数理解析研究所講究録
1064巻 1998年 6-14

On the space problem of Helmholtz

By Mitsuo Sugiura

In section 1, we shall give a brief history of Helmholtz's space problem. In section 2, we shall give four characterizations of the orthogonal group of a positive definite quadratic form. One of them is that of free mobility. Our characterization is deeply connected with the Iwasawa decomposition of $GL(n,\mathbb{R})$.

1. A brief history of Helmholtz's space problem.

A space problem is a problem which characterizes some classical spaces in a class of spaces.

Riemann introduced the concept of manifold and Riemannian metric and their (sectional)curvature in his Habilitationsvortrag [10]. Particularly he studied the spaces of constant curvature and remarked that insuch a space, a figur can be moved freely (without deformation).

Under the influence of Riemann's Vortrag, Helmholtz studied the foundation of geometry in [5]. His models were Euclidean and non Euclidean geometries. His starting point was the concept of congruence and his central assumption was the free mobility of the rigid bodies. He proved that the line element in his geometry is a quadratic form of the differentials of the coordinates. Helmholtz did not use the concept of groups explicitly.

Lie also studied the foundation of geometry in [7],[8] but he used his theory of continuous transformation groups. Lie raised the question which properties characterize the groups of isometries in Euclidean and non-Euclidean spaces. He called the problem the Riemann-Helmholtz problem. His answer to the problem was the infinitesimalfree mobility.

Weyl [14] gave a characterization of the Lie algebra $\mathfrak{g}$ of the orthogonal group of a non-degenerate symmetric bilinear form B on n-dimensional real (or complex) vector space V. Of course $\mathfrak{g}$ is the Lie algebra of the linear transformation X on V which keep invariant B infinitesimally (${}^{t}XB + BX = 0$). But Weyl gave the characterization of $\mathfrak{g}$ which does not use the bilinear form B. His result is as follows:

Theorem (Weyl [14]). Let K be the field $\mathbb{R}$ or $\mathbb{C}$. A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n,K)$ is the Lie algebra of the orthogonal group of a non-degenerate symmetric bilinear form if and only if $\mathfrak{g}$ satisfies the following 3 conditions (a),(b),(c):

(a) $\dim \mathfrak{g} = n(n-1)/2$, (b) $\mathrm{Tr}\, X = 0$ for all $X \in \mathfrak{g}$.

(c) Let $A_1, \cdots, A_n$ be n elements in $\mathfrak{g}$ satisfying the conditions

$$\text{the k-th column of } A_i = \text{the i-th row of } A_k,$$

then necessarily we have $A_1 = \cdots = A_n = 0$.

Weyl proved the Theorem by a rather lengthy calculation of matrices. E.Cartan [2] gave another proof using his classification theory of simple Lie algebras and their representations.

The mathematical meaning of the condition (c) is not so clear and Weyl's theorem seems to be isolated in mathematics.

Let R be an n-dimensional affine space over an ordered field K and A_n be the affine transformation group of R. Let K^+ be the set of all positive elements in K. If an r-dimensional affine subspace S_r contains (r-1)-dimensional affine subspace S_{r-1}, then S_r is a disjoint union of S_{r-1} and two halfsubspaces S_{r-1}' and S_{r-1}'' with respect to S_{r-1}. A decreasing sequence

(1) $$S:\ S_n' \supset S_{n-1}' \supset \cdots \supset S_1' \supset S_0\ ,\quad S_n = R.$$

of halfspaces of R is called a flag of R. Let g be an element of the group A_n. Then the sequence

$$gS:\ gS_n' \supset gS_{n-1}' \supset \cdots \supset gS_1' \supset gS_0$$

is another flag of R. So the group A_n acts on the set $\mathbb{F}$ of all flags of R. It is easily seen that the action of A_n on F is transitive.

Definition. A subgroup G of A_n is called to satisfy the axiom of free mobility if G acts on $\mathbb{F}$ simply transitively.

