

## Towards a classification of distance-transitive graphs

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### Abstract

We outline the programme of classifying all finite distance-transitive graphs. We mention the most important classification results obtained so far and give special attention to the so called affine graphs.

### 1. Introduction

The graphs in this paper will be always assumed to be finite, connected, undirected and without loops or multiple edges. The edge set of a graph can thus be identified with a subset of the set of unordered pairs of vertices.

Let  $\Gamma = (V\Gamma, E\Gamma)$  be a graph and  $x, y \in V\Gamma$ . With  $d(x, y)$  we will denote the usual distance in  $\Gamma$  between the vertices  $x$  and  $y$  (i.e., the length of the shortest path connecting  $x$  and  $y$ ) and with  $d$  we will denote the diameter of  $\Gamma$ , the maximum of all possible values of  $d(x, y)$ . Let  $\Gamma_i(x) = \{y \mid y \in V\Gamma, d(x, y) = i\}$  be the set of all vertices at distance  $i$  of  $x$ . An *automorphism* of a graph is a permutation of the vertex set that maps edges to edges.

Let  $G$  be a group acting on a graph  $\Gamma$  (i.e. we are given a morphism  $G \rightarrow \text{Aut}(\Gamma)$ ). For a vertex  $x \in V\Gamma$  and  $g \in G$  the image of  $x$  under  $g$  will be denoted by  $x^g$ . The set  $\{g \in G \mid x^g = x\}$  is a subgroup of  $G$ , called that stabilizer in  $G$  of  $x$  and will be denoted by  $G_x$ . We say that  $G$  acts *distance-transitively* on  $\Gamma$  if, for each  $i \in \{1, \dots, d\}$  its induced action on each of the set

$$\{(x, y) \mid x, y \in V\Gamma, d(x, y) = i\}$$

is transitive.

It is an easy to show that the definition is equivalent to the following two conditions:

- (i)  $G$  is transitive on  $V\Gamma$ ;
- (ii) for any vertex  $x$ , the stabilizer  $G_x$  acts transitively on the sets  $\Gamma_i(x)$ , for all  $i \in \{1, \dots, d\}$ .

If  $\Gamma$  admits a distance-transitive group of automorphisms we say that  $\Gamma$  is a *distance-transitive graph*.

### Examples

- (i) A graph  $\Gamma$  of diameter 1 is a clique, for  $G$  to be a distance-transitive group of automorphism is equivalent to  $G$  being a 2-transitive group acting on  $V\Gamma$ .
- (ii) Let  $\Gamma$  be a polygon, i.e. a regular graph of valency 2, on  $n$  vertices. The dihedral group of order  $2n$  acts as distance-transitive group of automorphism  $\Gamma$ .

(iii) Let  $X = \{1, \dots, n\}$  be a set and  $d \in \mathbb{N}$ . The *Hamming graph*  $H(n, d)$  is the graph  $\Gamma = (V\Gamma, E\Gamma)$  where  $V\Gamma$  is the set of the  $d$ -tuples in  $X$ . Two  $d$ -tuples  $(x_1, \dots, x_d)$  and  $(y_1, \dots, y_d)$  are adjacent if and only if they differ at exactly one place. The graph is of diameter  $d$  and the wreath product of  $Sym_n$  (and in fact any 2-transitive group on  $X$ ) with  $Sym_d$  acts as a distance-transitive group of automorphisms on  $\Gamma$ .

From this last example it follows that a distance-transitive graph can admit many groups acting distance-transitively on it.

The goal of this paper is to outline the classification project of distance-transitive graphs. This paper does not claim to give a detailed overview, for this we refer the reader to the excellent survey article by A.A. Ivanov [23] and the book by A.E. Brouwer, A.M. Cohen & A. Neumaier [14], but just to give a brief overview of the current situation and the strategy followed.

The organization of this paper is as follows (see the next section for explanation of terminology). In section 2 we give results that show that the classification of distance-transitive graphs can be reduced to the classification of primitive distance-transitive graphs, which falls into two cases depending on the structure of the group  $G$ . In Section 3 we will discuss the case where  $G$  is almost simple. Section 4 discusses the case where  $G$  is affine. This case itself falls into two subcases which will be discussed in some more detail in the two corresponding subsections.

