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On Projective Planes of Order 15 Admitting a Collineation of Order 7

CHIHIRO SUETAKE (末竹 千博)
Amagasaki-minami High School, Amagasaki, Hyogo 660-0804, Japan

1. Introduction

In his paper [3], Ho showed that a collineation group of a projective plane of order 15 is solvable whose order divides $2^6, 2^3 \cdot 3^3$, $2 \cdot 5$, $3 \cdot 5$, $2^3 \cdot 3 \cdot 7$ or $2^6 \cdot 7$.

In the paper we prove the following result:

THEOREM 1. *There is no projective plane of order* 15 *admitting a collineation group of order* 21.

This theorem and Ho's theorem (Theorem B in [3]) yield

COROLLARY. *Let* G *be a collineation group of a projective plane of order* 15. *If* $7||G|$, *then* $|G||2^6 \cdot 7$.

The terminology we use is standard and can be found in [1] and [4]. Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a projective plane and G a subset of $Aut\ \Pi$. Then set $\mathcal{P}(G) = \{X | X \in \mathcal{P}, X^\sigma = X$ for all $\sigma \in G\}$, $\mathcal{L}(G) = \{x | x \in \mathcal{L}, x^\sigma = x$ for all $\sigma \in G\}$ and $Fix(G) = (\mathcal{P}(G), \mathcal{L}(G))$. When G is a collineation group of Π, for a point orbit Ω and a line orbit Δ of G, set $(\Omega\Delta) = |\Omega \cap (l)|$, where l is a line in Δ. Here we remark that the number $(\Omega\Delta)$ depends only on Ω and Δ, not on l.

2. Projective Planes of Order 15

The following proposition will be frequently used in the paper.

PROPOSITION 1(see [1] or Proposition 4.1 of [3]). *Let* $\Pi = (\mathcal{P}, \mathcal{L})$ *be a pro-*

be a projective plane of order n and G a collineation group of Π. Let $\mathcal{L}_1, \cdots, \mathcal{L}_w$ be the line orbits of G. Then for any two point orbits $\mathcal{P}_1$ and $\mathcal{P}_2$ of G,

$$\sum_{1\leq j\leq w} |\mathcal{L}_j|(\mathcal{P}_1\mathcal{L}_j)(\mathcal{P}_2\mathcal{L}_j) = |\mathcal{P}_1||\mathcal{P}_2| + n|\mathcal{P}_1 \cap \mathcal{P}_2|.$$

The following condition will be assumed in the rest of the paper.

HYPOTHESIS 1. *Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a projective plane of order 15 and G a collineation group of order 21 of Π.*

PROPOSITION 2([2], [3]). (i) $G = \langle \varphi, \tau | \varphi^7 = \tau^3 = 1, \tau^{-1}\varphi\tau = \varphi^2 \rangle$.
(ii) *τ is either a generalized elation or planar.*
(iii) *If τ is a generalized elation, then $|\mathcal{P}(\tau)| = 1$ or 7.*
(iv) *If τ is planar, then $Fix(\tau)$ is a subplane of order 3.*
(v) *φ is either triangular or a homology.*
(vi) *If φ is triangular, then τ is either planar or $|\mathcal{P}(\tau)| = 1$.*

For a subset H of $\langle\varphi\rangle$, set $\widehat{H} = \sum_{\mu \in H} \mu (\in Z[\langle\varphi\rangle])$, $H^{-1} = \{\mu^{-1} | \mu \in H\}$.

3. The Case That φ is Triangular

In this section we assume that φ is triangular. Then one of the following 3 cases occurs by Proposition 2 (iv,vi):
Case 1. $\mathcal{P}(\varphi) \cap \mathcal{P}(\tau) = \phi, |\mathcal{P}(\tau)| = 1$.
Case 2. $\mathcal{P}(\varphi) \cap \mathcal{P}(\tau) = \phi, |\mathcal{P}(\tau)| = 13$.
Case 3. $\mathcal{P}(\tau) \supseteq \mathcal{P}(\varphi), |\mathcal{P}(\tau)| = 13$.

