

# TIGHT DISTANCE-REGULAR GRAPHS

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## 1 Introduction

In this talk we study tight distance-regular graphs. We show an inequality for distance-regular graphs and we call a non-bipartite distance-regular graph tight when equality holds in this inequality. We give some characterizations of those graphs and give all examples known to us. At last we will study tight distance-regular with diameter 3 and 4.

This talk is based on joint work with Aleksandar Jurišić (Ljubjana) and Paul Terwilliger (Madison).

In the remainder of this section we introduce some basic definitions and notation. An **equitable partition** of a graph  $\Gamma$  is a partition of its vertices into cells  $C_1, C_2, \dots, C_s$  such that for all  $i$  and  $j$  the number  $c_{ij}$  of neighbours, which a vertex in  $C_i$  has in the cell  $C_j$ , is independent of the choice of the vertex in  $C_i$ . In other words each cell  $C_i$  induces a regular graph of valency  $c_{ii}$ , and between any two cells  $C_i$  and  $C_j$  there is a biregular graph, with vertices of the cells  $C_i$  and  $C_j$  having valencies  $c_{ij}$  and  $c_{ji}$  respectively.

A graph  $\Gamma = (X, R)$  with diameter  $d$  is **distance-regular** when the distance partition corresponding to any vertex  $x \in X$  is equitable and the parameters of the equitable partition do not depend on  $x$ . In a distance-regular graph for a pair of vertices  $(x, y)$  at distance  $h$  the number  $p_{ij}^h$  of vertices at distance  $i$  from  $x$  and  $j$  from  $y$  depends only on integers  $i, j, h$ , and not on  $(x, y)$ . We denote the intersection numbers  $p_{ii}^i, p_{i,i+1}^i, p_{i,i-1}^i$  and  $p_{ii}^0$  respectively by  $a_i, b_i, c_i$  and  $k_i$ , for  $i = 0, 1, \dots, d$ , note  $b_0 = a_i + b_i + c_i$  is the valency of the graph  $\Gamma$  and call  $\{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$  the intersection array of  $\Gamma$ . For a detailed treatment and all the terms which we do not

define see [1]. A graph is *i-homogeneous* when a distance partition corresponding to any pair of vertices at distance  $i$  is equitable, see Nomura [5]. A graph  $\Gamma$  of diameter  $d$  is **antipodal** if the vertices at distance  $d$  from a given vertex are all at distance  $d$  from each other. Then ‘being at distance  $d$  or zero’ induces an equivalence relation on the vertices of  $\Gamma$ , and the equivalence classes are called **antipodal classes**. For an antipodal graph  $\Gamma$  we define the **antipodal quotient** of  $\Gamma$ , to be the graph with the antipodal classes as vertices, where two classes are adjacent if they contain adjacent vertices.

## 2 Tight graphs

We show that strongly regular graphs are special kind of extremal graphs. From this one quickly derives an inequality for distance-regular graphs, see (3). A graph  $\Gamma$  on  $n$  vertices is called **strongly regular** with parameters  $(k, \lambda, \mu)$  if and only if its adjacency matrix  $A$  satisfies  $A^2 = kI + \lambda A + \mu(J - I - A)$  and  $AJ = kJ$  for some integers  $k, \lambda$  and  $\mu$ , i.e., when it is  $k$ -regular and has at most three eigenvalues. A connected strongly regular graph is distance-regular and has diameter two. The nontrivial eigenvalues  $r$  and  $s$  (whose eigenvectors can be assumed to be orthogonal to the all ones vector, which corresponds to the trivial eigenvalue  $k$ ) are the roots of the quadratic equation  $x^2 - (\lambda - \mu)x + (\mu - k) = 0$  and thus

$$\lambda - \mu = r + s, \quad \mu - k = rs. \quad (1)$$

The above relations show that the parameter  $(k, \lambda, \mu)$  could be expressed also by the eigenvalues  $(k, r, s)$  of the strongly regular graph. By counting the edges between the neighbours and non-neighbours of a vertex in a connected strongly regular graph we obtain:  $\mu(n - 1 - k) = k(k - \lambda - 1)$ , and so in the case when the graph is not complete graph we derive, by (1),

$$n = \frac{(k - r)(k - s)}{k + rs}. \quad (2)$$

We will now show that the right side of the equality (2) is an upper bound on the number of vertices of a  $k$ -regular graph with the eigenvalues other than  $k$  from the interval  $[s, r]$ .

