On Strongly Regular Graphs with Parameters (k,0,2) and Their Antipodal Double Covers (Algebraic Combinatorics)

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On Strongly Regular Graphs with Parameters 
(k,0,2) and Their Antipodal Double Covers

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Abstract
Let $\Gamma$ be a strongly regular graph with parameters $(k,\lambda,\mu) = (q^2 + 1,0,2)$ admitting $G(\cong PGL(2,q^2))$ as one point stabilizer for odd prime power $q$. We show that if $G$ stabilizes a vertex $\infty$ of $\Gamma$ and acts on $\Gamma_2(\infty)$ transitively, then $q = 3$ holds and $\Gamma$ is the Gewirtz graph. Moreover it is shown that an antipodal double cover whose diameter 4 of a strongly regular graph with parameters $(k,0,2)$ is reconstructed from a symmetric association scheme of class 6 with suitable parameters.

1 Introduction

We are interested in the classification problems of distance regular graphs with $b_2 = 1$. Let $\Gamma$ be a distance regular graph with $b_2 = 1$ and valency $k > 2$. If the diameter $d$ of $\Gamma$ is larger more than 4, then $\Gamma$ is isomorphic to the dodecahedron ([3, pp.182]). In [1], M.Araya,A.Hiraki and A.Jurisic showed that if $d = 4$, then $\Gamma$ is an antipodal double cover of a strongly regular graph with parameters $(k,\lambda,\mu) = (n^2 + 1,0,2)$ for an integer $n$ not divisible by four and if $d = 3$, then $\Gamma$ is an antipodal cover of a complete graph. Obviously an antipodal cover of a complete graph is a distance regular graph with $b_2 = 1$ if it’s diameter is three.

The classification problems of antipodal covers of complete graphs are very difficult. Because the existence of an antipodal distance regular $(n-2)$-fold cover of the complete graph $K_n$ claims the existence of a projective plane of order $(n-1)$ for an odd positive integer $n$, moreover an antipodal distance regular $(n-1)$-fold cover of $K_n$ is equivalent to the existence of a Moore graph with diameter two and valency $n$ ([6],[7]).

The strongly regular graphs with parameters $(k,\lambda,\mu) = (5,0,2)$ and $(k,\lambda,\mu) = (10,0,2)$ are known, the former one has an antipodal double cover with $d = 4$, namely the Wells graph, the latter one (the Gewirtz graph) does not have an antipodal double cover.
with $d = 4([3, \text{pp.}372])$. The existence or nonexistence of strongly regular graphs with 
$(n^2 + 1, 0, 2)$ for $n \geq 5$ are not known up to date. We have studied these graphs.

2 Strongly regular graphs with $(q^2 + 1, 0, 2)$ admitting
\(PGL(2, q^2)\) for $q = p^e$

The following theorem is proved by using the character table of the association scheme 
corresponding to the permutation group \((O(3, q), O(3, q)/O^+(2, q))\) which W.M.Kwok 
gave in [5]. We note that $O(3, q) \cong \{\pm 1\} \times SO(3, q)$ and $SO(3, q) \cong PGL(2, q)$.

**Theorem 2.1** Let $\Gamma$ be a strongly regular graph with parameters $(q^2 + 1, 0, 2)$ and $G$ be a group isomorphic to $PGL(2, q^2)$ for an odd prime power $q$. If $G$ acts on $\Gamma$ as $G$ stabilizes 
a vertex $\infty$ of $\Gamma$ and $G$ is transitive on $\Gamma_2(\infty)$, then $q = 3$ and $\Gamma$ is the Gewirtz graph.

Sketch of the proof)

Any two involutions of $G$ are conjugate each other in $G$. We denote the centralizer of 
an involution $z$ in $G$ by $H$. Character table of association scheme $\mathcal{X}$ corresponding to 
the permutation group $(G, G/H)$ is given from Kwok's results. Then we obtain sevral 
informations concerning eigenvalues and their multiplicities of the graph $\Gamma_2(\infty)$ admitting $G$ as a transitive automorphism group from the character table of $\mathcal{X}$.

Comparing these informations with eigenvalues and their multiplicities of $\Gamma_2(\infty)$ as 
the second neighbourhood of a strongly regular graph with parametars $(q^2 + 1, 0, 2)$, we 
can lead a contradiction if $q > 3$.

3 Reconstruction of the graph $\Gamma$ and the antipodal 
double cover $\Gamma^*$ of $\Gamma$

Let $\Gamma$ be a strongly regular graphs with parameters $(k, 0, 2)$. In this section we study 
about the structure of the second neighbourhood of $\Gamma$ and antipodal double covers of 
them with $d = 4$. E.R.van.Dam and A.Munemasa proved the following theorem 3.1 
indeedently. ([4, pp.13-14],[8])

**Theorem 3.1** Let $\Gamma$ be a strongly regular graph with $\lambda = 0$, $\mu = 2$ and degree $k$ with $k > 5$. Then the second neighbourhood of $\Gamma$ with respect to any vertex generates a $3$-class association scheme. Furthermore any scheme with the same parameters can be constructed 
in this way from a strongly regular graph with the same parameters as $\Gamma$.

The intersection numbers $p_{h,i}$ of the association scheme of theorem 3.1 are the following.
Let $B_h(0 \leq h \leq 3)$ be the intersection matrices which $(B_h)_{i,j} = p_{h,i}^j(0 \leq i \leq 3, 0 \leq j \leq 3)$. 
\[ B_0 = I, \]
\[
B_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
k-2 & 0 & 2 & 1 \\
0 & k-5 & k-8 & k-5 \\
0 & 2 & 4 & 2
\end{pmatrix},
B_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
k-5 & (k-5)(k-8) & k-5 \\
(k-2)(k-5) & k-5 & k-5 \\
2 & 2 & 2
\end{pmatrix},
\]
\[
B_3 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 2 & 4 & 2 \\
0 & 2k-10 & 2k-12 & k-5 \\
2k-4 & 4 & 4 & k-2
\end{pmatrix}.
\]

Now we consider an antipodal double cover \( \Gamma^* \) of \( \Gamma \). The intersection array of \( \Gamma^* \) is the following.
\[
\iota(\Gamma^*) = \begin{pmatrix}
0 & 1 & 1 & k-1 & k \\
0 & 0 & k-2 & 0 & 0 \\
k & k-1 & 1 & 1 & 0
\end{pmatrix}
\]