4. The Case That φ is a Homology

In this section we assume that φ is a homology. Let P_0 be the center of φ and l_0 the axis of φ. Since τ fixes the point P_0 and normalizes $\langle\varphi\rangle$, there exists a line l_1 through P_0 such that $|\mathcal{P}(\tau) \cap (l_1)| \geq 2$. Then from Proposition 2 (ii, iii) the following 2 cases occur:
Case 4. $(l_1) \supseteq \mathcal{P}(\tau), |\mathcal{P}(\tau)| = 7$.

Case 5. $|(l_1)\cap\mathcal{P}(\tau)|=4, |\mathcal{P}(\tau)|=13$.

5. Case 5

In this section we consider Case 5. Clearly φ is a homology and $Fix(\tau)$ is a plane of order 3. Let $\mathcal{P}(\varphi)=\{P_0,P_1,P_2,P_3,P_4,P_{13},P_{14},\cdots,P_{24}\}$ and $\mathcal{L}(\varphi)=\{l_0,l_1,l_2,l_3,l_4,l_{13},l_{14},\cdots,l_{24}\}$, where P_0 is a center of φ and l_0 is an axis of φ. Let $\mathcal{P}_0,\cdots,\mathcal{P}_{48}$ be 49 $\langle\varphi\rangle$-orbits on $\mathcal{P}$ and $\mathcal{L}_0,\cdots,\mathcal{L}_{48}$ be 49 $\langle\varphi\rangle$-orbits on $\mathcal{L}$. Then we may assume the following:

$$\mathcal{P}_i=\{P_i\},\mathcal{L}_i=\{l_i\}(0\le i\le 4, 13\le i\le 24),$$

$$|\mathcal{P}_i|=7,|\mathcal{L}_i|=7(5\le i\le 12, 25\le i\le 48),$$

$$(l_0)=\{P_1,P_2,P_3,P_4,P_{13},P_{14},\cdots,P_{24}\},\quad (P_0)=\{l_1,l_2,l_3,l_4,l_{13},l_{14},\cdots,l_{24}\},$$

$$(l_1)=\mathcal{P}_5\cup\mathcal{P}_6\cup\{P_0,P_1\},\quad (P_1)=\mathcal{L}_5\cup\mathcal{L}_6\cup\{l_0,l_1\},$$

$$(l_2)=\mathcal{P}_7\cup\mathcal{P}_{10}\cup\{P_0,P_2\},\quad (P_2)=\mathcal{L}_7\cup\mathcal{L}_{10}\cup\{l_0,l_2\},$$

$$(l_3)=\mathcal{P}_8\cup\mathcal{P}_{11}\cup\{P_0,P_3\},\quad (P_3)=\mathcal{L}_8\cup\mathcal{L}_{11}\cup\{l_0,l_3\},$$

$$(l_4)=\mathcal{P}_9\cup\mathcal{P}_{12}\cup\{P_0,P_4\},\quad (P_4)=\mathcal{L}_9\cup\mathcal{L}_{12}\cup\{l_0,l_4\},$$

$$(l_{13})=\mathcal{P}_{25}\cup\mathcal{P}_{37}\cup\{P_0,P_{13}\},\quad (P_{13})=\mathcal{L}_{25}\cup\mathcal{L}_{37}\cup\{l_0,l_{13}\},$$

$$(l_{14})=\mathcal{P}_{26}\cup\mathcal{P}_{38}\cup\{P_0,P_{14}\},\quad (P_{14})=\mathcal{L}_{26}\cup\mathcal{L}_{38}\cup\{l_0,l_{14}\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(l_{24})=\mathcal{P}_{36}\cup\mathcal{P}_{48}\cup\{P_0,P_{24}\},\quad (P_{24})=\mathcal{L}_{36}\cup\mathcal{L}_{48}\cup\{l_0,l_{24}\}.$$

Set $m_{ij}=(\mathcal{P}_j\mathcal{L}_i)$ for $i,j,0\le i,j\le 48$ and $M=(m_{ij})_{0\le i,j\le 48}$. Choose a line $l_i\in\mathcal{L}_i$ and a point $P_i\in\mathcal{P}_i$ for $i,5\le i\le 12,25\le i\le 48$. Set $D_{ij}=\{\mu|\mu\in\langle\varphi\rangle,$ $P_j{}^{\mu}\in(l_i)\}$. Clearly $|D_{ij}|=m_{ij}$ for $i,j,5\le i,j\le 12,25\le i,j\le 48$.

