1 for some $q \in \mathbf{C} \backslash \{0, 1, -1\}$.

(iii) *$\mathcal{X}$ is Q-polynomial and T is generated by elements X^-, X^+, Y, and Y^{-1} of the form*

$$X^- = \sum_{i=0}^{D-1} x_i^- E_i A^* E_{i+1}, \quad X^+ = \sum_{i=1}^{D} x_i^+ E_i A^* E_{i-1}, \quad Y = \sum_{i=0}^{D} y_i E_i, \quad Y^{-1} = \sum_{i=0}^{D} y_i^{-1} E_i,$$

where $y_i \neq 0$ $(0 \leq i \leq D)$, which satisfy the defining relations of $U_q(sl(2))$ given in Definition 3.1 for some $q \in \mathbf{C}\backslash\{0, 1, -1\}$.

Suppose (i) – (iii) hold. Then in (ii) and (iii)

$$y_i = \epsilon q^{D-2i} \qquad (0 \leq i \leq D), \tag{15}$$

$$x_i^- x_{i+1}^+ = \epsilon \frac{(q^D + q^{2i+2})(q^D + q^{2i})}{q^{2i-1}(q^D + q^2)^2} \qquad (0 \leq i \leq D-1) \tag{16}$$

for some $\epsilon \in \{1, -1\}$.

In the previous section, we proved that (i) implies (ii) and (iii) for particular values of X^- and X^+. It is now routine to verify that, given a 2-homogeneous bipartite distance-regular graph, any X^- and X^+ of the form described in (ii) and (iii) (satisfying (15) and (16)) still satisfy the $U_q(sl(2))$ relations of Definition 3.1. Thus we admit the forms of parts (ii) and (iii). The factor ϵ in (15) and (16) appears because of the automorphism $Y \mapsto -Y$, $X^+ \mapsto -X^+$ of $U_q(sl(2))$.

Lemma 3.9 *Let $\mathcal{X} = (X, \{A_i\}_{i=0,1,\ldots,D})$ denote a scheme with $D \geq 3$. Fix $x \in X$, and write $T = T(x)$. Suppose Theorem 3.8(ii) holds. Then $\mathcal{X}$ is a 2-homogeneous bipartite distance-regular graph other than a Hamming cube. Let Φ be as in Lemma 3.4. Then one of the following holds.*

(i) $q \in \Phi$.

(ii) *D is odd and $-q \in \Phi$.*

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