Put \( \Omega = \{1, 2, \cdots, k\} \). Let \( \infty^+ \) be a vertex of \( \Gamma^* \) and \( \infty^- \) be a unique vertex in \( \Gamma^* \) such that \( d(\infty^+, \infty^-) = 4 \). We may set \( \Gamma^*(\infty^+) = \{1^+, 2^+, \cdots, k^+\} \) and \( \Gamma^*(\infty^-) = \{1^-, 2^-, \cdots, k^-\} \) and we may consider that \( d(i^+, i^-) = 4 \) is satisfied for any element \( i \in \Omega \). Obviously \( \Gamma^*(\infty^+) = \Gamma_3^*(\infty^-) \), \( \Gamma^*(\infty^-) = \Gamma_3^*(\infty^+) \) and \( \Gamma_2^*(\infty^+) = \Gamma_2^*(\infty^-) \). We denote the subgraph \( \Gamma_2^*(\infty^+) \) by \( \Delta \) and the set of vertices of \( \Delta \) by \( X \). For each \( x \in X \), \( |\Gamma^*(\infty^+) \cap \Gamma^*(x)| = 1 \) and \( |\Gamma^*(\infty^-) \cap \Gamma^*(x)| = 1 \) because of \( c_2 = b_3 = 1 \). Suppose that \( |\Gamma^*(\infty^+) \cap \Gamma^*(x)| = \{i^+\} \) and \( |\Gamma^*(\infty^-) \cap \Gamma^*(x)| = \{j^-\} \). Then there exists a bijection \( \varphi \) from \( X \) onto \( (\Omega \times \Omega) \setminus \{(i, i) | i \in \Omega\} \) defined by \( \varphi(x) = (i, j) \). Then we put \( i = \varphi(x)_1 \) and \( j = \varphi(x)_2 \). We denote by \( x' \) the element of \( X \) such that \( d(x, x') = 4 \), then \( \varphi(x)_1 = \varphi(x')_2 \) and \( \varphi(x)_2 = \varphi(x')_1 \) as we show in the sequel. Moreover we set as follows.

\[ A(x) = \{y \in X | d(x, y) = 1\}, \]
\[ B(x) = \{y \in X | \varphi(y)_1 = \varphi(x)_2 \text{ or } \varphi(y)_2 = \varphi(x)_1, y \neq x'\} \]
\[ A'(x) = \{y \in X | d(x', y) = 1\}, \]
\[ B'(x) = \{y \in X | \varphi(y)_1 = \varphi(x)_1 \text{ or } \varphi(y)_2 = \varphi(x)_2, x \neq y\} \]
\[ C(x) = X \setminus (A(x) \cup B(x) \cup A'(x) \cup B'(x) \cup \{x, x'\}) \]

We have the following theorem.

**Theorem 3.2** We define relations on \( X \) as follows.

\[ R_0 = \{(x, x) | x \in X\}, \]
\[ R_1 = \{(x, y) | y \in A(x)\}, \]
\[ R_2 = \{(x, y) | y \in B(x)\}, \]
\[ R_3 = \{(x, y) | y \in C(x)\}, \]
\[ R_4 = \{(x, y) | y \in B'(x)\}, \]
\[ R_5 = \{(x, y) | y \in A'(x)\}, \]
\[ R_6 = \{(x, x') | x \in X\} \]
Then \( \mathcal{X} = (X, R_i(0 \leq i \leq 6)) \) is a symmetric association scheme whose parameters are \( p_{h,i}^j(0 \leq h, j, i \leq 6) \) in the following matrices.

Here \( B_h \) is a \( 7 \times 7 \)-matrix whose rows and columns are indexed by \( \{0, 1, 2, 3, 4, 5, 6\} \) satisfying \( (B_h)_{i,j} = p_{h,i}^j \) for each \( h \) such that \( 0 \leq h \leq 6 \).

\[
B_0 = I, B_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
k - 2 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 2 & 1 & 0 & 0 \\
0 & k - 5 & k - 5 & k - 8 & k - 5 & k - 5 & 0 \\
0 & 0 & 1 & 2 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & k - 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
B_2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 2 & 1 & 0 & 0 \\
2k - 4 & 2 & 1 & 2 & k - 3 & 2 & 0 \\
0 & 2k - 10 & k - 5 & 2k - 12 & k - 5 & 2k - 10 & 0 \\
0 & 2 & k - 3 & 2 & 1 & 2 & 2k - 4 \\
0 & 0 & 1 & 2 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix},
\]

\[
B_3 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & k - 5 & k - 5 & k - 8 & k - 5 & k - 5 & 0 \\
0 & 2k - 10 & k - 5 & 2k - 12 & k - 5 & 2k - 10 & 0 \\
(k - 2)(k - 5) & (k - 5)(k - 8) & (k - 5)(k - 6) & k^2 - 13k + 48 & (k - 5)(k - 6) & (k - 5)(k - 8) & (k - 2)(k - 5) \\
0 & 2k - 10 & k - 5 & 2k - 12 & k - 5 & 2k - 10 & 0 \\
0 & k - 5 & k - 5 & k - 8 & k - 5 & k - 5 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix},
\]

\[
(B_4)_{i,j} = (B_2)_{i,(6-j)}, (B_5)_{i,j} = (B_1)_{i,(6-j)}; (B_6)_{i,j} = (B_0)_{i,(6-j)} \quad \text{for } 0 \leq i \leq 6, 0 \leq j \leq 6.
\]

Proof.
It is immediately shown that

\[
|A(x)| = |A'(x)| = k - 2, |B(x)| = |B'(x)| = 2(k - 2). \quad \text{We have}
\]

\[
\{\varphi(x), \varphi(x)\} \cap \{\varphi(y), \varphi(y)\} = \emptyset \quad \text{if } d(x, y) = 1 \quad (3.1)
\]

from \( a_1(\Gamma^*) = 0 \), and

\[
\{\varphi(y) \mid y \in A(x)\} = \{\varphi(y) \mid y \in A(x)\} = \Omega \setminus \{\varphi(x), \varphi(x)\} \quad (3.2)
\]
from $c_2(\Gamma^*) = 1$. Hence for any $i \in \Omega \setminus \{\varphi(x)_1, \varphi(x)_2\}$, there exists a unique element $y \in A(x)$ such that $\varphi(y)_1 = i$ and $z \in A(x)$ such that $\varphi(z)_2 = i$ because of $|A(x)| = |\Omega \setminus \{\varphi(x)_1, \varphi(x)_2\}|$.