LEMMA 5.1. $\sum_{j\in\{5,6,\cdots 12\}\cup\{25,26,\cdots,48\}}\widehat{D_{ij}}\widehat{D_{i'j}}^{-1}=$

$$\begin{cases} 15 & \textit{if } i=i'\in\{5,6,\cdots,12\}\cup\{25,26,\cdots,48\},\\ 0 & \textit{if } \{i,i'\}\in\{\{5,6\},\{7,10\},\{8,11\},\{9,12\},\{25,37\},\{26,38\},\cdots,\{36,48\}\},\\ \widehat{\langle\varphi\rangle} & \textit{otherwise.}\end{cases}$$

Proof. We only show the case $i=i'$ (one can obtain the others by a similar argument). Set $A=\{5,6,\cdots,12\}\cup\{25,26,\cdots,48\}$, $l=l_i$ and $\Phi=$

$\sum_{j\in A}\sum_{P_j^\mu,P_j^\xi(\neq)\in(l)}\mu\xi^{-1}$. For example, let $i=5$. Then since $(l)\cap(l^{\alpha^{-1}})=\{P_1\}$ for all $\alpha\in\langle\varphi\rangle-\{1\}$, $\Phi=0$. Therefore $\sum_{j\in A}\widehat{D_{ij}}\widehat{D_{ij}}^{-1}=\sum_{j\in A}\sum_{P_j^\mu\in(l),\mu\in\langle\varphi\rangle}\mu\mu^{-1}$ $=|(\mathcal{P}_7\cup\mathcal{P}_{10}\cap(l)|+|(\mathcal{P}_8\cup\mathcal{P}_{11})\cap(l)|+|(\mathcal{P}_9\cup\mathcal{P}_{12})\cap(l)|+|(\mathcal{P}_{25}\cup\mathcal{P}_{37})\cap(l)|$ $+|(\mathcal{P}_{26}\cup\mathcal{P}_{38})\cap(l)|+\cdots+|(\mathcal{P}_{36}\cup\mathcal{P}_{48})\cap(l)|=15$.

LEMMA 5.2. (i) $\sum_{j\in\{5,6,\cdots,12\}\cup\{25,26,\cdots,48\}}m_{ij}=15$ *for* $i\in\{5,6,\cdots,12\}\cup\{25,26,\cdots,48\}$.
(ii) $\sum_{j\in\{5,6,\cdots,12\}\cup\{25,26,\cdots,48\}}m_{ij}m_{i'j}=$

$$\begin{cases}15 & \text{if } i=i'\in\{5,6,\cdots,12\}\cup\{25,26,\cdots,48\},\\ 0 & \text{if } \{i,i'\}\in\{\{5,6\},\{7,10\},\{8,11\},\{9,12\},\{25,37\},\{26,38\},\cdots,\{36,48\}\},\\ 7 & \text{otherwise.}\end{cases}$$

Proof. (i) follows immediately from the definition of m_{ij}'s. We get by considering the action of the trivial character of $\langle\varphi\rangle$ on the equations of Lemma 5.1.

We consider the action of τ. We may assume that $\mathcal{P}(\tau)=\{P_0,P_1,\cdots,P_{12}\}$ and $\mathcal{L}(\tau)=\{l_0,l_1,\cdots,l_{12}\}$. Moreover we may assume that τ induces the actions on both $\{P_i|0\le i\le 48\}$ and $\{l_i|0\le i\le 48\}$, and they are $(P_0)(P_1)\cdots(P_{12})$ $(P_{13},P_{14},P_{15})(P_{16},P_{17},P_{18})\cdots(P_{22},P_{23},P_{24})(P_{25},P_{26},P_{27})(P_{28},P_{29},P_{30})\cdots$ (P_{46},P_{47},P_{48}) and $(l_0)(l_1)\cdots(l_{12})(l_{13},l_{14},l_{\{15\}})(l_{16},l_{17},l_{18})\cdots(l_{22},l_{23},l_{24})(l_{25},l_{26},l_{27})$ $(l_{28},l_{29},l_{30})\cdots(l_{46},l_{47},l_{48})$ respectively.

