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# Rigidity and Nonrigidity of the Geometric structure on the boundary of Quaternionic (Complex) Hyperbolic space

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## Introduction

This paper is a sequel of our result in the symposium #1022-"Analysis of Discrete Groups II" in 1996. The isometry group of quaternionic hyperbolic space $\mathbb{H}_{\mathbb{F}}^{n+1}$ acts transitively on the boundary sphere as projective transformations. The action on the boundary gives rise to a geometry $(\mathrm{PSp}(n+1,1), S^{4n+3})$. A $(4n+3)$-manifold locally modelled on this geometry is said to be a *spherical pseudo-quaternionic* manifold. We studied *rigidity* of compact spherical pseudo-quaternionic manifolds and proved the following result which was announced in the above symposium that

**Theorem A** *Let $M$ be a compact spherical pseudo-quaternionic $(4n+3)$-manifold whose fundamental group $\pi_1(M)$ is isomorphic to a discrete uniform subgroup of $\mathrm{PSp}(m,1)$ for some $m$ where $2 \leqq m \leqq n$. Then $M$ is pseudo-quaternionically isomorphic to the locally homogeneous space $S^{4n+3} - S^{4m-1}/\rho(\pi)$.*

The restricted spherical pseudo-quaternionic structure on the sphere complement $S^{4n+3} - S^{4m-1}$ coincides canonically with the homogeneous spherical pseudo-quaternionic structure compatible with the automorphism group $\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1)$.
If $\rho : \pi_1(M) \longrightarrow \mathrm{PSp}(n+1,1)$ is the holonomy map, then it maps the fundametal group $\pi$ isomorphicaly onto the discrete uniform subgroup $\rho(\pi)$ in $\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1)$.
In the present paper we examine *non-rigidity* of a compact spherical pseudo-quaternionic $(4n+3)$-manifold.

**Theorem B** *There exists a compact non-locally homogeneous spherical pseudo-quaternionic $(4n+3)$-manifold $M_1$ $(n \geqq 1)$. Let $\mathrm{Sp}(n-m+1)$ be a symplectic group where $(1 \leqq m \leqq n)$.*

*For $t \in \mathrm{Sp}(n-m+1)$ such that $|t|$ is sufficiently close to 1, there exists a nontrivial family of distinct spherical pseudo-quaternionic structures $\{\rho_t, \mathrm{dev}_t\}$ on $M_1$.*

In §1, we prove Theorem B. In §2, we examine the properties of developing maps dev for the geometric structures obtained from the boundary of hyperbolic spaces. In §3, we prove Theorem A. Our proof of Theorem A requires not only using the results of §2 but also to know a Carnot-Carathéodory structure on spherical pseudo-quaternionic manifolds in connection with the Sasakian 3-structure. However the Carnot-Carathéodory structure has been developed in its own right. From the organization of our paper, it is not suitable to discuss it in the present context. So we shall find another time to examine the Carnot-Carathéodory structure on odd dimensional manifolds. (Compare [13].)

# 1 Nonrigidity of spherical pseudo-quaternionic structure

Let $\mathbb{H}_{\mathbb{R}}^{m+1}$ be a totally geodesic subspace of $\mathbb{H}_{\mathbb{F}}^{n+1}$ $(0 \leqq m \leqq n)$. The subgroup of $\mathrm{PSp}(n+1,1) = \mathrm{Iso}(\mathbb{H}_{\mathbb{F}}^{n+1})$ preserving $\mathbb{H}_{\mathbb{R}}^{m+1}$ is isomorphic to $(\mathrm{O}(m+1,1)\cdot\mathrm{Sp}(1))\cdot\mathrm{Sp}(n-m)$. Let $\pi$ be a discrete torsionfree cocompact subgroup of $(\mathrm{O}(m+1,1)\cdot\mathrm{Sp}(1))\cdot\mathrm{Sp}(n-m)$. Then it is isomorphic to $\mathrm{Iso}(\mathbb{H}_{\mathbb{R}}^{m+1}) = \mathrm{PO}(m+1,1)$. Since $S^m = \partial\mathbb{H}_{\mathbb{R}}^{m+1}$, $\pi$ leaves invariant the complement $S^{4n+3} - S^m$ so that we have a spherical pseudo-quaternionic manifold $S^{4n+3} - S^m/\pi$. Since $S^{4n+3} - S^m$ is homeomorphic to $\mathbb{H}_{\mathbb{R}}^{m+1} \times S^{4n-m+2}$, $S^{4n+3} - S^m/\pi$ is compact. Note that the compact symplectic group $\mathrm{Sp}(n-m)$ does not act transitively on $S^{4n-m+2}$. So the group $(\mathrm{O}(m+1,1)\cdot\mathrm{Sp}(1))\cdot\mathrm{Sp}(n-m)$ is not transitive on $S^{4n+3} - S^m$.