Therefore the following also holds.

$$(A(y) \setminus \{x\}) \cap (A(z) \setminus \{x\}) = \emptyset \quad (y \neq z \in A(x)) \quad (3.3)$$

From (3.2) and (3.3), $|\{z \mid z \in A(y), z \neq x, y \in A(x)\}| = (k - 2)(k - 3)$ and so we have this set is equal to $B(x) \cup C(x)$. Thus $|C(x)| = (k - 2)(k - 5)$. Moreover we obtain

$|A(z) \cap B(x)| = 2$, $|A(z) \cap C(x)| = k - 5 \quad (\forall z \in A(x)) \quad (3.4)$

$|A(z) \cap B'(x)| = 2$, $|A(z) \cap C(x)| = k - 5 \quad (\forall z \in A'(x)) \quad (3.5)$

$|A(z) \cap A(x)| = 1$, $|A(z) \cap B(x)| = 1$, $|A(z) \cap C(x)| = k - 5$

$|A(z) \cap B'(x)| = 1 \quad (\forall z \in B(x)) \quad (3.6)$

$|A(z) \cap A'(x)| = 1$, $|A(z) \cap B'(x)| = 1$, $|A(z) \cap C(x)| = k - 5$

$|A(z) \cap B(x)| = 1 \quad (\forall z \in B'(x)) \quad (3.7)$

$|A(z) \cap A(x)| = 1$, $|A(z) \cap A'(x)| = 1$, $|A(z) \cap B(x)| = 2$

$|A(z) \cap B'(x)| = 2$, $|A(z) \cap C(x)| = k - 8 \quad (\forall z \in C(x)) \quad (3.8)$

Moreover about the neighbourhoods of $\Delta$ it is easy shown that $\Delta_1(x) = A(x)$, $\Delta_2(x) = B(x) \cup C(x)$, $\Delta_3(x) = B'(x) \cup A'(x)$ and $\Delta_4(x) = \{x'\}$ for any $x \in X$.

About the neighbourhoods of $\Omega$ we have $\Gamma_1(x) = A(x) \cup \{\varphi(x)_1^+, \varphi(x)_2^+\}$, $\Gamma_2(x) = B(x) \cup C(x) \cup B'(x) \cup \{i^+, i^- \mid i \neq \varphi(x)_1, i \neq \varphi(x)_2\} \cup \{\infty^+, \infty^-\}$, $\Gamma_3(x) = A'(x) \cup \{\varphi(x)_1^+, \varphi(x)_2^-\}$ and $\Gamma_4 = \{x'\}$ for any $x \in X$.

Therefore it follows that $(x, y) \in R_i$ if and only if $(y, x) \in R_i$ for $0 \leq i \leq 6$. We also have $p_{i,j} = p_{i,h}^j$ and $p_{h,i}^j = p_{6-h,i}^j$ since $(x, y) \in R_j$ if and only if $(x', y) \in R_{6-j}$. Then $p_{i,h}^j = p_{6-j}^{6-h}$.

Now we prove that $p_{3,0}^3 = p_{3,6}^3 = 1$, $p_{3,1}^3 = p_{3,5}^3 = k - 8$ and $p_{3,2}^3 = p_{3,4}^3 = 2k - 12$ which means that $p_{3,3}^3 = k^2 - 13k + 48$ because of $\sum_{i=0}^6 p_{3,i}^3 = |C(x)| = (k - 2)(k - 5)$.

It is trivial that $p_{3,0}^3 = p_{3,6}^3 = 1$. Let $x, y$ be elements of $X$ such that $(x, y) \in R_3$, namely $y \in C(x)$. Then $|C(x) \cap A(y)| = k - 8$ from (3.8) and this implies $p_{3,1}^3 = k - 8$.

Considering $x'$ instead of $x$, similarly above we have $p_{3,5}^3 = k - 8$. Let $z$ be an element of $X$ such that $(x, z) \in R_3$ and $(z, y) \in R_2$. Set $\varphi(x) = (i, j)$, $\varphi(y) = (k, \ell)$ and $\varphi(z) = (s, t)$, then $\varphi(x') = (j, i)$ and $s = \ell$ or $t = k$ holds because of $(z, y) \in R_2$. Suppose that $s = \ell$ holds. From (3.2) there is a unique element $u \in A(x)$ such that $\varphi(u)_1 = \ell$ and $v \in A'(x)$ such that $\varphi(v)_1 = \ell$. Then we can take any element of $\Omega$ except $\{i, j, k, \ell, \varphi(u)_2, \varphi(v)_2\}$ as a number $t$ satisfying $\varphi(z) = (\ell, t)$ and $z \in C(x)$, namely $|\{t \mid \varphi(z) = (\ell, t), (z, y) \in R_2, (x, z) \in R_3\}| = k - 6$. 
Similarly at the case \( t = k \), we get
\(|\{s \mid \varphi(z) = (s, k), (z, y) \in R_2, (x, z) \in R_3\}| = k - 6\). Hence \( p^2_{3,2} = 2(k - 6) \) holds. By the same arguments above we have \( p^2_{3,4} = 2(k - 6) \). Similarly we can decide other parameters \( p^i_{h,i} \) from (3.1) \( \sim \) (3.8). Thus the theorem is proved.

At the following theorem we prove that the inverse of the statement in theorem 3.2 is also true.

**Theorem 3.3** Let \( \mathcal{X} = (X, R_i(0 \leq i \leq 6)) \) be a symmetric 6-association scheme with same parameters as \( p^i_{h,i} \) in Theorem 3.2 for \( k > 5 \). Then the antipodal double cover \( \hat{\Gamma} \) with \( \mathcal{d}(\hat{\Gamma}^*) = 4 \) of a strongly regular graph with parameters \((k, 0, 2)\) can be constructed from \( \mathcal{X} \). Moreover the graph \((X, R_1)\) is isomorphic to the second neighbourhood of \( \hat{\Gamma} \) with respect to any vertex.

We now start with a short sketch of the proof. At first we consider the graph \( \hat{\Gamma} = (X, R_4) \). The parameters of this graph is that of the graph deleting the diagonal vertices of \( k \times k\)-grid. We reconstruct the graph \( \hat{\Gamma} \) isomorphic to \( k \times k\)-grid from \( \hat{\Gamma} \) by adding a set of some pairs of maximal cliques as new vertices to the vertices of \( \hat{\Gamma} \). Next using the graph \( \hat{\Gamma} \), an extended graph \( \Gamma^* \) of the graph \((X, R_1)\) is constructed. This \( \Gamma^* \) is the graph to be constructed in this theorem.