Let $\Omega_0,\cdots,\Omega_{24}$ be G-orbits on $\mathcal{P}$ and $\Delta_0,\cdots,\Delta_{24}$ be G-orbits on $\mathcal{L}$. Then we may assume the following:

$$\Omega_0=\mathcal{P}_0,\quad \Delta_0=\mathcal{L}_0,$$
$$\Omega_1=\mathcal{P}_1,\quad \Delta_1=\mathcal{L}_1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Omega_{12}=\mathcal{P}_{12},\quad \Delta_{12}=\mathcal{L}_{12},$$
$$\Omega_{13}=\mathcal{P}_{13}\cup\mathcal{P}_{14}\cup\mathcal{P}_{15},\quad \Delta_{13}=\mathcal{L}_{13}\cup\mathcal{L}_{14}\cup\mathcal{L}_{15},$$
$$\Omega_{14}=\mathcal{P}_{16}\cup\mathcal{P}_{17}\cup\mathcal{P}_{18},\quad \Delta_{14}=\mathcal{L}_{16}\cup\mathcal{L}_{17}\cup\mathcal{L}_{18},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Omega_{24}=\mathcal{P}_{46}\cup\mathcal{P}_{47}\cup\mathcal{P}_{48},\quad \Delta_{24}=\mathcal{L}_{46}\cup\mathcal{L}_{47}\cup\mathcal{L}_{48}.$$

Set $l_{ij}=(\Omega_j\ \Delta_i)$ for $0\le i,j\le 24$ and $L=(l_{ij})_{0\le i,j\le 24}$. Without loss of generality we may assume that $(l_{5\ 7}\ l_{5\ 8}\ l_{5\ 9})=(111)$, $(l_{7\ 5}\ l_{8\ 5}\ l_{9\ 5})=(111)$ and $l_{17\ 5}\ l_{18\ 5}\ l_{19\ 5}\ l_{20\ 5})=(1111)$. Then we have the following.

LEMMA 5.3. *There are exactly the following two possibilities, L_1, L_2 for L up to equivalence and transposition.*

$$
L_1 = \begin{pmatrix}
0&1&1&1&1&0&0&0&0&0&0&0&0&3&3&3&3&0&0&0&0&0&0&0&0\\
1&1&0&0&0&7&7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&7&0&0&7&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&7&0&0&7&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&1&0&0&0&0&7&0&0&7&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&3&3&3&3\\
0&1&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&3&3&3&3&0&0&0&0\\
0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0&0&3&3&3&3&0&0\\
0&0&0&1&0&1&0&0&0&1&1&0&0&0&0&0&0&0&3&0&3&3&0&3&0\\
0&0&0&0&1&1&0&1&0&0&0&1&0&0&0&0&0&0&3&3&0&3&0&0&3\\
0&0&1&0&0&0&1&0&0&1&0&1&0&0&0&0&0&3&3&0&0&0&0&3&3\\
0&0&0&1&0&0&1&1&0&0&0&0&1&0&0&0&0&3&0&3&0&0&3&0&3\\
0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&3&0&0&3&0&3&3&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&7&0&0&0&7&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&7&0&0&0&7&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&7&0&0&0&7&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&7&0&0&0&7\\
0&0&0&0&0&1&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&1&2&2&2\\
0&0&0&0&0&1&0&0&1&1&1&0&0&0&1&0&0&2&1&2&1&1&1&1&2\\
0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&1&0&2&1&1&2&1&2&1&1\\
0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&1&2&2&1&1&1&1&2&1\\
0&0&0&0&0&0&1&1&1&1&0&0&0&1&0&0&0&1&2&2&2&1&1&1&1\\
0&0&0&0&0&0&1&1&0&0&0&1&1&0&1&0&0&1&1&1&2&2&1&2&1\\
0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&1&0&1&2&1&1&2&1&1&2\\
0&0&0&0&0&0&1&0&0&1&1&1&0&0&0&0&1&1&1&2&1&2&2&1&1
\end{pmatrix},
$$

$$
L_2 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 3 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 0 & 3 & 3 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 3 & 3 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 3 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 3 & 0 & 0 & 3 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 7 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 7 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 7 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 7 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1
\end{pmatrix}.
$$

It follows that each $L_i (i = 1, 2)$ of Lemma 5.3 can be extended 162 M's which satisfy the conditions of Lemma 5.2, using a computer. Therefore we get the following lemma.