**Proposition 1** *There exists a compact non-locally homogeneous spherical pseudo-quaternionic manifold $S^{4n+3} - S^m/\pi$ for $0 \leqq m \leqq n$.*

Let $\mathbb{H}_{\mathbb{R}}^m \subset \mathbb{H}_{\mathbb{R}}^{m+1} \subset \mathbb{H}_{\mathbb{F}}^{n+1}$ be the canonical inclusion of totally geodesic real hyperbolic subspaces where $1 \leqq m \leqq n$. As above the subgroup of $\mathrm{PSp}(n+1,1)$ preserving $\mathbb{H}_{\mathbb{R}}^m$ is isomorphic to $(\mathrm{O}(m,1)\cdot\mathrm{Sp}(1))\cdot\mathrm{Sp}(n-m+1) = \mathrm{Iso}(\mathbb{H}_{\mathbb{F}}^{n+1}, \mathbb{H}_{\mathbb{R}}^m)$. We have also the embedding : $\mathbb{H}_{\mathbb{R}}^m \subset \mathbb{H}_{\mathbb{C}}^m \subset \mathbb{H}_{\mathbb{F}}^m$ by taking its span. The subgroup $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n-m+1)$ leaves $\mathbb{H}_{\mathbb{R}}^m$ fixed pointwisely. The subgroup $\mathrm{Sp}(n-m+1)$ leaves fixed its span $\mathbb{H}_{\mathbb{F}}^m$ $(\supset \mathbb{H}_{\mathbb{C}}^m)$ as well. However, letting $\mathbb{H}_{\mathbb{R}}^m$ as an axis, each element of $\mathrm{Sp}(n-m+1)$ rotates $\mathbb{H}_{\mathbb{R}}^{m+1}$ around $\mathbb{H}_{\mathbb{R}}^m$.

Let $(\mathrm{PO}(m,1), \mathbb{H}_{\mathbb{R}}^m) \subset (\mathrm{PO}(m+1,1), \mathbb{H}_{\mathbb{R}}^{m+1}) \subset (\mathrm{PSp}(n+1,1), \mathbb{H}_{\mathbb{F}}^{n+1})$ be the canonical inclusion as above. Suppose that a compact hyperbolic $(m+1)$-manifold $\mathbb{H}_{\mathbb{R}}^{m+1}/\pi$ contains a totally geodesic closed $m$-dimensional submanifold at least one, say $\mathbb{H}_{\mathbb{R}}^m/\pi'$. Then according to Thurston, Apanasov, we can *bend* $\mathbb{H}_{\mathbb{R}}^{m+1}/\pi$ along $\mathbb{H}_{\mathbb{R}}^m/\pi'$ inside $\mathbb{H}_{\mathbb{F}}^{n+1}/\pi$.
More directly suppose that $\mathbb{H}_{\mathbb{R}}^{m+1}/\pi$ is two-sided. Then $\pi = \pi_1 \underset{\pi'}{*} \pi_2$. Choose a 1-parameter

family $\{g_t\} \subset \mathrm{Sp}(n-m+1)$ and define a representation $\rho_t : \pi \rightarrow \mathrm{PSp}(n+1,1)$:

$$\begin{aligned}\rho_1 &= id,\\ \rho_t(\gamma) &= \gamma \quad (\gamma \in \pi_1),\\ \rho_t(\gamma) &= g_t \cdot \gamma \cdot g_t^{-1} \quad (\gamma \in \pi_2).\end{aligned}$$