## 2. Preliminaries

Distance-transitive graphs have many combinatorial properties. Before stating some of them we introduce some more notation. Let  $i \in \{0, \dots, d\}$  and  $x, y \in V\Gamma$ , with  $d(x, y) = i$ . Define,  $k_i = |\Gamma_i(x)|$ ,  $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$ ,  $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$  and  $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$ .

From the definitions one deduces easily that these numbers do not depend on the choice of  $x$  and  $y$  but only on the number  $i$ . Observe that, for each  $0 \leq i \leq d$ , we have  $a_i + b_i + c_i = k_1$  which is called the *valency* of  $\Gamma$ ,  $b_d = c_0 = 0$ ,  $b_0 = k_1$  and  $c_1 = 1$ . The set of invariants  $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$  is called the *intersection array* of  $\Gamma$ . If for a graph there exists an intersection array, that is the numbers  $a_i$ ,  $b_i$  and  $c_i$  do not depend on the particular choices of  $x$  and  $y$ , then the graph is called *distance-regular*. Thus every distance-transitive graph is distance-regular, but the converse is not true. We refer the reader to Bannai & Ito [2] and Brouwer, Cohen & Neumaier [14] for facts concerning distance-regular graphs.

For our purpose we mention some properties which we will need later on. Proofs can be found for example in [14].

**Lemma 2.1** *Let  $\Gamma$  be a distance-regular graph of diameter  $d \geq 3$ , with intersection array  $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$ .*

(i) *There are numbers  $i, j \in \mathbb{N}$  with  $1 \leq i \leq j \leq d$  such that  $1 < k_1 < \dots < k_i = \dots = k_j > \dots > k_d$ .*

(ii) *If  $i \leq j$  and  $i + j \leq d$ , then  $k_i \leq k_j$ .*

(iii) *If  $k_i = k_{i+1}$ , then  $k_i \geq k_j$  for all  $j$ .*

(iv) *For  $1 \leq i \leq d$ ,  $c_{i+1}k_{i+1} = b_i k_i$*

Let  $\Gamma$  be a distance-transitive graph of diameter  $d$  with distance-transitive group of automorphisms  $G$  and  $x \in V\Gamma$ . The set  $\{k_1, \dots, k_d\}$  is the set of orbit lengths of  $G_x$  on  $V\Gamma - \{x\}$ . Observe that the previous lemma implies that the number  $k_1$  is the smallest or second smallest orbit length that occurs.

Imprimitivity of the action of  $G$  on  $V\Gamma$  can be completely described in terms of the graph. For  $j \in \{1, \dots, d\}$  let  $\Gamma_j$  denote the graph with vertex set  $V\Gamma$  whose edges are the unordered pairs of vertices at distance  $j$ . Thus  $\Gamma$  is *bipartite* if and only if  $\Gamma_2$  is disconnected. In this case the connected components of  $\Gamma_2$  are called the halved graphs of  $\Gamma$ . We say that  $\Gamma$  is *antipodal* if  $\Gamma_d$  is disconnected and defines an equivalence relation. Clearly,  $G$  is imprimitive on  $V\Gamma$  if  $\Gamma$  is bipartite or antipodal. In fact, for a distance-transitive graph, the converse is also true, see Smith [38].

**Theorem 2.2** *An imprimitive distance-transitive graph is bipartite or antipodal or both.*

### Example

The Hamming graph  $H(2, d)$ , also called  $d$ -cube or just cube, is antipodal and bipartite. For  $d \geq 3$  the corresponding halved graph is not bipartite and only for  $d = 3$  or even values for  $d$  it is still antipodal.

A distance-transitive graph which is neither bipartite nor antipodal is called *primitive*. If a distance-transitive graph  $\Gamma$  is bipartite or antipodal, then there is a natural process to obtain an new distance-transitive graph from  $\Gamma$  which is primitive.

The programme of classifying all finite distance graphs turns thus into a two stage process. First stage is to classify all primitive distance-transitive graphs. The second stage is to determine for each primitive distance-transitive graph the related imprimitive distance-transitive graphs.

From the first examples it will be clear that we may and will assume that the diameter of a distance-transitive graph is not equal to 1 and valency is not equal to 2.

We refer the reader to [14, 23] for more information about the second step and content ourself here by remarking that since then Ivanov, Liebler, Penttilla & Praeger [24] completed the classification of the distance-transitive antipodal covers of complete bipartite graphs.