**Theorem 2.1** *Let  $\Gamma = (X, R)$  denote a  $k$ -regular graph on  $n$  vertices,  $n > k + 1$ , with eigenvalues  $k = \eta_1, \dots, \eta_n$  (not necessarily distinct). Let  $r$  and  $s$  be such numbers*

that  $r \leq \eta_i \leq s$ , for  $i = 2, \dots, n$ . Then  $n(k + rs) \leq (k - r)(k - s)$ . Equality holds if and only if  $\Gamma$  is strongly regular with eigenvalues in  $\{k, r, s\}$ .

*Proof.* The trace of the adjacency matrix  $A$  equals the sum of its eigenvalues and is zero. The trace of  $A^2$  equals the sum of squares of eigenvalues and is  $nk$ , i.e., the number of walks of length two which start and end in the same vertex. Summing the inequalities  $(\eta_i - r)(\eta_i - s) \leq 0$  for  $i = 2, \dots, n$ , and using the above two facts we obtain the desired inequality, which holds with equality if and only if  $\eta_i \in \{r, s\}$  for  $i = 2, \dots, n$ . It follows that in the case of equality the graph  $\Gamma$  has at most three eigenvalues, namely  $k$ ,  $s$  and  $r$ , and is therefore strongly regular. ■

We will now apply this result to distance-regular graphs. Let  $\Gamma = (X, R)$  be a distance-regular graph with diameter  $d$ , and eigenvalues  $k = \theta_0 > \theta_1 > \dots > \theta_d$ . For a vertex  $x \in X$  let  $\Gamma_i(x)$  denote the set of vertices at distance  $i$  from  $x$ , and for a vertex  $y \in X$  let  $D_j^i(x, y) := \Gamma_i(x) \cap \Gamma_j(y)$ . The graph induced on the vertices  $\Gamma_i(x)$  is called the  $i$ -th **subconstituent graph** of  $x$ . It is the regular graph on  $k_i$  vertices and with valency  $a_i$ . The first subconstituent graph of  $x$  will be called also the **local graph** of  $x$ , and will be denoted by  $\Delta = \Delta(x)$ . Let  $\partial(x, y)$  denote the distance between the vertices  $x$  and  $y$ . Then for  $\partial(x, y) = 2$  the graph induced on  $D_1^1(x, y)$  is called the  $\mu(x, y)$ -graph, or just the  $\mu$ -graph.

For  $d \geq 2$ , an easy eigenvalue interlacing argument guarantees  $\theta_1 \geq 0$  and  $\theta_d \leq -\sqrt{2}$ , so we can define

$$b^- := -1 - \frac{b_1}{\theta_1 + 1} \quad \text{and} \quad b^+ := -1 - \frac{b_1}{\theta_d + 1}.$$

Suppose the graph  $\Gamma$  is nonbipartite with diameter  $d \geq 3$ , and let  $a_1 = \eta_1 \geq \eta_2 \geq \dots \geq \eta_k$  be the eigenvalues of the local graph  $\Delta(x)$ . Then, by Terwilliger's result [1, Thm. 4.4.3 and Thm. 4.4.4]  $b^+ \geq \eta_i \geq b^-$ , for  $i = 2, \dots, d$ , and therefore, by Theorem 2.1, we have

$$k(a_1 + b^+b^-) \leq (a_1 - b^+)(a_1 - b^-). \quad (3)$$

Equality holds in (3) if and only if  $\eta_i \in \{b^+, b^-\}$  for  $i = 2, \dots, k$ , i.e., the local graph  $\Delta$  is strongly regular with eigenvalues  $a_1$ ,  $b^-$  and  $b^+$ . The nonbipartite distance-regular graphs for which the equality holds are called **tight graphs**.

In the following theorem we will give some characterizations of tight graphs.

**Theorem 2.2** *Let  $\Gamma = (X, R)$  be a non-bipartite distance-regular graph with diameter  $d \geq 3$ . The following are equivalent:*

- (i)  $\Gamma$  is tight,
- (ii)  $\Gamma$  is 1-homogeneous and  $a_d = 0$ ,
- (iii) For each vertex  $x$  the local graph  $\Delta(x)$ , i.e. the subgraph induced by  $\Gamma(x)$ , is strongly regular with eigenvalues  $a_1, b^+, b^-$ .
- (iv) For some vertex  $x$  the local graph  $\Delta(x)$ , i.e. the subgraph induced by  $\Gamma(x)$ , is strongly regular with eigenvalues  $a_1, b^+, b^-$ .