Suppose that $1 \leqq m \leqq n$. By Proposition 1, a compact manifold $M_1 = S^{4n+3} - S^m/\pi$ admits a (non-locally homogeneous) spherical pseudo-quaternionic structure for which the developing pair $(\mathrm{dev}_1, \rho_1)$ is the inclusion. We have a nontrivial deformation $\rho_t : \pi \rightarrow \mathrm{PSp}(n+1,1)$ starting at $\rho_1 = id$. Then by the Thurston's nearby structure we obtain a spherical pseudo-quaternionic structure $(\mathrm{dev}_t, \rho_t)$ $(t \in \mathrm{Sp}(n-m+1))$ on $M_1$. For $t$ sufficiently close to 1, the holonomy representation $\rho_t : \pi \longrightarrow \mathrm{PSp}(n+1,1)$ is discrete faithful and so the developing image becomes $\mathrm{dev}(\tilde{M}_1) = S^{4n+3} - L(\rho_t(\pi))$ where $L(\rho_t(\pi))$ is the limit set of $\rho_t(\pi)$. The limit set $L(\rho_t(\pi))$ is not homeomorphic to the geometric sphere $S^m$. (See §2.)

**Theorem 2** *There are examples of compact non-locally homogeneous spherical pseudo-quaternionic manifolds, which are not mutually geometrically rigid.*

The result of this type has been obtained in [1]. I was taught by Apanasov about the bending of this type.

# 2 Rigidity of developing maps and correction

Recall that a geometric structure on a smooth $n$-manifold is a maximal collection of charts modeled on a simply connected $n$-dimensional homogeneous space $X$ of a Lie group $\mathcal{G}$ whose coordinate changes are restrictions of transformations from $\mathcal{G}$. We call such a structure a $(\mathcal{G}, X)$-structure and a manifold with this structure is called a $(\mathcal{G}, X)$-manifold. In the paper [9], we have used the following lemma to show the uniqueness of developing maps in compact conformally flat manifolds.

**Lemma** *Let $A$ be a $\Gamma$-invariant closed subset in $X$. Suppose that in the complement of $A$ in $X$ there exists a component $U$ which admits a $\Gamma$-invariant complete Riemannian metric. Then the developing map* $\mathrm{dev} : V \longrightarrow U$ *on each component $V$ of* $\mathrm{dev}^{-1}(U)$ *is a covering map.*

However we recognized that the statement of the above lemma is not valid in some geometric structure, which is shown by the example by Kapovich (Compare [5].) And under some additional condition on $X$, Choi and Lee [5] have shown that the lemma is true for any geometric structure. On the other hand, we have noticed that our results in [9] can be proved more directly without use of the above lemma. So the purpose of this section is to show that the geometric uniqueness of developing maps are true in compact conformally flat manifolds, compact spherical $CR$ manifolds, and spherical pseudo-quaternionic manifolds. That is, our previous results of [9] will be generalized into the geometry on the boundary of Rank 1 noncompact symmetric spaces.

Let $\mathbb{K}$ stand for the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ or the field of quaternions $\mathbb{F}$. Denote $|K| = 1, 2$, or 4 respectively. Let $\mathbb{K}^{n+2}$ denote the vector space, equipped with the Hermitian pairing over $\mathbb{K}$ ; $\mathcal{B}(z, w) = -\bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_{n+2} w_{n+2}$. Define the $(n+2)|K|$-dimensional cone $V_-$ to be the subspace $\{z \in \mathbb{K}^{n+2} \mid \mathrm{Re}(z) > 0,\ \mathcal{B}(z,z) < 0\}$. If $P : \mathbb{K}^{n+2} - \{0\} \longrightarrow \mathbb{KP}^{n+1}$ is the canonical projection onto the $\mathbb{K}$-projective space, then the image $P(V_-)$ is defined to be the $\mathbb{K}$-hyperbolic space $\mathbb{H}_{\mathbb{K}}^{n+1}$ of dimension $(n+1)|K|$. (cf. [3]).
Let $\mathrm{O}(n+1, 1; \mathbb{K})$ be the subgroup of $\mathrm{GL}(n+2, \mathbb{K})$ whose elements preserve the Hermitian form $\mathcal{B}$. Since $\mathrm{O}(n+1, 1; \mathbb{K})$ leaves $V_-$ invariant, it induces an action on $\mathbb{H}_{\mathbb{K}}^{n+1}$ whose kernel is the center $\mathcal{Z}(n+1, 1; \mathbb{K})$. It is isomorphic to $\{\pm 1\}$ if $\mathbb{K} = \mathbb{R}$ or $\mathbb{F}$ or the circle $S^1$ if $\mathbb{K} = \mathbb{C}$. Denote by $\mathrm{PO}(n+1, 1; \mathbb{K})$ the quotient group $\mathrm{O}(n+1, 1; \mathbb{K})/\mathcal{Z}(n+1, 1; \mathbb{K})$. We usually write $\mathrm{PO}(n+1, 1), \mathrm{PU}(n+1, 1)$ or $\mathrm{PSp}(n+1, 1)$, which are known as the full group of isometries of complete simply connected $\mathbb{K}$- hyperbolic space $\mathbb{H}_{\mathbb{K}}^{n+1}$ respectively.