So far we have mainly looked at the graph  $\Gamma$ , let us now therefore pay some attention to the group  $G$ . From the definition of distance-transitivity it readily follows that  $G$  acts *flag-transitively* on  $\Gamma$ , i.e. is transitive on ordered pairs of adjacent vertices. The following construction of flag-transitive graphs is standard. Let  $G$  be a group,  $H$  a subgroup of  $G$  define

$$\text{core}_G(H) = \bigcap_{g \in G} H^g.$$

Let  $r \in G \setminus H$  with  $HrH = Hr^{-1}H$  and  $\langle H, r \rangle = G$ . We can define a graph  $\Gamma(G, H, r)$  as follows. As vertex set we take the  $H$ -cosets in  $G$ . Two cosets  $Hg_1$  and  $Hg_2$  will be called adjacent if and only if  $g_2g_1^{-1} \in HrH$ . The conditions on  $r$  and  $H$  now imply that  $\Gamma(G, H, r)$  is an undirected connected graph without loops or multiple edges on which  $G$  acts flag-transitively by right multiplication and, if  $\text{core}_G(H) = \{1\}$ , we also have  $G \leq \text{Aut}\Gamma(G, H, r)$ .

Suppose  $\Gamma$  is a distance-transitive graph with distance-transitive group  $G$  of automorphisms. Let  $x, y \in V\Gamma$  with  $y$  adjacent to  $x$ . The hypothesis of distance-transitivity implies the existence

of an element  $g \in G$  with  $x^g = y$  and  $y^g = x$ . From this follows that  $G_x g G_x = G_x g^{-1} G_x$ . Clearly  $\langle G_x, g \rangle = G$  and  $\text{core}_G(G_x) = \{1\}$ . It is well known that the resulting graph  $\Gamma(G, G_x, g)$  is isomorphic to  $\Gamma$ .

The following result of Praeger, Saxl & Yokoyama [35] determines the structure of the automorphism group in the primitive case. The proof depends on the classification of finite simple groups, though see also [3]. Below, a group is called *almost simple* if there is a normal non abelian simple group  $S$  such that  $S \leq G \leq \text{Aut}(S)$ , it is called *affine* if there exists a normal elementary abelian  $p$ -group which is regular on  $V\Gamma$ .

**Theorem 2.3** *Let  $\Gamma$  be a primitive distance-transitive graph, of valency at least 3 and diameter at least 2, with a distance-transitive group  $G$  of automorphisms of  $\Gamma$ . Then we have one of*

- (i)  $\Gamma$  is a Hamming graph, or in case  $d = 2$  its complement, and  $G$  is a wreath product;
- (ii)  $G$  is almost simple;
- (iii)  $G$  is affine.

In the second case we can invoke the classification of finite simple groups and their (large) maximal subgroups, this will be discussed in section 3. In the third case  $H$  has to be the complement of the regular normal subgroup, but here the full structure of  $G$  is not determined. This problem will be discussed in section 4.

As is already clear from the second case there are several possibilities for  $G$  even if given its socle  $S$ . However under mild hypotheses, see [4], one can extend automorphisms of  $G$  to automorphisms of  $\Gamma$ , whence in case (ii) assume that  $G$  is the full automorphism group of  $S$ .

### 3. The almost simple groups

In this section we describe the state of the art in case  $G$  is almost simple. Let  $S$  denote the socle of  $G$ , thus  $S \triangleleft G \leq \text{Aut}(S)$ , with  $S$  a non abelian simple group.

#### Example

Set  $V\Gamma$  the set of all  $d$ -dimensional linear subspaces of a vector space  $V$  of dimension  $n$  over  $GF(q)$ , and  $E\Gamma = \{\{x, y\} \mid \dim(x \cap y) = d - 1\}$ . The Grassmann graph  $G(n, d, q) = (V\Gamma, E\Gamma)$  is of diameter  $d$  if  $d \leq 2n$  and its isomorphic to  $G(n, n - d, q)$ . The projective special linear group  $L_n(q)$  acts distance-transitively on  $G(n, d, q)$ .

As outlined in the the previous section a distance-transitive graph  $\Gamma$  with distance-transitive group of automorphisms  $G$  is isomorphic to  $\Gamma(G, H, r)$  for a suitable choice of  $H$  and  $r$ . Since  $\Gamma$  is supposed to be primitive,  $H$  is a maximal subgroup from which immediately follows that  $\langle H, r \rangle = G$ , and since  $G$  is almost simple we also have  $\text{core}_G(H) = \{1\}$ .