Iyanaga and Abe [6] characterized the group of isometries of an n-dimensiomal Euclidean space by the free mobility. Their results were as follows:

Theorem (Iyanaga and Abe [6]). Let the ground field K be the field $\mathbb{R}$ of all real numbers. Then a subgroup G of the affine transformation group A_n is the group of all isometries on R with respect to a distance defined by a positive definite quadraticform of the coordinates in R if and only if G satisfies the axiom of the free mobility.

Iyanaga and Abe also considered the case where the ground fieldis a Pythagorean field. In this case some additional axioms were needed. Later Baer [1] gave a proof of the Theorem for a Pythagorean field without the additional axioms.

The proof for the real case are simplified by Wilker [15]. The characterization of the special orthogonal group was given by Pickert [9].

2. The free mobility and the Iwasawa decomposition.

Let $\mathbb{R}^n$ be the real vector space of the n-dimensional column vectors $x = {}^t(x_1, \cdots, x_n)$ and (x,y) be the standard inner product defined by

(1) $$(x,y) = \sum_{i=1}^{n} x_i y_i \quad \text{for } x = {}^t(x_1,\cdots,x_n) \text{ and } y = {}^t(y_1,\cdots,y_n).$$

Let $B = (b_{ij})$ be a posotive definite real symmetric matrix of degree n. Then the bilinear form

(2) $$\langle x,y \rangle = (Bx,y)$$

is symmetric and positive definite. Conversely every positive definite symmetric bilinear form on $\mathbb{R}^n$ is given by the formula (2) for s certain positive definite symmetric matrix B. The correspondence between a matrix B and a bilinear form (2) is bijective. So we identify the space of all positive definite symmetric matrices with the space of all positive definite symmetric bilinear forms on $\mathbb{R}^n$.

<u>Theorem 1</u>. Let T be the subgroup of $G = GL(n,\mathbb{R})$ consisting of all lower triangular matrices with positive diagonal elements. T is called the Iwasawa subgroup of G. Then the only compact subgroup of T is the identity subgroup $\{I\}$.

Proof. Let K be a compact subgroup of T and

(3) $$t = \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ t_{ij} & & t_n \end{bmatrix}, \quad t_i > 0, \quad t_{ij} \in \mathbb{R}.$$

be an arbitrary element of T. For any integer m, the matrix t^m has the form

$$t^m = \begin{bmatrix} t_1{}^m & & 0 \\ & \ddots & \\ * & & t_n{}^m \end{bmatrix}.$$

The subgroup $A_i = \{t_i{}^m | m \in \mathbb{Z}\}$ of the multiplicative group of positive reals is bounded. Hence the real number t_i must be equall to 1, because A_i is not bounded if $t_i > 1$ or $1 > t_i > 0$. Therefore t has the form

$$t = \begin{bmatrix} 1 & & 0 \\ t_{ij} & \ddots & \\ & & 1 \end{bmatrix}.$$

For any positive integer m, the (i+1,i)-element of t^m is $mt_{i+1,i}$. Since K is compact, the set $B_i = \{mt_{i+1,i} \mid m \in Z\}$ ($1 \leq i \leq n-1$) is bounded hence $t_{i+1,i} = 0$ ($1 \leq i \leq n-1$). Repeating this process, we reach the conclusion that all $t_{i,j} = 0$ in (3) and t = I. Since t is an arbitrary element of K, we get $K = \{I\}$.

Theorem 2. Let $K_o = O(n) = \{g \in GL(n,\mathbb{R}) \mid {}^t gg = I\}$ be the real orthogonal group of degree n and T be the Iwasawa subgroup of $G = GL(n,\mathbb{R})$ defined in Theorem 1. Then we have the following two proposition 1) and 2):

1) $G = K_oT$ and $K_o \cap T = \{I\}$. (the standard Iwasawa decomposition)

2) $K_o = O(n)$ is a maximal compact subgroup of $G = GL(n,\mathbb{R})$.