LEMMA 5.4. *There are exactly* 324 *possibilities* $M_1, \cdots, M_{324}$ *for* M *up to equivalence and transposition. Actually,*

$$M_i = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q & Q & Q & Q & P & P & P & P & P & P & P & P \\
1 & 1 & 0 & 0 & 0 & 7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & P & P & P & P & P & P & P & P & P & P & P & P \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & 0 & 0 & P & P & P & P & P & P & P & P & P & P & P & P \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & 0 & P & P & P & P & P & P & P & P & P & P & P & P \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & P & P & P & P & P & P & P & P & P & P & P & P \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & P & P & P & P & P & P & P & P & Q & Q & Q & Q \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & P & P & P & P & Q & Q & Q & Q & P & P & P & P \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & P & P & P & P & P & P & Q & Q & Q & Q & P & P \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & P & P & P & P & P & Q & P & Q & Q & P & Q & P \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & P & P & P & P & P & Q & Q & P & Q & P & P & Q \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & P & P & P & P & Q & Q & P & P & P & P & Q & Q \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & P & P & P & P & Q & P & Q & P & P & Q & P & Q \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & P & P & P & P & Q & P & P & Q & P & Q & Q & P \\
S & R & R & R & R & R & R & R & R & R & R & R & R & A & E & E & E & D & E & E & E & D & E & E & E \\
S & R & R & R & R & R & R & R & R & R & R & R & R & E & A & E & E & E & D & E & E & E & D & E & E \\
S & R & R & R & R & R & R & R & R & R & R & R & R & E & E & A & E & E & E & D & E & E & E & D & E \\
S & R & R & R & R & R & R & R & R & R & R & R & R & E & E & E & A & E & E & E & D & E & E & E & D \\
R & R & R & R & R & S & R & R & R & R & S & S & S & A & E & E & E & & & & & & & & \\
R & R & R & R & R & S & R & R & S & S & S & R & R & E & A & E & E & & & & & & & & \\
R & R & R & R & R & S & R & S & R & S & R & S & R & E & E & A & E & & & & & & & & \\
R & R & R & R & R & S & R & S & S & R & R & R & S & E & E & E & A & & & & * & & & & \\
R & R & R & R & R & R & S & S & S & S & R & R & R & A & E & E & E & & & & & & & & \\
R & R & R & R & R & R & S & S & R & R & R & S & S & E & A & E & E & & & & & & & & \\
R & R & R & R & R & R & S & R & S & R & S & R & S & E & E & A & E & & & & & & & & \\
R & R & R & R & R & R & S & R & R & S & S & S & R & E & E & E & A & & & & & & & &
\end{pmatrix}$$