Suppose we are given  $G, H$  and  $r$  such that  $\Gamma(G, H, r)$  is distance-transitive, with  $H = G_x$ . The numbers  $k_i$  can be found as  $H$ -orbit sizes. The remark after Lemma 2.1 shows that, if we know all these orbit sizes, there are only 2 possibilities for  $HrH$ .

The situation of the classification of primitive distance-transitive graphs with an almost simple group of automorphism was already outlined in the paper van Bon & Cohen [7]. Since then is the classification of distance-transitive graphs with an sporadic group has been completed by A.A. Ivanov, S.A. Linton, K. Lux, J.Saxl & L.H. Soicher [25], using (heavy) computer calculations.

For sake of completeness we mention the other results obtained so far and outline briefly the strategy envisioned and which has already been successfully employed in case of the linear groups. For more details we refer the reader to [7].

The alternating groups have been dealt with by Saxl [36] (when  $n > 18$ ) and Liebeck, Praeger & Saxl [31] and independently by Ivanov [22], who also does the imprimitive case. The linear groups have been treated in van Bon & Cohen [7, 8] and also in Inglis [21] for  $n \geq 13$  and Faradžev & Ivanov [19] for  $n = 2$ . For all Chevalley groups the distance-transitive graphs with vertex stabilizer a maximal parabolic subgroup are determined in Brouwer, Cohen & Neumaier [14].

Remains the classification of the distance-transitive graphs with  $G$  an almost simple group whose socle is isomorphic to a Chevalley group, but not the linear group. Besides the combinatorial information there are two techniques that seem to be of great use. Recall that a character is called *multiplicity free* if, written as the sum of different irreducible characters, each irreducible character occurs with a multiplicity of at most one.

**Lemma 3.1** *Let  $\Gamma$  be a distance-transitive graph with distance-transitive group of automorphisms  $G$ . Let  $\pi$  denote the permutation character of  $G$  on  $V\Gamma$ , then  $\pi$  is multiplicity free.*

This limits the possible point stabilizers considerably. Since  $\pi = 1_H^G$ , where  $H$  is the stabilizer of a vertex, we have in particular that the index  $[G : H]$  is less or equal to the sum of all irreducible character degrees.

One of the problems encountered is that even given  $G$  and  $H$  it is not always easy to find the two smallest orbit sizes. The next proposition, proved in [4], partly solves that problem by using an ordering on the kernels of action of a vertex stabilizer. For a vertex  $x \in V\Gamma$ , denote by  $G_x^i$  the kernel of the action of  $G_x$  on  $\Gamma_i(x)$ .

**Proposition 3.2** *Let  $\Gamma$  be a distance-transitive graph of diameter  $d$  with distance-transitive group of automorphisms  $G$ . If, for some vertex  $x \in V\Gamma$  and  $i \geq 1$  we have  $G_x^i \neq 1$ , then  $G_x^i \subset G_x^{i-1} \subset \dots \subset G_x^1$  or  $G_x^i \subset G_x^{i+1} \subset \dots \subset G_x^d$ .*

Thus, if a (normal) subgroup of  $G_x$  fixes all vertices at distance  $i$  from  $x$ , then it fixes all vertices at distance at most  $i$  from  $x$  or all vertices at distance at least  $i$  from  $x$ . The lemma is useful in case the stabilizer of a vertex is a  $p$ -local subgroup, a case that often occurs when the group  $S$  is a Chevalley group. In the particular case of the centralizer of an involution there is a stronger version of this proposition, see [4].

The strategy most fruitful to solve the remaining open cases now seems to first to eliminate the groups with a non multiplicity free permutation characters. For the classical groups one can extend results of N. Inglis [21] on multiplicity free permutation characters of classical groups of dimension at least 13 to smaller dimensions. In case of a exceptional Chevalley groups one uses the determination of the large maximal subgroups in Liebeck & Saxl [34] (some strengthening is needed in some cases). In the cases the permutation character is multiplicity free or undecided one can often apply proposition 3.2 or use geometric interpretations of  $H$ . Both these parts of the classification of primitive distance-transitive graphs are in progress [12, 16]. It is hoped that both projects will finish within reasonable time.