We use the following notation here. Let \( \Gamma' = (V(\Gamma'), E(\Gamma')) \) be a finite connected graph and \( \mathcal{d} \) be the metric of \( \Gamma' \). For two vertices \( x, y \) of \( \Gamma' \) such that \( \mathcal{d}(x, y) = i \), we denote by \( c_i(x, y), b_i(x, y) \) and \( a_i(x, y) \) the cardinalities of the sets \( \{z \in V(\Gamma') \mid d(x, z) = i - 1, d(z, y) = 1\} \), \( \{z \in V(\Gamma') \mid d(x, z) = i + 1, d(z, y) = 1\} \) and \( \{z \in V(\Gamma') \mid d(x, z) = i, d(z, y) = 1\} \) respectively.

We state four lemmas to prove the theorem. We note that \( k_0 = k_6 = 1, k_1 = k_5 = k - 2, k_2 = k_4 = 2k - 4 \) and \( k_3 = (k - 2)(k - 5) \) hold. Therefore we have \(|X| = k(k - 1)\). For any element \( x \in X \) there exists a unique element \( x' \in X \) such that \( (x, x') \in R_6 \) because of \( p^6_{0,6} = 1 \). We consider a bijective mapping \( \psi \) on \( X \) defined by \( \psi(x) = x' \) for any \( x \in X \). It is clear that \( \psi^2 = id_X \). We denote by \( \tilde{\Gamma} \) the graph \((X, R_4)\) and by \( \tilde{\mathcal{d}} \) the metric of \( \tilde{\Gamma} \).

**Lemma 3.1** The graph \( \tilde{\Gamma} \) is a regular graph with the valency \( 2k - 4 \) and \( \mathcal{d}(\tilde{\Gamma}) = 3 \). Moreover it follows that \( a_1(\tilde{\Gamma}) = k - 3, b_1(\tilde{\Gamma}) = k - 2, a_2(\tilde{\Gamma}) = 2k - 6, c_4(x, y) = 2 \) for \( x, y \in X \) such that \( \tilde{\mathcal{d}}(x, y) = 2 \) and \( y \not\in \tilde{\Gamma} (\psi(x)) \) and \( c_2(x, y) = 1 \) for \( x, y \in X \) such that \( \tilde{\mathcal{d}}(x, y) = 2 \) and \( y \in \tilde{\Gamma} (\psi(x)) \). We have also \( \tilde{\Gamma}_3(x) = \{\psi(x)\} \) for any \( x \in X \).

**Proof**
It is easily verified that \( \tilde{\Gamma} \) is a regular graph of the valency \( 2k - 4 \) as \( p^4_{0,4} = 2k - 4 \). We take elements \( x, y \in X \) such that \( (x, y) \in R_i \) for \( i \in \{1, 2, 3, 4, 5\} \). Then \( p^i_{0,4} \neq 0 \) holds and this implies that there is an element \( z \in X \) such that \( \tilde{\mathcal{d}}(x, z) = 1 \) and \( \tilde{\mathcal{d}}(z, y) = 1 \). Moreover \( \tilde{\mathcal{d}}(x, \psi(x)) = 3 \) holds. Therefore we have \( \mathcal{d}(\tilde{\Gamma}) = 3 \) and \( \tilde{\mathcal{d}}(x, y) = 3 \) holds if and
only if $y = \psi(x)$. Here we note that

$$(x, y) \in R_4 \text{ if and only if } (\psi(x), y) \in R_2.$$  (3.9)

because of $p_{4,i}^6 = 0$ for any $i(0 \leq i \leq 6, i \neq 2)$ and $p_{2,i}^6 = 0$ for any $i(0 \leq i \leq 6, i \neq 4)$. Therefore it follows that

$$\rho(x, y) = 1 \text{ if and only if } \rho(\psi(x), \psi(y)) = 1.$$  (3.10)

We now get $a_1(\bar{\Gamma}) = k - 3$ and $b_1(\bar{\Gamma}) = k - 2$ because of $p_{4,4}^4 = k - 3$ and $\sum_{1 \leq h \leq 5(h \neq 4)} p_{h,4}^4 = k - 2$. For any elements $x, y \in X$ such that $\rho(x, y) = 2$ and $y \notin \tilde{\Gamma}(\psi(x))$, we get $c_2(x, y) = 2, a_2(x, y) = 2k - 6$ and $b_2(x, y) = 0$ because of $p_{4,4}^4 = 2, \sum_{1 \leq h \leq 5(h \neq 4)} p_{h,4}^4 = 2k - 6$ for $i = 1, 3$ and 5 and from (3.9). Next let $x, y$ be elements of $X$ such that $\rho(x, y) = 2$ and $y \in \tilde{\Gamma}(\psi(x))$. Then $(x, y) \in R_2$ from (3.9). We get $c_2(x, y) = 1, b_2(x, y) = 1$ and $a_2(x, y) = 2k - 6$ because of $p_{4,4}^4 = p_{6,4}^2 = 1$ and $\sum_{1 \leq h \leq 5(h \neq 4)} p_{h,4}^4 = 2k - 6$. We also have $c_3(x, \psi(x)) = 2k - 4$ for any $x \in X$. This completes the proof of the lemma.

**Lemma 3.2** Let $x$ be an element of $X$. Then $\tilde{\Gamma}(x)$ is a disjoint union of two cliques of the same cardinalities $k - 2$.

Proof). Let $x \in X$ and $y \in \tilde{\Gamma}(x)$. Since $a_1(x, y) = k - 3$ and $k(\tilde{\Gamma}) = 2k - 4$ hold, we may set $\tilde{\Gamma}(x) = \{y, y_1, y_2, \ldots, z_{k-3}, z_1, z_2, \ldots, z_{k-2}\}$ for $\{y_1, y_2, \ldots, z_{k-3}\} \subset \tilde{\Gamma}(y)$ and $\{z_1, z_2, \ldots, z_{k-2}\} \subset \tilde{\Gamma}(y)$. We set $S = \{y, y_1, y_2, \ldots, y_{k-3}\}$ and $T = \{z_1, z_2, \ldots, z_{k-2}\}$. Let $z$ be any element of $T$. Then $\rho(y, z) = 2$. Since $c_2(y, z) = 2$, $\rho(x, y) = \rho(x, z) = 1$ and $S \cap \tilde{\Gamma}(z) \subset \tilde{\Gamma}(y) \cap \tilde{\Gamma}(z)$ hold, it follows that $|S \cap \tilde{\Gamma}(z)| \leq 1$. Then we have $|T \cap \tilde{\Gamma}(z)| \leq k - 4$ since $a_1(x, z) = k - 3$. But $T$ contains only $k - 3$ elements except $z$, therefore $|T \cap \tilde{\Gamma}_2(z)| \leq 1$ holds. Suppose that $T \cap \tilde{\Gamma}_2(z) \neq \emptyset$. Then there exists an element $u \in T$ where $\rho(z, u) = 2$, and every other elements of $T$ except $z$ and $u$ are adjacent to $z$. Moreover $|T \cap \tilde{\Gamma}_2(u)| \leq 1$ holds as same as $|T \cap \tilde{\Gamma}_2(z)| \leq 1$. Hence we get $T \cap \tilde{\Gamma}_2(u) = \emptyset$. Therefore it follows that $x$ and every elements of $T$ except $z$ and $u$ are contained in $\tilde{\Gamma}(z) \cap \tilde{\Gamma}(u)$, which implies $k - 3 \leq 2$. Thus $k \leq 5$, which contradicts $k > 5$. Hence we get $T \cap \tilde{\Gamma}_2(z) = \emptyset$ and any element of $T$ except $z$ is adjacent to $z$. However since $z$ is any element of $T$, $T$ is a clique. Applying the same arguments to a fixed element of $T$ instead of $y$, we also have $S$ is a clique. Thus the lemma is proved.