for $i, 1 \leq i \leq 162$ *and*

$$M_i = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q & Q & Q & Q & P & P & P & P & P & P & P & P \\
1 & 1 & 0 & 0 & 0 & 7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & P & P & P & P & P & P & P & P & P & P & P & P \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & 0 & 0 & P & P & P & P & P & P & P & P & P & P & P & P \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & 0 & P & P & P & P & P & P & P & P & P & P & P & P \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & P & P & P & P & P & P & P & P & P & P & P & P \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & P & P & P & P & P & P & P & P & Q & Q & Q & Q \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & P & P & P & P & Q & Q & Q & Q & P & P & P & P \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & P & P & P & P & P & P & Q & Q & Q & Q & P & P \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & P & P & P & P & P & Q & P & Q & Q & P & Q & P \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & P & P & P & P & Q & P & P & Q & P & Q & Q & P \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & P & P & P & P & Q & Q & P & P & P & P & Q & Q \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & P & P & P & P & Q & P & Q & P & P & Q & P & Q \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & P & P & P & P & P & Q & Q & P & Q & P & P & Q \\
S & R & R & R & R & R & R & R & R & R & R & R & R & A & E & E & E & D & E & E & E & D & E & E & E \\
S & R & R & R & R & R & R & R & R & R & R & R & R & E & A & E & E & E & D & E & E & E & D & E & E \\
S & R & R & R & R & R & R & R & R & R & R & R & R & E & E & A & E & E & E & D & E & E & E & D & E \\
S & R & R & R & R & R & R & R & R & R & R & R & R & E & E & E & A & E & E & E & D & E & E & E & D \\
R & R & R & R & R & S & R & R & R & S & S & S & R & A & E & E & E & & & & & & & & \\
R & R & R & R & R & S & R & R & S & R & S & R & S & E & A & E & E & & & & & & & & \\
R & R & R & R & R & S & R & S & R & R & R & S & S & E & E & A & E & & & & & & & & \\
R & R & R & R & R & S & R & S & S & S & R & R & R & E & E & E & A & & & & * & & & & \\
R & R & R & R & R & R & S & S & S & R & R & R & S & A & E & E & E & & & & & & & & \\
R & R & R & R & R & R & S & S & R & S & R & S & R & E & A & E & E & & & & & & & & \\
R & R & R & R & R & R & S & R & S & S & S & R & R & E & E & A & E & & & & & & & & \\
R & R & R & R & R & R & S & R & R & R & S & S & S & E & E & E & A & & & & & & & &
\end{pmatrix}$$

for $i, 163 \le i \le 324$, *where* $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P = (0\ 0\ 0), Q = (1\ 1\ 1), R = {}^tP$ *and* $S = {}^tQ$.

Next we examine whether there are projective planes of order 15 corresponding to each M_i or not. For example

$$M_1 = \begin{pmatrix} & * & & & & ** & & & \\ & & & & & & & & \\ & & C & C & C & C & B & Z & Z & Z \\ & & Y & C & Z & B & B & B & C & Y \\ & & Y & B & B & Y & B & Y & C & B \\ & & Y & Z & C & B & B & C & Z & C \\ & *** & B & Z & Z & Z & C & C & C & C \\ & & B & B & C & Y & Y & C & Z & B \\ & & B & Y & C & B & Y & B & B & Y \\ & & B & C & Z & C & Y & Z & C & B \end{pmatrix},$$

where $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

and $Z = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. To construct projective planes corresponding to M_1, we must determine D_{ij}'s satisfying the conditions of Lemma 5.1. We identify an element φ^i of $\langle\varphi\rangle$ with an element i of the additive group F^+ of the ring $F = \{0, 1, \cdots, 6\}$ (mod 7). Set $\Omega = \{5, 6, 25, 26, \cdots, 33\}$ and $\Delta = \{5, 6, \cdots, 12\}$ $\cup\{25, 26, \cdots, 48\}$. Then by Lemma 5.1, $\sum_{j\in\Delta} \widehat{D_{ij}}\widehat{D_{i'j}}^{-1} = \widehat{\langle\varphi\rangle}$ for distinct $i, i' \in \Omega$. We want to investigate whether D_{ij}'s ($i \in \Omega, j \in \Delta$) satisfying these conditions exist or not. Set $\Delta' = \{5, 6, \cdots, 12\}$, $\Delta'' = \{25, 26, \cdots, 36\}$ and $\Delta''' = \{37, 38, \cdots, 48\}$. Changing of $P_5, P_7, P_8, \cdots, P_{12}, P_{25}, P_{28}, P_{31}, P_{34}, P_{37}, P_{40}$,