If a distance-transitive graph has diameter 2, then its complement is distance-transitive too and the group  $G$  is a rank 3 group. To end this section we also mention that a classification of these groups exists, see Kantor & Liebler [27] and Liebeck & Saxl [33].

#### 4. The affine groups

Now we turn to the affine groups. The vertices of the graph  $\Gamma$  can be identified with the vectors of a finite dimensional vector space  $V$  over a finite field. The group of translations  $N$  is an elementary abelian  $p$ -group, where  $p$  is the characteristic of the finite field, and acts as a regular group of automorphisms of  $\Gamma$ . We can write  $G$  as the semi-direct product of  $N$  with the stabilizer of the 0 vector  $G_0$ . Thus  $G = N : G_0$  and  $G_0 \leq \Gamma L(V)$ . The sets  $\Gamma_i(0)$  are  $G_0$ -orbits and two vectors  $u, v \in V$  are adjacent if and only if  $u - v \in \Gamma_1(0)$ . One can now reformulate the condition for  $\Gamma$  to be distance-transitive as follows:

There exists an ordering on the  $G_0$ -orbits  $0, \mathcal{O}_1, \dots, \mathcal{O}_d$  on  $V$  such that for each  $1 \leq i \leq d$  a vector of  $\mathcal{O}_i$  can be written as the sum of  $i$ , but not less, vectors of  $\mathcal{O}_1$ .

Indeed consider the graph with  $\Gamma_1(0) = \mathcal{O}_1$ .

#### Examples

(i) Let  $V$  be the vector space of all  $m \times n$  matrices with entries over  $GF(q)$ , a finite field. The bilinear forms graph  $\Gamma = B(m, n, q)$  is the graph on  $V$  obtained by joining two matrices  $A$  and  $B$  by an edge if and only if  $\text{rank}(A - B) = 1$ . The central product of the general linear groups  $GL(m, q)$  with  $GL(n, q)$  acts naturally and transitively on each  $\Gamma_i(0)$ .

(ii) A completely different example can be obtained from  $\widehat{E}_6(q)$ , the universal Chevalley group of type  $E_6$  over  $GF(q)$ , and  $V$  is a 27-dimensional  $GF(q)H$ -module by taking  $\Gamma_1(0)$  the highest weight orbit of  $H$  on  $V$ . This graph has diameter 3.

In [1] M. Aschbacher determines the structure of a maximal subgroup of a classical group. This theorem together with the classification of solvable rank 3 groups obtained by D.A. Foulser and M.J. Kallaher in [20] was used by Liebeck [29] to classify the affine rank 3 groups and by van Bon [3] to determine the graphs or the structure of  $G_0$ . Combining these two results we obtain least one of the following cases occurs:

- $\Gamma$  is an explicitly given graph known from the literature;
- $G \leq \Gamma_1(q)$ ;
- $H = F^*(G_0)$  is a central extension of a non-abelian simple group whose representation on  $V$  is absolutely irreducible and can be realized over no proper subfield of  $GF(q)$  and  $d \geq 3$ .

Any graph occurring in the first case is either of diameter 2, so come from affine rank 3 groups, or is either a Hamming graph or Bilinear forms graph. Recently A.M. Cohen and A.A. Ivanov [15] proved that in the second case the graph is either of diameter two or isomorphic to the Hamming graph  $H(4, 3)$ ,  $q = 64$  and  $G_0 \cong Z_9 : Z_3$  or  $G_0 \cong Z_9 : Z_6$ . There remains the third case and again we can invoke the classification of finite simple groups. This time a list of absolutely irreducible modules is needed instead of maximal subgroups.

The first problem is to bound the order of  $V$  in terms of  $G_0$ . The following bound first appeared in a predecessor of [10] and was slightly improved by S. Shpectorov.

**Lemma 4.1** *Let  $\Gamma$  be an affine distance-transitive graph with vertex set  $V$  and vertex stabilizer  $G_0$ , then  $|V| \leq 5|G_0|$ .*

The next result is standard in the theory of affine distance-transitive graphs.