We denote by $C_1(x)$ and $C_2(x)$ the set $S \cup \{x\}$ and $T \cup \{x\}$ for $S, T$ in lemma 3.2. We note $|C_1(x)| = |C_2(x)| = k - 1$. Obviously $C_i(x)$ is a maximal clique of $\bar{\Gamma}$ for $i = 1, 2$ and any maximal clique of $\bar{\Gamma}$ is equal to $C_i(x)$ for an element $x \in X$ and $i \in \{1, 2\}$. We denote by $MC(\bar{\Gamma})$ the set of maximal cliques of $\bar{\Gamma}$ and put $D = \{C \cup \psi(C) \mid C \in MC(\bar{\Gamma})\}$. We note that $C \cap \psi(C) = \emptyset$ for any $C \in MC(\bar{\Gamma})$. For index $i \in \{1, 2\}$ we have $y \in C_i(x)$ if and only if $C_i(x) = C_j(y)$ for some $j \in \{1, 2\}$, as we saw in the proof of lemma 3.2. Hence we have $|MC(\bar{\Gamma})| = \frac{2|X|}{k - 1} = 2k$ and $|D| = k$. For $i \in \{1, 2\}$ we have $\psi(C_i(x)) = C_j(\psi(x))$.
for some $j \in \{1, 2\}$ from (3.10). Hence we may put $\psi(C_i(x)) = C_i(\psi(x))$ without loss of generality. We have the following lemma about $\mathcal{D}$.

**Lemma 3.3** (1) Let $x$ be any element of $X$. Then there exists exactly two elements of $\mathcal{D}$ containing $x$.

(2) Let $x, y$ be any elements of $X$ such that $\tilde{\rho}(x, y) = 1$. Then there exists exactly one element of $\mathcal{D}$ containing $x$ and $y$.

(3) Let $x, y$ be any elements of $X$ such that $\tilde{\rho}(x, y) = 2$ and $y \in \tilde{\Gamma}(\psi(x))$. Then there exists exactly one element of $\mathcal{D}$ containing $x$ and $y$.

(4) Let $D_1$ and $D_2$ be distinct elements of $\mathcal{D}$. Then $|D_1 \cap D_2| = 2$.

(5) Let $D$ be an element of $\mathcal{D}$ and $x$ be an element of $X$ such that $x \notin D$. Then $|\tilde{\Gamma}(x) \cap D| = 2$.

**Proof.**

(1): For $x \in X$, $C_1(x) \cup \psi(C_1(x))$ and $C_2(x) \cup \psi(C_2(x))$ are distinct elements of $\mathcal{D}$ containing $x$. Let $D$ be an element of $\mathcal{D}$ such that $x \in D$. Then there is an element $a \in X$ such that $D = C_i(a) \cup \psi(C_i(a))$ for some $i \in \{1, 2\}$. We may suppose $x \in C_i(a)$ because of $\psi(C_i(a)) = C_i(\psi(a))$. Then we have $C_i(a) = C_j(x)$ for some $j \in \{1, 2\}$ and $D = C_j(x) \cup \psi(C_j(x))$. Thus (1) is proved.

(2): Let $x, y$ be any elements of $X$ such that $\tilde{\rho}(x, y) = 1$. Then there is a unique maximal clique $C$ of $\tilde{\Gamma}$ containing $x$ and $y$ from lemma (3.2). Then $C \cup \psi(C)$ is a unique element of $\mathcal{D}$ containing $x$ and $y$. Thus (2) is proved.

(3): Let $x, y$ be any elements of $X$ such that $\tilde{\rho}(x, y) = 2$ and $y \in \tilde{\Gamma}(\psi(x))$. Then $\tilde{\rho}(\psi(x), y) = 1$. Therefore from (2) there exists exactly one element $D$ of $\mathcal{D}$ containing $\psi(x)$ and $y$. Then obviously $x \in D$ holds. Thus (3) is proved.

(4): Let $D_1$ and $D_2$ be distinct elements of $\mathcal{D}$. Then there are elements $a$ and $b$ of $X$ such that $D_1 = C_i(a) \cup \psi(C_i(a))$ and $D_2 = C_j(b) \cup \psi(C_j(b))$ for some $i, j \in \{1, 2\}$. We set $\{i, i'\} = \{j, j'\} = \{1, 2\}$. We will prove that $D_1 \cap D_2 \neq \emptyset$.

Suppose that $a \in \tilde{\Gamma}(b)$. If $a \in C_j(b)$ or $b \in C_i(a)$, then $D_1 \cap D_2 \neq \emptyset$. Hence we may assume $a \in C_{i'}(b)$ and $b \in C_{i'}(a)$.