The projective compactification of $\mathbb{H}_{\mathbb{K}}^{n+1}$ is obtained by taking the closure $\bar{\mathbb{H}}_{\mathbb{K}}^{n+1}$ of $\mathbb{H}_{\mathbb{K}}^{n+1}$ in $\mathbb{KP}^{n+1}$. If we put an $(n+2)|K| - 1$ dimensional subspace $V_0 = \{z \in \mathbb{K}^{n+2} \mid \mathcal{B}(z,z) = 0\}$, then $\bar{\mathbb{H}}_{\mathbb{K}}^{n+1} = \mathbb{H}_{\mathbb{K}}^{n+1} \cup P(V_0)$ so that the boundary $\partial \mathbb{H}_{\mathbb{K}}^{n+1} = P(V_0)$ is the standard sphere of dimension $n$, $2n+1$, $4n+3$ according to that $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{F}$. Put $\partial \mathbb{H}_{\mathbb{K}}^{n+1} = S^{(n+1)|K|-1}$. Then the group of isometries $\mathrm{PO}(n+1, 1; \mathbb{K})$ extends to a transitive action of projective transformations of $S^{(n+1)|K|-1}$. Thus we obtain the geometry $(\mathrm{PO}(n+1, 1; \mathbb{K}), S^{(n+1)|K|-1})$. In each case note that the geometry $(\mathrm{PO}(n+1, 1), S^n)$ is called *conformally flat geometry*, the geometry $(\mathrm{PU}(n+1, 1), S^{2n+1})$ is called *spherical CR geometry*, and we call $(\mathrm{PSp}(n+1, 1), S^{4n+3})$ a *spherical pseudo-quaternionic geometry.*
If $\mathbb{H}_{\mathbb{K}}^{m+1}$ $(1 \leqq m \leqq n-1)$ is the totally geodesic subspace of $\mathbb{H}_{\mathbb{K}}^{n+1}$, then the geometric subsphere $S^{(m+1)|K|-1}$ of $S^{(n+1)|K|-1}$ is defined to be $\partial \mathbb{H}_{\mathbb{K}}^{m+1}$. Put $\mathbb{Y} = S^{(n+1)|K|-1} - S^{(m+1)|K|-1}$ and denote by $\mathrm{Aut}(\mathbb{Y})$ the subgroup of $\mathrm{PO}(n+1, 1; \mathbb{K})$ whose elements preserve $S^{(m+1)|K|-1}$. Then $\mathrm{Aut}(\mathbb{Y})$ is isomorphic to the subgroup $\mathrm{P}(\mathrm{O}(m+1, 1; \mathbb{K}) \times \mathrm{O}(n-m; \mathbb{K}))$ (cf. [11],[3]). Moreover $\mathbb{Y}$ is a Riemannian homogeneous space

$$\mathrm{P}(\mathrm{O}(m+1, 1; \mathbb{K}) \times \mathrm{O}(n-m; \mathbb{K}))/\mathrm{P}(\mathrm{O}(m+1; \mathbb{K}) \times \mathrm{O}(1; \mathbb{K}) \times \mathrm{O}(n-m-1; \mathbb{K})).$$