**Lemma 4.2** *The  $GF(p)$ -dimension  $m$  of  $V$  is less than or equal to the diameter  $d$  of  $\Gamma$ , which is equal to the number of  $G_0$ -orbits on the non-zero vectors of  $V$ .*

The project of classifying affine primitive distance-transitive graph now falls naturally into several subcases depending whether the group  $H/Z(H)$  is an alternating group, one of the 26 sporadic simple groups, a Chevalley group defined over a field of characteristic different from  $p$  and finally the generic case where  $H/Z(H)$  is a Chevalley group defined over a field of the same characteristic as  $p$ . The strategy here differs from the almost simple case significantly since Lemma 3.1 and Lemma 3.2 do not put any restrictions on  $G_0$  and  $V$ . We will describe the current situation in the next two subsections.

#### 4.1. The non-generic case

In this section we discuss the cases where  $H/Z(H)$  is an alternating or sporadic group or a Chevalley group defined over a field of different characteristic as  $p$ , the so called cross characteristic case.

Recently a complete classification of this case has been obtained. To be more precise, the alternating groups have been studied by Liebeck & Praeger [30], the sporadic groups by van Bon, Ivanov & Saxl [10] and finally the cross characteristic case by Cohen, Magaard & Shpectorov [17].

The strategy of these papers is to first use Lemma 4.1, though [30] uses a weaker version, to obtain a list of groups with possible modules. Information about the dimension of modules can be found in the papers by Landazuri & Seitz [28], Seitz & Zalesskii [37], the Atlas [18] and the Modular Atlas [26]. Though for some groups additional work had to be done.

The next step is the study the action of  $G_0$  on  $V$ . Like in the almost simple case, where one had some results on the choice of adjacency, we have a general theorem, due to van Bon [5], which deals with a large class of possibilities at once.

**Theorem 4.3** *Let  $G$  be a primitive affine distance-transitive automorphism group of  $\Gamma$ . Let  $V$  be the normal subgroup in  $G$  identified with the vertex set of  $\Gamma$ . Suppose that  $V$  carries a  $GF(q)$ -structure preserved by  $G_0$ . Assume further that the group of scalars  $GF(q)^*$  is contained in  $G_0$ , and that  $G_0$  preserves a non-degenerate quadratic form on  $V$  up to scalar multiplication and field automorphisms. Then either  $d \leq 2$  or  $\Gamma$  is one of a Hamming graph, a half cube, a folded cube or a folded half cube.*

Notice that the above theorem also includes the case of invariant unitary form since it can be considered as an orthogonal form over the prime field.

Since all distance-transitive graphs of diameter 2 are known and we may assume that the graph is neither a Hamming graph, a half cube, a folded cube or folded half cube, one can use Theorem 4.3 to dispose of the ones that leave a form invariant.

If the character tables are available then one can use Burnside's lemma to estimate the number of orbits from below by analyzing some of the conjugacy classes of  $G_0$  and compare it with Lemma 4.2. The small list of remaining groups and modules are then dealt with by using ad hoc arguments and in some cases with help of a computer.

To illustrate this approach we outline the proof of the cross characteristic case. First the authors use Lemma 4.1 and the knowledge of minimal dimensions of representations to obtain

a list of 39 groups together with various possibilities for the characteristic of  $V$ . They excluded groups like  $L_2(8) \cong G_2(3)'$  over fields of characteristic 3 and 2 respectively since they belong to the generic case. After taking in to account various isomorphisms between these 39 groups and also isomorphisms to alternating groups the list reduces further to 31 groups. Besides the groups  $S_4(7)$  and  $S_8(3)$  of the remaining 29 groups the character tables are known which with Lemma 4.1 lead to a total of 87 pairs of groups with an explicit module, of which 42 are excluded by Lemma 4.3. Of the remaining 45 cases, 5 correspond to known examples, 27 are disposed by using the character argument and 10 of them are small enough to be disposed of by a short calculation, sometimes with help of a computer. This leaves the authors to consider the 3 cases  $S_4(5)$  on  $GF(4)^{12}$ ,  $U_3(5).S_3$  on  $GF(2)^{20}$  and  $G_2(4)$  on  $GF(5)^{12}$  and the two above mentioned groups. They then use a mixture of ad hoc arguments and computer calculations to finish the proof.