Moreover since $\tilde{\rho}(a, \psi(b)) = 2$ and $\tilde{\rho}(\psi(a), \psi(b)) = 1$, there is a unique element $u \in X$, which is adjacent to $a$ and $\psi(b)$ from lemma (3.1). If $u \in C_i(a) \cap C_{i'}(b)$, then $D_1 \cap D_2 \neq \emptyset$. Hence we may assume $u \in C_{i'}(a)$ or $u \in C_{i'}(b)$. If $u \in C_{i'}(a)$, then $u$ is adjacent to $b$ because of $b \in C_{i'}(a)$, which means $\tilde{\rho}(b, \psi(b)) = 2$. This is a contradiction. If $u \in C_{i'}(b)$, then $\psi(u) \in C_{i'}(b)$, then $\psi(u)$ is adjacent to $a$ because of $a \in C_{i'}(b)$, which means $\tilde{\rho}(u, \psi(u)) = 2$. This is also a contradiction. Thus we may assume that $a$ is not adjacent to $b$. Similarly we may assume $a$ is not adjacent to $\psi(b)$. Hence $\tilde{\rho}(a, b) = 2$ and $\tilde{\rho}(a, \psi(b)) = 2$, and there are exactly two elements $u, v \in X$ which are adjacent to both $a$ and $b$ and there are exactly two elements $u', v' \in X$ which are adjacent to both $a$ and $\psi(b)$ from lemma 3.1. If $u$ is adjacent to $v$ then $a$ is adjacent to $b$ from (2). This contradicts our assumption. Therefore it does not occur.
that both \( u \) and \( v \) are contained in one of \( \{C_i(a), C_i'(a), C_j(b), C_j'(b)\} \). For \( u', v' \), the same arguments hold. If \( u \in C_i(a) \cap C_j(b) \) or \( v \in C_i(a) \cap C_j(b) \), then \( D_1 \cap D_2 \neq \emptyset \). Hence we may assume that \( u \in C_i(a), v \in C_i'(a), u \in C_j'(b) \) and \( v \in C_j(b) \). Similarly we may assume that \( u' \in C_i(a), v' \in C_i'(a), u' \in C_j'(\psi(b)) \) and \( v' \in C_j(\psi(b)) \). Then \( u \) and \( u' \) are adjacent because of \( u, u' \in C_i(a) \) and \( \psi(u) \) and \( u' \) are adjacent because of \( \psi(u), u' \in C_j'(\psi(b)) \). Therefore we have \( \hat{\rho}(u, \psi(u)) = 2 \), which is a contradiction. Thus it follows that \( D_1 \cap D_2 \neq \emptyset \).

Now suppose that \( C_i(a) \cap C_j(b) \) contains at least two elements \( u, z \). Then from (2) there exists a unique \( C \in CM(\tilde{\Gamma}) \) containing \( u, z \), and we have \( C = C_i(a) = C_j(b) \), which implies \( D_1 = D_2 \). Therefore \( |C_i(a) \cap C_j(b)| \leq 1 \).

Similarly \( |C_i(a) \cap C_j'(\psi(b))| \leq 1, |C_i(\psi(a)) \cap C_j(b)| \leq 1 \) and \( |C_i(\psi(a)) \cap C_j'(\psi(b))| \leq 1 \). Since \( D_1 \cap D_2 = (C_i(a) \cap C_j(b)) \cup (C_i(a) \cap C_j'(\psi(b))) \cup (C_i(\psi(a)) \cap C_j(b)) \cup (C_i(\psi(a)) \cap C_j'(\psi(b))) \), \( \psi(C_i(a) \cap C_j(b)) = C_i(\psi(a)) \cap C_j'(\psi(b)) \) and \( \psi(C_i(a) \cap C_j(b)) = C_i(\psi(a)) \cap C_j'(b) \), we have \( |D_1 \cap D_2| = 2 \), if it is proved that \( C_i(a) \cap C_j(b) \neq \emptyset \) is not compatible with \( C_i(a) \cap C_j'(\psi(b)) \neq \emptyset \).

Suppose that there are elements \( u, v \) such that \( u \in C_i(a) \cap C_j(b) \) and \( v \in C_i(a) \cap C_j'(\psi(b)) \). Then \( u \) and \( v \) are adjacent because of \( u, v \in C_i(a) \). Moreover \( \psi(u) \) and \( v \) are adjacent because of \( \psi(u), v \in C_j'(\psi(b)) \). Therefore \( \hat{\rho}(u, \psi(u)) = 2 \), a contradiction. Thus (4) is proved.

(5): Let \( D \) be an element of \( D \) and \( y \) be an element of \( X \) such that \( y \not\in D \). For fix any \( j \in \{1, 2\}, D \neq C_j(y) \cup \psi(C_j(y)) \) because of \( y \not\in D \). Therefore \( |D \cap (C_j(y) \cup \psi(C_j(y)))| = 2 \) from (4). Hence \( |D \cap C_j(y)| = 1 \) under consideration \( \psi(D) = D \), which means that \( |D \cap \tilde{\Gamma}(y)| = 2 \). Thus (5) is proved, and the lemma was verified.

We now construct a graph isomorphic to the Hamming graph \( H(2, k) \) from \( \tilde{\Gamma} \) adding some vertices to \( X \). We define the graph \( \hat{\Gamma} \).

The set of vertices of \( \hat{\Gamma} \) is \( X \cup D \). The adjacency is defined by \( x, y \in X \) are adjacent if \( \hat{\rho}(x, y) = 1 \), \( x \in X \) and \( D \in D \) are adjacent if \( x \in D \).

The metric of the graph \( \hat{\Gamma} \) is denoted by \( \hat{\rho} \).

Lemma 3.4 The graph \( \hat{\Gamma} \) is isomorphic to the Hamming graph \( H(2, k) \)

Proof).

Let \( x \) be any element of \( X \), then there exists exactly two elements of \( D \) containing \( x \) and \( \psi(x) \). Therefore \( \hat{\rho}(x, \psi(x)) = 2 \) by the definition above. Hence we have the diameter of \( \hat{\Gamma} \) is two. For any \( x \in X \), there exists exactly two elements of \( D \) containing \( x \) from (1) of lemma 3.3. Moreover, since \( k(\tilde{\Gamma}) = 2k - 4 \), the valency of \( x \) in the graph \( \hat{\Gamma} \) is \( 2k - 2 \). For any \( D \in D \), since \( D \) contains exactly \( 2(k-1) \) elements of \( X \), the valency of \( D \) in \( \hat{\Gamma} \) is \( 2k - 2 \). Thus the valency of \( \hat{\Gamma} \) is \( 2k - 2 \). Let \( x, y \) be elements of \( X \) such that \( \hat{\rho}(x, y) = 1 \).

Then there exists exactly one element of \( D \) containing \( x \) and \( y \) from (2) of lemma 3.3.
On the other hand exactly $k - 3$ elements of $X$ are adjacent to $x$ and $y$ because of $a_1(\hat{\Gamma}) = k - 3$. Hence it follows $a_1(x, y) = k - 2$ in $\hat{\Gamma}$. Let $x \in X$ and $D \in \mathcal{D}$ be adjacent in $\hat{\Gamma}$. Then $x \in D$ and $|D \cap \hat{\Gamma}(x)| = k - 1$. Hence it follows $a_1(x, D) = k - 2$ in $\hat{\Gamma}$. Thus $a_1(\hat{\Gamma}) = k - 2$ holds. Let $x, y$ be elements of $X$ such that $\hat{\rho}(x, y) = 2$. If $y = \psi(x)$, then obviously $c_2(x, y) = 2$ in $\hat{\Gamma}$. If $y \notin \hat{\Gamma}(\psi(x))$, then there exists exactly one element of $\mathcal{D}$ containing $x$ and $y$ from (3) of lemma 3.3.