Proof. 1) Let g be an arbitrary element of G and $g_1, \cdots, g_n$ be the column vectors of the matrix g. Since $g_1, \cdots, g_n$ are linearly independent, we can construct an orthonormal base $(k_1, \cdots, k_n)$ from g_i's by the Schmidt orthogonalization process. Let $k = (k_1, \cdots, k_n)$ be the matrix with the column vectors k_i's. Then k belongs to the orthogonal group $K_o = O(n)$ and can be written as

(4) $k = gs$ for a certain $s \in T$,

by the Schmidt's process. Here $s^{-1} = t$ belongs to T and the equality (4)is rewritten as

$$g = kt, \quad k \in K_o, \quad t \in T.$$

Hence we have

(5) $G = K_oT.$

Since K_o is compact, a compact subgroup $K_o \cap T$ of T must be equal to $\{I\}$ by Theorem 1. So we have

(6) $K_o \cap T = \{I\}.$

Thus 1) is proved.

2) K_o is bounded and closed in $G = GL(n, \mathbb{R})$, and is a compact subgroup of G. Let K be any compact subgroup of G satisfying

(7) $K_o \subset K \subset G.$

Then $K \cap T$ is compact subgroup of T. Hence we have

(8) $K \cap T = \{I\},$

by Theorem 1.

Let k be any element of K. By 1), k can be written uniquely as

(9) $k = k_ot, \quad k_o \in K_o, \quad t \in T.$

Then we have

$$k_o^{-1}k = t \in K \cap T = \{I\} \quad \text{and}$$
$$k = k_o \in K_o.$$

So we have proved that

(10) $K \subset K_o$.

(7) and (10) show that

(11) $K = K_o$.

Hence we have proved that K_o is a maximal compact subgroup of G.

Theorem 3. Let B be a positive definite real symmetric matrix and $O(B) = \{g \in GL(n, \mathbb{R}) \mid {}^t gBg = B\}$ be the orthogonal group of B. Then O(B) is conjugate to $O(n) = O(I) = K_o$. More precisely, there exists an element t of the Iwasawa subgroup T such that

(12) $O(B) = t^{-1} K_o t$.

Proof. Since B is a positive definite real symmetric matrix, there exists a positive definite real symmetric matrix H such that $H^2 = B$.

By the standard Iwasawa decomposition (Theorem 2, 1)), H can be written as

(13) $H = k_o t, \quad k_o \in K_o, \quad t \in T$.

Hence B can be written as

(14) $B = H^2 = {}^t HH = {}^t t\, {}^t k_o k_o t = {}^t tt$.

So we have the following equivalences:

$$g \in O(B) \Longleftrightarrow {}^t gBg = B \Longleftrightarrow {}^t g\, {}^t ttg = {}^t tt \Longleftrightarrow {}^t (tgt^{-1}) \cdot (tgt^{-1}) = I$$
$$\Longleftrightarrow tgt^{-1} \in K_o \Longleftrightarrow g \in t^{-1} K_o t.$$

So we have proved (12).

Definition. Let $\mathbb{R}^+$ be the set of all positive real numbers, V_k a k-dimensional vector subspace of $\mathbb{R}^n$ and V_{k-1} a (k-1)-dimensional vector subspace of V_k and v_k be an element of V_k not belonging to V_{k-1}. Then the set

$$V_k' = V_{k-1} + \mathbb{R}^+ v_k$$

is called a k-dimensional half-space of V_k with respect to V_{k-1}.

Another half-space of V_k w.r.t. V_{k-1} is given by

$$V_{k-1}'' = V_{k-1} + (-\mathbb{R}^+) v_k.$$

An increasing sequence V of incident half-spaces V_k' of $\mathbb{R}^n$

(15) $V: \quad V_1' \subset V_2' \subset \cdots \subset V_n'$

is called a flag of $\mathbb{R}^n$ where V_k' is a k-dimensional half-space w.r.t. V_{k-1}.

The set of all flags of $\mathbb{R}^n$ is called the flag manifold of $\mathbb{R}^n$ and is denoted by $\mathbb{F}$.