## 4.2. The generic case

In this section we discuss the remaining case of a Chevalley group defined over the same characteristic as  $V$ . Let us write  $GF(q)$  for the field of definition of the group,  $p$  for the characteristic, and let  $GF(r)$  denote the field of definition of  $V$ . To determine the possibilities of  $V$  we follow a calculation along the lines of Liebeck [29]. The bound given by Lemma 4.1 is only slightly weaker and only in a few cases it is necessary to redo the calculation. For a twisted Chevalley group we either have  $r = q$  or  $GF(r)$  is an extension of  $GF(q)$  of the same order as the twisting automorphism. For an untwisted Chevalley group we have  $r = q$  or we have one of a small list of exceptions. Thus, generically, the module  $V$  is defined over a natural field and closely related to the corresponding representation of the algebraic group.

We follow a strategy which was already outlined in an unpublished manuscript of van Bon and has been further developed in van Bon & Cohen [9]. The idea is to use a special orbit which often occurs.

Assume now that  $\Gamma$  is an affine distance-transitive graph with distance-transitive group  $G$ . For a  $G_0$ -orbit  $\mathcal{O}$  of vectors in  $V$  consider the following two properties:

- (O1) if  $v \in \mathcal{O}$ , then  $\lambda v \in \mathcal{O}$  for all  $\lambda \in GF(q)^*$ .
- (O2) for each  $v, w \in \mathcal{O}$  with  $w \notin \langle v \rangle$  there exists a  $g \in G_{0,v}$  with  $w^g - w \in \mathcal{O}$ .

The following two theorems of [9] show that such an orbit must either occur close to  $\mathcal{O}$  or at maximal distance.

**Theorem 4.4** *Suppose that  $\Gamma$  is an affine distance-transitive graph with distance-transitive group  $G$ . Let  $\mathcal{O}$  be a  $G_0$ -orbit satisfying O1 and O2. If  $a_1 \neq 0$ , then one of the following holds for any  $v \in \mathcal{O}$ .*

- (i)  $d(0, v) = 1$ .
- (ii)  $d(0, v) = 2$  and there exists  $w \in \mathcal{O}$  with  $v - w \in \Gamma_1(0)$ .
- (iii)  $d(0, v) = d \leq 4$  and either  $d = 2$  or there exists a  $w \in \mathcal{O}$  with  $v - w \in \Gamma_2(0)$ .
- (iv)  $d(0, v) = d$ ,  $a_d = 0$ ,  $b_{d-1} = 1$ , and  $G_0 = G_{0,v}G_{0,w}$  for some  $w \in \Gamma_1(0)$ .

Since we may assume that the diameter of the graph is at least three, it follows from [6] that multiplicative group of the prime field acts on  $\Gamma$ . Thus if  $p \neq 2$  there will always be a triangle. In case the graph does not contain a triangle the situation becomes slightly worse.

**Theorem 4.5** *Suppose that  $\Gamma$  is an affine distance-transitive graph with distance-transitive group  $G$ . Let  $\mathcal{O}$  be a  $G_0$  orbit satisfying  $O1$  and  $O2$ . Let  $v \in \mathcal{O}$ . Suppose that  $a_1 = 0$  and set  $d(0, v) = i$ . Then one of the following holds:*

- $a_i \neq 0$ ,  $d(0, v) = 2$  and there exists a  $w \in \mathcal{O}$  with  $v - w \in \Gamma_1(0)$ .
- $a_i = 0$  and one of the following holds:
  - (i)  $d(0, v) = 1$ .
  - (ii)  $d(0, v) \leq 4$  and either  $i = 2$  or there exists a  $w \in \mathcal{O}$  with  $v - w \in \Gamma_2(0)$ .
  - (iii)  $d(0, v) = d$  and  $G_0 = G_{0,v}G_{0,w}$  for some  $w \in \Gamma_1(0)$ .

At this point we make some observations.

Suppose that the conditions  $O1$  and  $O2$  are satisfied. If  $\mathcal{O}$  is the smallest orbit then either it is the orbit adjacent to 0, or the diameter is at most 4, or there exist a factorization with a maximal, usually parabolic, subgroup containing the stabilizer of a vector in  $\mathcal{O}$ .

If  $\mathcal{O}$  is not the smallest orbit, but the orbits different from  $\mathcal{O}$  that are representable by the difference of two vectors of  $\mathcal{O}$  are all strictly larger, then we have the same possibilities as before or we have  $a_1 = 0$  and one of the following situations:

$\mathcal{O}$  is at distance at most 4 from 0 and the diameter is at most 6 or  $\mathcal{O}$  is at distance 2 and  $a_2 = 0$  and there are at most two orbits different from  $\mathcal{O}$  that can be represented as the difference of two vectors in  $\mathcal{O}$  (representing distance 3 and/or distance 4 from 0).