Moreover there exists exactly one element of $X$ which is adjacent to $x$ and $y$ because of $c_2(x, y) = 1$ in $\hat{\Gamma}$ from lemma 3.1. Therefore $c_2(x, y) = 2$ in $\hat{\Gamma}$. If $y \notin \hat{\Gamma}(\psi(x))$, then there is no element of $\mathcal{D}$ containing $x$ and $y$ since $y$ is not adjacent to $x$ or $\psi(x)$.

However there exists exactly two element of $X$ which are adjacent to $x$ and $y$ because of $c_2(x, y) = 2$ in $\hat{\Gamma}$. Therefore $c_2(x, y) = 2$ in $\hat{\Gamma}$. Let $D_1, D_2$ be distinct elements of $\mathcal{D}$. Then $|D_1 \cap D_2| = 2$ from (4) of lemma 3.3. Therefore $c_2(D_1, D_2) = 2$ in $\hat{\Gamma}$. Let $D$ be an element of $\mathcal{D}$ and $x$ be an element of $X$ such that $x \notin D$. Then $|\hat{\Gamma}(x) \cap D| = 2$ from (5) of lemma 3.3. Therefore $c_2(D, x) = 2$ in $\hat{\Gamma}$. Thus $c_2(\hat{\Gamma}) = 2$ holds. Hence the graph $\hat{\Gamma}$ has the same parameters as those of the Hamming graph $H(2, k)$. Thus the graph $\hat{\Gamma}$ is isomorphic to the Hamming graph $H(2, k)(\text{cf. } [9])$. This completes the proof of the lemma.

From lemma 3.4 there exists a bijection $\varphi$: $X \cup \mathcal{D} \rightarrow \Omega \times \Omega$ such that $\varphi(\mathcal{D}) = \{(i, i) \mid i \in \Omega\}$ and for any distinct elements $x, y \in X$, $(x, y) \in R_4$ if and only if $\varphi(x) \varphi(y) = \varphi(x) \varphi(y)$. We can now construct the antipodal double cover $\Gamma^*$ of a strongly regular graph with parameters $(k, 0, 2)$.

The set of vertices of $\Gamma^*$ is $V(\Gamma^*) = X \cup \Omega^+ \cup \Omega^- \cup \{\infty\}^\pm$ where $\Omega^+ = \{1^+, 2^+, \cdots, k^+\}$ and $\Omega^- = \{1^-, 2^-, \cdots, k^-\}$.

The adjacency of $\Gamma^*$ is defined by

\begin{align*}
\Gamma^*(\infty^+) &= \Omega^+, \quad \Gamma^*(\infty^-) = \Omega^-; \\
\text{for } x, y \in X, \text{ } x \text{ and } y \text{ are adjacent if } (x, y) \in R_1; \\
x \in X \text{ and } i^+ \in \Omega^+ \text{ are adjacent if } \varphi(x)_1 = i, \\
x \in X \text{ and } j^- \in \Omega^- \text{ are adjacent if } \varphi(x)_2 = j.
\end{align*}

The metric of the graph $\Gamma^*$ is denoted by $\rho$. Then we get the following.

$$\rho(x, y) = 2 \text{ if } (x, y) \in R_4 \quad (3.11)$$

We can verify that $\Gamma^*$ is a distance regular graph whose intersection array is $(k, k - 1, 1, 1, 1; 1, 1, k - 1, k)$ in the sequel. For any $x \in \{\pm \infty\} \cup \Omega^+ \cup \Omega^-$, it is clear that the valency of $x$ is $k$. For any $x \in X$, there are exactly $k - 2$ elements of $X$ which are adjacent to $x$ because of $p_{1, 1}^\circ = k - 2$. Moreover $x$ is adjacent to only one element $\varphi(x)^\uparrow$ in $\Omega^+$ and $\varphi(x)^\downarrow$ in $\Omega^-$ respectively. Therefore the valency of $x$ is $k$. Thus the valency of $\Gamma^*$ is $k$.

We note the bijection $\varphi$ is a graph isomorphism from $\hat{\Gamma}$ onto the Hamming graph $H(2, k)$ on $\Omega \times \Omega$ such that $\varphi(\mathcal{D}) = \{(i, i) \mid i \in \Omega\}$. Moreover in the subgraph of $H(2, k)$ being deleted the vertices $\{(i, i) \mid i \in \Omega\}$, there exists exactly one vertex at distance 3 from a vertex $(i, j)$ in the subgraph, namely $(j, i)$. This implies the following.

$$\varphi(x) = (i, j) \text{ if and only if } \varphi(\psi(x)) = (j, i) \text{ for } x \in X \quad (3.12)$$
Now we have the following lemma.

**Lemma 3.5** Let $x, y$ be elements of $X$ such that $\varphi(x) = (i, j)$ and $\varphi(y) = (\ell, h)$. Then the following (1) and (2) hold.

1. If $\rho(x, y) = 1$, then $\{i, j\} \cap \{\ell, h\} = \emptyset$.
2. If $t \in \Omega$ and $t \notin \{i, j\}$, then there exists exactly one element $u$ of $X$ such that $\rho(x, u) = 1$ and $\varphi(u)_{1} = t$ and exactly one element $v$ of $X$ such that $\rho(x, v) = 1$ and $\varphi(v)_{2} = t$.

Proof. 

(1): Suppose that $\rho(x, y) = 1$. Then $(x, y) \in R_{1}$. If $i = \ell$ or $j = h$, then $(x, y) \in R_{4}$, a contradiction. If $i = h$ or $j = \ell$, then $(x, \psi(y)) \in R_{4}$ from (3.12), therefore $(x, y) \in R_{2}$ from (3.9), a contradiction. Therefore $\{i, j\} \cap \{\ell, h\} = \emptyset$. Thus (1) holds.

(2): For any distinct elements $u, v \in X$ such that $\rho(x, u) = 1$ and $\rho(x, v) = 1$, we have $\varphi(u)_{1} \neq \varphi(v)_{1}$ and $\varphi(u)_{2} \neq \varphi(v)_{2}$ because of $p_{1,1}^{i} = 0$.