To a flag V Of (15) and all integers $k = 1.\cdots,n$, choose an element $v_k \in V_k'$ not belonging to V_{k-1}. Then the set $(v_1, \cdots, v_n)$ is a base of $\mathbb{R}^n$ which is called to be associated with the flag V. Conversely every flag V can be constructed from a base $(v_1,\cdots,v_n)$ by defining

$$V_k' = \sum_{i=1}^{k-1} \mathbb{R}\cdot v_i + \mathbb{R}^+ v_k.$$

This flag V is called to be associated with the base $(v_1,\cdots,v_n)$.

For any flag V of (15) and an arbitrary element g in $G = GL(n, \mathbb{R})$, the sequence of half-spces

(16) $\qquad gV: \; gV_1' \subset gV_2' \subset \cdots \subset gV_n'$

is another flag of $\mathbb{R}^n$. Hence the group G acts on the flag manifold $\mathbb{F}$.

Let $e_1 = {}^t(1,0,\cdots,0),\cdots, e_n = {}^t(0,\cdots,0,1)$ be the natural base of $\mathbb{R}^n$ and E be the flag associated with the natural base. E is called the natural flag of $\mathbb{R}^n$.

Theorem 4.

1) $G = GL(n, \mathbb{R})$ acts on the flag manifold $\mathbb{F}$ transitively.

2) The Iwasawa subgroup T of G is the stationary subgroup of G at the natural flag E.

3) Let K be a subgroup of G. Then K acts on $\mathbb{F}$. The following two conditions (a) and (b) for K are mutually equivalent.

(a) K acts on $\mathbb{F}$ simply transitively.

(b) $G = KT$ and $K \cap T = \{I\}$.

Proof. 1) Let V and W be two arbitrary elements of $\mathbb{F}$ and $(v_1,\cdots,v_n)$ and $(w_1,\cdots,w_n)$ be the two bases of $\mathbb{R}^n$ associated with V and W respectively. Then there exists a unique element g in G satisfying $gv_i = w_i$ $(1 \leq i \leq n)$. Then we have $gV = W$. Hence G acts on $\mathbb{F}$ transitively.

2) Each element t in T has the form (3). Hence

$$tE_k' = E_k' \quad (1 \leq k \leq n) \quad \text{and} \quad tE = E.$$

Conversely every element t in G satisfying $tE = E$ belongs to T.

3) By 2), if $G = KT$ then $GE = KTE = KE$.On the other hand, the condition that G(or K) acts on $\mathbb{F}$ transitively is equivalent to one that $GE = \mathbb{F}$ (or $KE = \mathbb{F}$). Hence the condition $G = KT$ implies that K acts on $\mathbb{F}$ transitively.

Conversely if K acts on $\mathbb{F}$ transitively then $G = KT$. Because in this case for any element g in G, the flag gE con be written as $gE = kE$ for a certain k in K. Hence we have

$$k^{-1}gE = E \text{ and hence } k^{-1}g = t \in T, \text{ and } g = kt,\; G = KT.$$

Thus we have proved that the condition $G = KT$ is equivalent to one that K acts on $\mathbb{F}$ transitively.

Assume that $K \cap T = \{I\}$ and $kE = k'E$ for two elements k and k' in K. Then we have

$$k'^{-1}kE = E,\ k'^{-1}k \in K \cap T = \{I\} \text{ and } k = k'.$$

Therefore the mapping $f: k \mapsto kE$ is injective. Conversely assume that the mapping f is injective and k = t be an arbitrary element of $K \cap T$. Then we have

$$kE = tE = E = IE.$$

Hence by the injectivity of f, we get k = I and $K \cap T = \{I\}$.

Thus we have proved that the condition $K \cap T = \{I\}$ is equivalent to one that the mapping $f: k \to kE$ is an injection.

So we have proved that the condition (a) is equivalent to the condition (b).

<u>Main Theorem.</u>

The following five conditions (1),(2),(3),(4) and (5) for a closed subgroup K of G = GL(n, $\mathbb{R}$) are mutually equivalent.

(1) A group K is equal to the orthogonal group O(B) of a certain positive definite real symmetric matrix B of degree n.

(2) $K = t^{-1}K_0t$ for an element t in T(the Iwasawa subgroup of G) and K_0 = O(I) = O(n).