Then the homogeneous Riemannian metric $h$ on $\mathbb{Y}$ induces a Riemannian submersion:

$$S^{(n-m)|K|-1} \rightarrow (\mathrm{Aut}(\mathbb{Y}), \mathbb{Y}, h) \xrightarrow{\nu} (\mathrm{PO}(m+1, 1; \mathbb{K}), \mathbb{H}_{\mathbb{K}}^{m+1}, h_0).$$

Here $h_0$ is the hyperbolic metric on $\mathbb{H}_{\mathbb{K}}^{m+1}$. (See [14], [13].)
Note that if $\mathrm{O}(n-m; \mathbb{K}) \rightarrow S^{(n+1)|K|-1} \xrightarrow{P} \mathbb{D}_{\mathbb{K}}^{m+1}$ is the projection onto the closed disk such that the fixed point set Fix $(\mathrm{O}(n-m; \mathbb{K}), S^{(n+1)|K|-1}) = S^{(m+1)|K|-1}$, then $P|\mathbb{Y} = \nu$ and $\nu$ maps the ideal boundary $S^{(m+1)|K|-1} = \partial(S^{(n+1)|K|-1} - S^{(m+1)|K|-1})$ identically onto $S^{(m+1)|K|-1} = \partial \mathbb{H}_{\mathbb{K}}^{m+1}$.

Recall that if a smooth connected manifold $M$ admits a $(\mathrm{PO}(n+1, 1; \mathbb{K}), S^{(n+1)|K|-1})$-structure, then there exists a developing pair $(\phi, \mathrm{dev})$, where $\mathrm{dev} : \tilde{M} \rightarrow S^{(n+1)|K|-1}$ is a *structure-preserving* immersion and $\phi : \pi_1(M) \rightarrow \mathrm{PO}(n+1, 1; \mathbb{K})$ is a homomorphism whose image $\phi(\pi_1(M))$ is called the holonomy group for $M$. We prove the following proposition.

**Proposition 3** *Let $M$ be a compact $(\mathrm{PO}(n+1,1;\mathbb{K}), S^{(n+1)|K|-1})$-manifold in dimension $(n+1)|K|-1$. Suppose that $\phi(\pi_1(M))$ leaves a geometric subsphere $S^{(m+1)|K|-1}$ $(0 \leqq m \leqq n-1)$. Then the restriction of the developing map*

$$\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}) \longrightarrow S^{(n+1)|K|} - S^{(m+1)|K|-1}$$

*is a covering map.*

***Proof.*** Put $\pi = \pi_1(M)$ and $\Gamma = \phi(\pi)$. Since the holonomy group $\Gamma$ leaves invariant a geometric subsphere $S^{(m+1)|K|-1}$, we have the restriction of the developing pair:

$$(\rho, \mathrm{dev}) : (\pi, \tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})) \longrightarrow (\mathrm{Aut}(\mathbb{Y}), \mathbb{Y}).$$

As above note that the Riemannian metric $h$ on $\mathbb{Y}$ induces a Riemannian submersion:

$$S^{(n-m)|K|-1} \rightarrow (\mathrm{Aut}(\mathbb{Y}), \mathbb{Y}, h) \xrightarrow{\nu} (\mathrm{PO}(m+1,1;\mathbb{K}), \mathbb{H}_{\mathbb{K}}^{m+1}, h_0).$$

Let $\mathrm{dev}^* h$ be the induced Riemannian metric on $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$, which is invariant under $\pi$.