Moreover since $|\{u \in X \mid \rho(x, u) = 1\}| = k - 2$, we have $\Omega = \{\varphi(u)_{1} \mid u \in X, \rho(x, u) = 1\} \cup \{i, j\}$ and $\Omega = \{\varphi(u)_{2} \mid u \in X, \rho(x, u) = 1\} \cup \{i, j\}$ from (1). Thus (2) holds.

Proof of Theorem 3.3:

Suppose that $x, y \in X$. Since $p_{1,1}^{i} \neq 0$ for $i \in \{2, 3\}$ and $p_{1,1}^{5} = 0$, the following holds.

$$\rho(x, y) = 2 \text{ if } (x, y) \in R_{2} \cup R_{3}$$

(3.13)

$$\rho(x, y) > 2 \text{ if } (x, y) \in R_{5}$$

(3.14)

For any $x \in X$, we set as follows.

$A(x) = \{y \in X \mid (x, y) \in R_{1}\}$,

$B(x) = \{y \in X \mid y \neq \psi(x), \varphi(y)_{1} = \varphi(x)_{2} \text{ or } \varphi(y)_{2} = \varphi(x)_{1}\}$,

$B'(x) = \{y \in X \mid y \neq x, \varphi(y)_{1} = \varphi(x)_{1} \text{ or } \varphi(y)_{2} = \varphi(x)_{2}\}$,

$A'(x) = \{y \in X \mid (x, y) \in R_{3}\}$ and

$C(x) = X \setminus (A(x) \cup B(x) \cup B'(x) \cup A'(x) \cup \{\psi(x)\})$.

We note that $y \in B'(x)$ if and only if $(x, y) \in R_{4}$ and $y \in B(x)$ if and only if $(x, y) \in R_{2}$ from (3.9) and (3.12). Hence it follows that $y \in C(x)$ if and only if $(x, y) \in R_{3}$.

Suppose that $x \in X$ and $\varphi(x) = (i, j)$. Then we have $\Gamma_{3}^{*}(x) = A(x) \cup \{i^{+}, j^{-}\}$ and $\Gamma_{3}^{*}(x) = B(x) \cup C(x) \cup B'(x) \cup (\Omega^{+} \setminus \{i^{+}, j^{+}\}) \cup (\Omega^{-} \setminus \{i^{-}, j^{-}\}) \cup \{\infty^{+}\}$ from (3.11), (3.13) and (2) of Lemma 3.5.

Moreover obviously $A(y) \cap \Gamma_{3}^{*}(x) \neq \emptyset$ for any $y \in A'(x)$. Hence we have $\Gamma_{3}^{*}(x) = A'(x) \cup \{i^{-}, j^{+}\}$ from (3.14) and $\Gamma_{4}^{*}(x) = \{\psi(x)\}$. On the other hand for any $i \in \Omega$, we have $\Gamma_{3}^{*}(i^{+}) = \{x \in X \mid \varphi(x)_{1} = i\} \cup \{\infty^{+}\}$, $\Gamma_{3}^{*}(i^{-}) = \{x \in X \mid \varphi(x)_{1} \neq i \text{ and } \varphi(x)_{2} \neq i\} \cup (\Omega^{+} \setminus \{i^{+}\}) \cup (\Omega^{-} \setminus \{i^{-}\})$, $\Gamma_{4}^{*}(i^{+}) = \{x \in X \mid \varphi(x)_{2} = i\} \cup \{\infty^{-}\}$ and $\Gamma_{4}^{*}(i^{-}) = \{\infty^{-}\}$. Therefore especially it follows that the diameter of $\Gamma^{*}$ is 4.

Now since $p_{3,6}^{i} = 0$ for $i \in \{0, 1, 2, 3, 4, 6\}$ and $p_{5,6}^{i} = 0$ for $i \in \{0, 2, 3, 4, 5, 6\}$, we obtain the following.

$$(x, y) \in R_{1} \text{ if and only if } (x, \psi(y)) \in R_{5}$$

(3.15)
This statement with (3.9) and (3.12) imply that $\Gamma^*(x) = \Gamma_3^*(\psi(x))$, $\Gamma_2^*(x) = \Gamma_2^*(\psi(x))$ and $\Gamma^*_3(x) = \Gamma^*(\psi(x))$ for any $x \in X$. Therefore we have $c_1(\Gamma^*) = b_3(\Gamma^*)$, $c_2(\Gamma^*) = b_2(\Gamma^*)$, $c_3(\Gamma^*) = b_1(\Gamma^*)$ and $c_4(\Gamma^*) = b_0(\Gamma^*)$.

Lastly we will prove that $a_1(\Gamma^*) = 0$ and $c_2(\Gamma^*) = 1$, which lead to a complete proof of Theorem 3.3. Since $p_{1,1}^1 = 0$, there are no triangle whose vertices are all in $X$. Moreover for any elements $x, y \in X$ such that $\rho(x, y) = 1$ it follows that $\psi(x)_1 \neq \psi(y)_1$ and $\psi(x)_2 \neq \psi(y)_2$ from (1) of Lemma 3.5. Thus there are no triangle in $\Gamma^*$. Hence we have $a_1(\Gamma^*) = 0$ and $b_1(\Gamma^*) = k - 1$.

Let $x, y$ be elements in $X$ and suppose that $\rho(x, y) = 2$. Then $y \in B(x) \cup C(x) \cup B'(x)$. If $y \in B(x) \cup C(x)$, then $c_2(x, y) = 1$ because of $p_{1,1}^2 = 1$ and $p_{1,1}^3 = 1$. If $y \in B'(x)$, then $c_2(x, y) = 1$ because of $p_1^4 = 0$ and either $\varphi(x)_1 = \varphi(y)_1$ or $\varphi(x)_2 = \varphi(y)_2$ occurs.

Next suppose that $\rho(x, i^+) = 2$ for $x \in X$ and $i \in \Omega$. Then from (2) of Lemma 3.5, we have $c_2(x, i^+) = 1$. Obviously $c_2(\infty^+, x) = 1$ for any $x \in X$. $c_2(i^+, j^+) = 1$ for any distinct $i, j \in \Omega$ and $c_2(i^+, j^-) = 1$ for any distinct $i, j \in \Omega$. Thus it is proved that $c_2(\Gamma^*) = 1$. This completes the proof of the theorem.

参考文献


[8] A.Munemasa, Strongly regular graphs with parameters $(k, \lambda, \mu) = (k, 0, 2)$, private communication.