P_{43}, P_{46} and generators of F^+, if necessary, we can write $(D_{ij})_{i\in\Omega, j\in\Delta'}=$

$$\begin{pmatrix}
 & 0 & 0 & 0 & & & \\
 & & & & 0 & 0 & 0 \\
0 & & & & 1 & d_2 & d_3 \\
0 & & & & 2 & 2d_2 & 2d_3 \\
0 & & & & 4 & 4d_2 & 4d_3 \\
0 & & e_1 & e_2 & e_3 & & \\
0 & & 2e_1 & 2e_2 & 2e_3 & & \\
0 & & 4e_1 & 4e_2 & 4e_3 & & \\
0 & f_1 & & f_2 & & f_3 & \\
0 & 2f_1 & & 2f_2 & & 2f_3 & \\
0 & 4f_1 & & 4f_2 & & 4f_3 &
\end{pmatrix}$$

, $(D_{ij})_{i\in\Omega, j\in\Delta''}=$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\
 & & d_4 & & & d_5 & & & d_6 & & & d_7 \\
2d_4 & & & 2d_5 & & & 2d_6 & & & 2d_7 & & \\
 & 4d_4 & & & 4d_5 & & & 4d_6 & & & 4d_7 & \\
e_4 & & e_5 & & & e_6 & e_7 & e_8 & & & e_9 & \\
2e_5 & 2e_4 & & 2e_6 & & & & 2e_7 & 2e_8 & & & 2e_9 \\
 & 4e_5 & 4e_4 & & 4e_6 & & 4e_8 & & 4e_7 & 4e_9 & & \\
f_4 & & f_5 & & f_6 & & & f_7 & & f_8 & & f_9 \\
2f_5 & 2f_4 & & & & 2f_6 & & & 2f_7 & 2f_9 & 2f_8 & \\
 & 4f_5 & 4f_4 & 4f_6 & & & 4f_7 & & & & 4f_9 & 4f_8
\end{pmatrix}$$

and $(D_{ij})_{i\in\Omega, j\in\Delta'''}=$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & \\
 & d_8 & & d_9 & d_{10} & & d_{11} & d_{12} & & d_{13} & d_{14} & \\
 & & 2d_8 & & 2d_9 & 2d_{10} & & 2d_{11} & 2d_{12} & & 2d_{13} & 2d_{14} \\
4d_8 & & & 4d_{10} & & 4d_9 & 4d_{12} & & 4d_{11} & 4d_{14} & & 4d_{13} \\
 & e_{10} & & & e_{11} & & & & e_{12} & e_{13} & & e_{14} \\
 & & 2e_{10} & & & 2e_{11} & 2e_{12} & & & 2e_{14} & 2e_{13} & \\
4e_{10} & & & 4e_{11} & & & & 4e_{12} & & & 4e_{14} & 4e_{13} \\
 & f_{10} & & f_{11} & & f_{12} & & & f_{13} & & f_{14} & \\
 & & 2f_{10} & 2f_{12} & 2f_{11} & & 2f_{13} & & & & & 2f_{14} \\
4f_{10} & & & & 4f_{12} & 4f_{11} & & 4f_{13} & & 4f_{14} & &
\end{pmatrix}.$$

Here we omit components D_{ij}'s if they are empty sets. For example, $\sum_{j\in\Delta}\widehat{D_{25\ j}D_{30\ j}}^{-1}=\widehat{\langle\varphi\rangle}$ means $\{0,1-4e_3,d_4-4e_4,d_6-4e_7,d_9-4e_{11}, d_{12}-4e_{12},d_{14}-4e_{14}\}=F$. But it follows that there is no $\{d_i\},\{e_i\}$ and $\{f_i\}$ satisfying these conditions, using programs of sortings. By a similar argument for each other M_i, we have the following. (It follows that it is sufficient to examine only 108 M_i's by considering the forms of M_i's.)

LEMMA 5.4. *There is no* $\{D_{ij}|i\in\{5,6,25,26,\cdots,33\},j\in\{0,1,\cdots,48\}\}$ *satisfying the conditions of Lemma 5.1.*

LEMMA 5.5. *There is no projective plane of order* 15 *satisfying Case 5.*

6. The Other Cases

For each case except Case 5, we can also define similar matrices M and L to those of case 5.

For Case 1, there are exactly 221 possibilities $L_1,\cdots,L_{221}$ for L up to equivalence, but we can not extend any L_i to M. For Case 2, Case 3 or Case 4, there is no L. Therefore by Lemma 5.5 we have Theorem 1.

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