We prove that $\mathrm{dev}^* h$ on $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ is complete. Let $\{x_i\}$ be a Cauchy sequence in $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ with respect to $\mathrm{dev}^* h$. Assume that $\mathrm{dev}^{-1}(S^{(m+1)|K|-1}) \neq \emptyset$. Let $\rho^*$ (resp. $\rho$) be the distance function on $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ (resp. $\mathbb{Y}$), and $\rho_0$ be the (hyperbolic) distance function on $\mathbb{H}_{\mathbb{K}}^{m+1}$. As $\mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ is invariant under $\pi$, $M$ decomposes into the union $(\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}))/\pi$ and $\mathrm{dev}^{-1}(S^{(m+1)|K|-1})/\pi$ where $\mathrm{dev}^{-1}(S^{(m+1)|K|-1})/\pi$ consists of a finite number of compact submanifolds. If $P : \tilde{M} \longrightarrow M$ is a covering map, then the sequence $\{P(x_i)\}$ has an accumulation point $y$ (after passing to a subsequence). Choose $\tilde{y} \in \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ with $P(\tilde{y}) = y$. There exists a neighborhood $W$ of $\tilde{y}$ in $\tilde{M}$ such that the closure $\bar{W}$ is compact. Moreover, $P : \bar{W} \longrightarrow P(\bar{W})$ and $\mathrm{dev} : \bar{W} \longrightarrow \mathrm{dev}(\bar{W})$ are diffeomorphic. As $y \in P(W)$, there exist elements $\{\gamma_i\} \in \pi$ such that $\{\gamma_i \cdot x_i\} \in W$ for $i \geqq L$ where $L$ is a sufficiently large number. We have $\lim \gamma_i \cdot x_i = \tilde{y}$. Since $\{x_i\}$ is Cauchy in $(\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}), \rho^*)$, associated with each integer $n$, there exists an integer $\lambda(n)$ satisfying that if $i, j \geqq \lambda(n)$, $\rho^*(x_i, x_j) < \dfrac{1}{n}$. Let $B_{\frac{1}{n}}(x_{\lambda(n)})$ be the ball of radius $\dfrac{1}{n}$ centered at $x_{\lambda(n)}$ in $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$. In particular,

$$\{x_i\} \in B_{\frac{1}{n}}(x_{\lambda(n)}) \quad \text{for } i \geqq \lambda(n).$$

As $\lambda(n)$ increases as $n$ does, we can assume that $\lambda(n) \geqq n$ for $n \geqq N$ where $N$ is a sufficiently large number with $N > L$. Note that $\{\gamma_{\lambda(n)} \cdot x_{\lambda(n)}\} \in W$ for $n \geqq N$ as above.
Then we show that there is an integer $m$ such that $B_{\frac{1}{m}}(\gamma_{\lambda(m)} \cdot x_{\lambda(m)}) \subset W$. Suppose not. Put $\partial' W = \partial \bar{W} \cap (\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}))$. Then for each $n \geqq N$, there is a point of $B_{\frac{1}{n}}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)})$ outside $W$. Thus we have that

$$(*) \qquad \rho^*(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}, \partial' W) \leqq \frac{1}{n}.$$

In general, for every $z \in \partial' W \subset \tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$,

$$\rho_0(\nu \circ \mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}), \nu \circ \mathrm{dev}(z)) \leqq \rho(\mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}), \mathrm{dev}(z)) \leqq \rho^*(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}, z).$$

Taking the infimum for all $z \in \partial' W$ and using $(*)$ imply that

$$(**) \qquad \rho_0(\nu \circ \mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}), \nu \circ \mathrm{dev}(z)) \leqq \frac{1}{n}.$$

On the other hand, as $\partial' W \subset \tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$, $\nu \circ \mathrm{dev}(z) \in \mathbb{H}_{\mathbb{K}}^{m+1}$.
Since $\nu \circ \mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}) \longrightarrow \nu \circ \mathrm{dev}(\tilde{y}) \in \nu(S^{(m+1)|K|-1}) = S^{(m+1)|K|-1} = \partial \mathbb{H}_{\mathbb{K}}^{m+1}$, it follows that

$$\lim_{n \to \infty} \rho_0(\nu \circ \mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}), \nu \circ \mathrm{dev}(z)) = \infty,$$