A complete list of maximal factorizations for the simple groups has been obtained by Liebeck, Praeger & Saxl [32]. The possibility of a factorization is very limited as the stabilizer of a vector in  $\mathcal{O}$  is usually contained in a maximal parabolic subgroup. Factorizations for the exceptional (twisted and untwisted) Chevalley groups involving a maximal parabolic do not exist.

The modules we need to study (c.f. Lemma 4.1) are almost always modules coming from fundamental highest weights. Let  $w$  be such a fundamental highest weight vector and let  $P$  be the parabolic subgroup stabilizing  $\langle w \rangle$  and  $r$  be the reflection corresponding to the weight. Let  $\mathcal{G}$  be the coset geometry on  $(H, P, r)$ . Under mild assumptions, see below, the orbit  $\mathcal{O}$  of highest weight vectors satisfies the conditions  $O1$  and  $O2$  and if we take the orbit  $\mathcal{O}$  as adjacency then we obtain an embedding of  $\mathcal{G}$  in  $V$ . We can use this geometry to find representatives of vectors that add up to vectors not in  $\mathcal{O}$ . To be more precise, points of the geometry that are in a different distance relation provide good candidates for vectors that add up to vectors in different orbits. In the cases that have to be studied the actions of  $G_0$  on the geometries  $\mathcal{G}$  all have relatively low permutation rank, so the number of orbits representable by the difference of two vectors in  $\mathcal{O}$  relatively small too.

Let us now return to the classification programme. The list pairs group and modules is relatively short. For example, of the 10 exceptional Chevalley groups the representations that can occur are either the smallest dimensional representation or the Lie algebras. Taking graph automorphisms in account this leads to a total of only 16 cases to consider.

The next result of [9] is that under the mild restrictions the orbit of the highest weight vector satisfies the conditions *O1* and *O2*. We follow the notation of Bourbaki [13]. Suppose  $G_0$  is an algebraic group over  $GF(q)$  with a split Tits system  $(B, N, W, R)$ . Put  $H = B \cap N$ ; this is a maximal torus of  $G_0$ . Suppose  $\lambda$  is a dominant weight of  $H$  with respect to the given system, and let  $P$  be the corresponding standard parabolic subgroup of  $G_0$ ; that is,  $P = BW_\lambda B$ , where  $W_\lambda$  is the stabilizer in  $W$  of  $\lambda$ . Then  $G_0/P$  has a projective embedding in the highest weight module  $V_\lambda$ , given by  $gP \mapsto GF(q)gv_\lambda$ .

**Lemma 4.6** *For  $G_0$ ,  $\lambda$  and further notation as above, let  $\mathcal{O}$  be the  $G_0$ -orbit of the highest weight vector  $v_\lambda$ . If there is a fundamental root  $\alpha$  such that  $\langle \lambda, \alpha \rangle = 1$ , then properties *O1* and *O2* are satisfied for  $\mathcal{O}$ .*

A similar result has been obtained for the twisted groups.

For the exceptional Chevalley groups all modules are fundamental highest weight modules so the conditions are satisfied. The following theorem is the main result of van Bon & Cohen [9].

**Theorem 4.7** *Suppose that  $\Gamma$  is an affine distance-transitive graph, of diameter  $d \geq 3$ , with affine distance-transitive group  $G$ . Assume that the generalized Frattini group  $H = F^*(G_0)$  is an exceptional (quasi simple) Chevalley group over  $GF(q)$  for some power of the prime  $p$  involved in  $r$ . If  $H$  cannot be realized over a proper subfield of  $GF(r)$ , then  $q = r$ ,  $H \cong \widehat{E}_6(q)$ , the universal Chevalley group of type  $E_6$  over  $GF(q)$ ,  $V$  is a 27-dimensional  $GF(q)H$ -module and  $\Gamma_1(0)$  is the highest weight orbit of  $H$  on  $V$ .*

The remaining case of the classical groups is also under study [11]. The modules that appear are related to the natural modules, their alternating powers and symmetric squares, Lie algebras and spin modules. Only a few modules escape the conditions *O1* and *O2*, but these modules are relatively well understood.

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