(3) G= KT and $K \cap T = \{I\}$. (the Iwasawa decomposition)

(4) K is a maximal compact subgroup of G.

(5) K acts on the flag manifold F of $\mathbb{R}^n$ simply transitively. (the condition of free mobility)

Proof. (3)$\Longleftrightarrow$(5) Theorem 4, 3).

(1)$\Longrightarrow$(2) Theorem 3.

(2)$\Longrightarrow$(3) By Theorem 2, we have

(a) $G = K_0T$ and $K_0 \cap T = \{I\}$.

By the assumption (2), $K = t^{-1}K_0t$. Hence applying the inner automorphism $A: x \mapsto t^{-1}xt$ on the both sides of the two equalities in (a), we get (3).

(3)$\Longrightarrow$(4) By Theorem 2, the natural mapping $f_0: K_0 \to G/T$ defined by $f_0(k)$ = kT is a bijection. Since the canonical projection $p: G \to G/T$ defined by p(g) = gT is continuous, the mapping f_0 is also continuous. Since K_0 is a closed and bounded set in G, K_0 is compact. Hence $G/T = f_0(K_0)$ is compact. Since T is closed in G, the quotient space G/T is a Hausdorff space.

Let K be a closed subgroup of G satisfying the condition (3). Then K is a locally compact group. Since G has a countable basis of open sets, a subgroup K (with the relative topology) has also a countable basis. Since K satisfies the condition (3), K acts on G/T simply transitively. Hence a locally compact group K acts continuously on a compact Hausdorff space G/T.

So by the category theorem of Bair, the natural mapping $f:K\to G/T$ defined by $f(K) = kT$ is a homeomorphism (cf. Helgason [4] Ch.II, Th.3.2. p.121). Therefore K is a compact subgroup of G.

Let K_1 be a compact subgroup of G satisfying

(b) $$K \subset K_1 \subset G.$$

Then $K_1 \cap T$ is a compact subgroup of T and hence $K_1 \cap T = \{I\}$ by Theorem 1.

Since $G = KT$ by the assumption (3), every k_1 in K_1 can be written as

$$k_1 = kt \quad \text{for } k\in K \text{ and } t\in T.$$

Hence $k^{-1}k_1 = t \in K_1 \cap T = \{I\}$ and $k_1 = k \in K$. Thus we have proved that $K_1 \subset K$ and $K_1 = K$ and K is a maximal compact subgroup of G.

(4)$\Longrightarrow$(1) Let K be a maximal compact subgroup of G. Since K is a closed subgroup of Lie group $G = GL(n, \mathbb{R})$, K is a Lie group (Chevalley [3] p. 135).

There exists a unique right invariant normalized Haar integration on K:

$$I(f) = \int_K f(k)dk \quad \text{for all } f \text{ in } C_R(K),$$

where $C_R(K)$ is the space of all real valued continuous functions on K (cf. Chevalley [3] p.161-170).

I is a continuous linear form on $C_R(K)$. $I(f) \geqq 0$ if $f \geqq 0$ and moreover $I(f) > 0$ if $f \geqq 0$ and $f \neq 0$. I is normalized as $I(1) = 1$.

Since I is right invariant, we have

$$\int_K f(kk_o)dk = \int_K f(k)dk.$$

Put

$$\langle x,y\rangle = \int_K (kx,ky)dk \quad \text{for any } x,y\in \mathbb{R}^n.$$

Then $\langle x,y\rangle$ is a positive definite symmetric bilinear form on $\mathbb{R}^n$. Hence there exists a positive definite real symmetric matrix B satisfying

$$\langle x,y\rangle = (Bx, y) \quad \text{for all } x,y\in \mathbb{R}^n.$$

By the right invariance of the Haar integration, we have

$$\langle kx,ky\rangle = \langle x,y\rangle \quad \text{for all } k \text{ in } K.$$

This equality implies that K is a subgroup of O(B). Since O(B) is a compact subgroup of G and Kis a maximal compact subgroup of G by the assumption (4), we get K = O(B). q.e.d.

Th results in 2 were published in [11] in Japanese.

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