### 3 Examples

The following examples (i)-(v) are all the known tight distance-regular graphs with diameter at least 3. In each case we give the intersection array, and the parameters and eigenvalues of the local graph.

(i) The Johnson graph  $J(2d, d)$  has diameter  $d$  and intersection numbers  $b_i = (d-i)^2$ ,  $c_i = i^2$  for  $i = 0, 1, \dots, d$ . It is locally the lattice graph  $K_d \times K_d$ , with parameters  $(d^2, 2(d-1), d-2, 2)$  and non-trivial eigenvalues  $r = d-2$ ,  $s = -2$ .

(ii) The halved cube  $\frac{1}{2}H(2d, 2)$  has diameter  $d$  and intersection numbers  $b_i = (d-i)(2d-2i-1)$ ,  $c_i = i(2i-1)$  for  $i = 0, 1, \dots, d$ . It is locally the Johnson graph  $J(2d, 2)$ , with parameters  $(d(2d-1), 4(d-1), 2(d-1), 4)$  and non-trivial eigenvalues  $r = 2d-4$ ,  $s = -2$ .

(iii) The Taylor graphs are the distance-regular graphs with  $k_3 = 1$ . See Taylor [8] and Seidel and Taylor [6] for more information.

(iv) The Conway-Smith graph has intersection array  $\{10, 6, 4, 1; 1, 2, 6, 10\}$ . It is locally the Petersen graph, with parameters  $(10, 3, 0, 1)$  and non-trivial eigenvalues  $r = 1$ ,  $s = -2$ .

(v) The Blokhuis-Brouwer graph with intersection array  $\{45, 32, 12, 1; 1, 6, 32, 45\}$ . It is locally the generalized quadrangle  $GQ(4, 2)$ , with parameters  $(45, 12, 3, 3)$  and

non-trivial eigenvalues  $r = 3, s = -3$ .

(vi) The graph  $3.O_7(3)$  with intersection array  $\{117, 80, 24, 1; 1, 12, 80, 117\}$ . It is locally strongly regular, with parameters  $(117, 36, 15, 9)$  and non-trivial eigenvalues  $r = 9, s = -3$ .

(vii) The graph  $3.Fi_{24}$  with intersection array  $\{31671, 28160, 2160, 1; 1, 1080, 28160, 31671\}$ . It is locally strongly regular, with parameters  $(31671, 3510, 693, 351)$  and non-trivial eigenvalues  $r = 351, s = -9$ .

(viii) The Soicher1 graph with intersection array  $\{56, 45, 16, 1; 1, 8, 45, 56\}$ , cf. [7]. It is locally strongly regular, with parameters  $(56, 10, 0, 2)$  and non-trivial eigenvalues  $r = 2, s = -4$ .

(ix) The Soicher2 graph with intersection array  $\{416, 315, 64, 1; 1, 32, 315, 416\}$ , cf. [7]. It is locally strongly regular, with parameters  $(117, 36, 15, 9)$  and non-trivial eigenvalues  $r = 9, s = -3$ .

(x) The Meixner1 graph with intersection array  $\{176, 135, 24, 1; 1, 24, 135, 176\}$ , cf. [4]. It is locally strongly regular, with parameters  $(176, 40, 12, 8)$  and non-trivial eigenvalues  $r = 8, s = -4$ .

(xi) The Meixner2 graph with intersection array  $\{176, 135, 36, 1; 1, 12, 135, 176\}$ , cf. [4]. It is locally strongly regular, with parameters  $(176, 40, 12, 8)$  and non-trivial eigenvalues  $r = 8, s = -4$ . It is a 2-cover of example (x).

(xii) The Patterson graph with intersection array  $\{280, 243, 144, 10; 1, 8, 90, 280\}$ . It is locally generalized quadrangle  $GQ(9, 3)$ , with parameters  $(280, 36, 8, 4)$  and non-trivial eigenvalues  $r = 8, s = -4$ .

For more information about the examples (i) and (ii), see [1, Chapter 9] and for examples (iii), (iv), (v), (vi), (vii), (xii), see [1, Chapter 13].

## 4 Tight graphs with small diameter

With the exception of Patterson graph all known tight graphs are antipodal, see [3]. For diameter larger than four there are only two examples known, the Johnson graph  $J(2d, d)$  and the halved cube  $\frac{1}{2}H(2d, 2)$ , both having diameter  $d$ .

In this section we focus on tight graphs of small diameter. The **Taylor graphs** are the distance-regular graphs with intersection array of the form  $\{k, c, 1; 1, c, k\}$ . We show that these are all the tight graphs with diameter three.

**Theorem 4.1** *Let  $\Gamma = (X, R)$  be a tight distance-regular graph with diameter three. Then  $\Gamma$  is a Taylor graph.*

In the following we will concentrate on antipodal graphs with diameter 4.

We say that a distance-regular graph  $\Gamma$  is an  $AT_4(p, q, r)$  ( $p, q, r$  real numbers) if it has intersection array

$$\{q(pq+p+q), (q^2-1)(p+1), \frac{(r-1)q(p+q)}{r}, 1; 1, \frac{q(p+q)}{r}, (q^2-1)(p+1), q(pq+p+q)\}.$$

**Theorem 4.2** *Let  $\Gamma = (X, R)$  be an antipodal distance-regular graph with diameter four. Then the following are equivalent.*

- (i)  $\Gamma$  is tight.
- (ii)  $\Gamma$  is an  $AT_4(p, q, r)$ , for some real numbers  $p, q$  and  $r$ .
- (iii) The antipodal quotient of  $\Gamma$  has the following parameters

$$(k, \lambda, \mu) = (q(pq + p + q), p(q + 1), q(p + q)).$$

for some real numbers  $p$ , and  $q$ .

- (iv) The graph  $\Gamma$  is locally strongly regular with parameters  $(k', \lambda', \mu') = (p(q + 1), 2p - q, p)$  for some real numbers  $p$ , and  $q$ .

If (i)-(iv) holds for some real numbers  $p, q, r$ , then  $p, q, r$  are integers with  $p \geq 1, q \geq 2, r \geq 2$ .

A graph with diameter at least two is called **Terwilliger graph** when every  $\mu$ -graph has the same number of vertices and is complete. We now give new feasibility conditions for the parameters of tight graphs with parameters  $(p, q, r)$  and group them with all previously known conditions in the following result.

**Theorem 4.3** Let  $\Gamma = (X, R)$  be an  $AT_4(p, q, r)$  for some real numbers  $p, q, r$ . Then

- (i)  $pq(p+q)/r$  is even.
- (ii)  $r(p+1) \leq q(p+q)$ , with equality if and only if  $\Gamma$  is Terwilliger graph.
- (iii)  $r \mid p+q$ .
- (iv)  $p \geq q-2$ .
- (v)  $p+q \mid q^2(q^2-1)$ .
- (vi)  $p+q^2 \mid q^2(q^2-1)(q^2+q-1)(q-2)$ .

In the next theorem we show when an  $AT_4(p, q, r)$  is a Terwilliger graph.

**Theorem 4.4** Let  $\Gamma = (X, R)$  be an  $AT_4(p, q, r)$  for some real numbers  $p, q, r$ . Then the following are equivalent.

- (i)  $\Gamma$  is a Terwilliger graph.
- (ii)  $p = 1$ .
- (iii)  $(p, q, r) = (1, 2, 3)$  and  $\Gamma$  is the Conway-Smith graph.
- (iv)  $p+q = r$ .

In the following we study the family  $AT_4(qs, q, q)$  where  $q$  and  $s$  are integers, with  $q, s \geq 2$ .

**Theorem 4.5** Let  $\Gamma = (X, R)$  be an  $AT_4(qs, q, q)$  for some real numbers  $q, s$ . Then one of the following holds.

- (i)  $(q, s) = (3, 1)$  and  $\Gamma$  is the Blokhuis-Brouwer graph.
- (ii)  $(q, s) = (2, 1)$  and  $\Gamma$  is the Johnson graph  $J(8, 4)$ .
- (iii)  $(q, s) = (2, 2)$  and  $\Gamma$  is the halved 8-cube.
- (iv)  $(q, s) = (3, 3)$ .
- (v)  $(q, s) = (4, 2)$ .

In case (iv) and (v) of the above theorem we are able to show that  $\Gamma$  is locally locally locally  $GQ(2, 2)$  and locally locally  $GQ(3, 3)$ , respectively. Note that the  $3.O_7(3)$ -graph and the Meixner2 graph are examples of case (iv) and (v) respectively. In the near future we hope to show that those two examples are unique.

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