which is impossible by $(**)$. Hence we obtain that $B_{\frac{1}{m}}(\gamma_{\lambda(m)} \cdot x_{\lambda(m)}) \subset W$ for some $m$. If we recall that $\{x_i\}_{i \geqq \lambda(m)} \in B_{\frac{1}{m}}(x_{\lambda(m)})$ and $\gamma_{\lambda(m)}$ is an isometry with respect to $\rho^*$, then $\{\gamma_{\lambda(m)} \cdot x_i\}_{i \geqq \lambda(m)} \in B_{\frac{1}{m}}(\gamma_{\lambda(m)} \cdot x_{\lambda(m)})$. As $\bar{W}$ is compact, there is a point $w \in \bar{W}$ such that $\lim_{i \to \infty} \gamma_{\lambda(m)} \cdot x_i = w$. Therefore $\lim_{i \to \infty} x_i = \gamma_{\lambda(m)}^{-1} \cdot w$ for which $\mathrm{dev}(\gamma_{\lambda(m)}^{-1} \cdot w) = \lim_{i \to \infty} \mathrm{dev}(x_i)$. Since the sequence of images $\{\mathrm{dev}(x_i)\}$ is also Cauchy in $\mathbb{Y}$, $\{\mathrm{dev}(x_i)\}$ has a limit point in $\mathbb{Y}$, which therefore implies that $\mathrm{dev}(\gamma_{\lambda(m)}^{-1} \cdot w) \in \mathbb{Y}$. Thus $\mathrm{dev}(\gamma_{\lambda(m)}^{-1} \cdot w)$ is not contained in $S^{(m+1)|K|-1}$, *i.e.*, $\gamma_{\lambda(m)}^{-1} \cdot w \in \tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$. This shows that the Cauchy sequence $\{x_i\}$ converges in $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ so that $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ is complete. As a consequence, the local isometry $\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}) \longrightarrow \mathbb{Y}$ is a covering map.

□

**Remark 4** (1) *For the induced Riemannian metric from an arbitrarily geometric structure, the above proof does not work with respect to the argument of* minimal geodesic; *the covering map $P : \tilde{M} \longrightarrow M$ induces a local isometry of $(\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}), \rho^*)$ onto $((\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})/\pi, \hat{\rho}^*)$. Given a Cauchy sequence $\{y_j\}$ lying in $P(W)$, choose a lift of sequence $\{\tilde{y}_j\}$ from $W$. Since $P : W \longrightarrow P(W)$ is diffeomorphic, $P : W - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}) \longrightarrow P(W) - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})/\pi$ is an isometry, however, note that given two points $y_i$, $y_j$ in $P(W)$, the minimal geodesic between $y_i$ and $y_j$ does not necessarily lie in $P(W) - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})/\pi$. So the equality $\hat{\rho}^*(y_i, y_j) = \rho^*(\tilde{y}_i, \tilde{y}_j)$ does not hold in general, which implies that the lift $\{\tilde{y}_j\}$ is not necessarily Cauchy. We did not check this point for an arbitrarily geometric structure, which is the mistake of the argument of the proof in Lemma B of [9] (also Lemma 4 of [10] ).*

As a consequence, Propositions 1.1.1 and 1.1.2 of [9] are valid for the conformally flat, spherical $CR$, and spherical pseudo-quaternionic structures respectively. More precisely, we obtain the following developing results in each case.

**Corollary 5** ($\mathbb{K} = \mathbb{R}$)*: Let $M$ be a closed conformally flat $n$-manifold.*

1. *If $0 \leqq m \leqq n - 3$, then $\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(S^m) \longrightarrow S^n - S^m$ is diffeomorphic.*

2. *If* $m = n-2$, *then* $\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(S^{n-2}) \longrightarrow S^n - S^{n-2}$ *is a covering map with fiber isomorphic to an infinite cyclic group.*
3. *If* $m = n-1$, *then* $\mathrm{dev}$ *maps each component of* $\tilde{M} - \mathrm{dev}^{-1}(S^{n-1})$ *diffeomorphically onto the real hyperbolic space* $\mathbb{H}^n_{\mathbb{R}}$.

($\mathbb{K} = \mathbb{C}$): *Let* $M$ *be a closed spherical CR-manifold of dimension* $2n+1$.

1. *If* $0 \leqq m \leqq n-2$, *then* $\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(S^{2m+1}) \longrightarrow S^{2n+1} - S^{2m+1}$ *is diffeomorphic.*
2. *If* $m = n-1$, *then* $\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(S^{2n-1}) \longrightarrow S^{2n+1} - S^{2n-1}$ *is a covering map with fiber isomorphic to an infinite cyclic group.*

($\mathbb{K} = \mathbb{F}$): *Let* $M$ *be a closed spherical pseudo-quaternionic manifold of dimension* $4n+3$. *Then,* $\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(S^{4m+3}) \longrightarrow S^{4n+3} - S^{4m+3}$ *is diffeomorphic for* $0 \leqq m \leqq n-1$.

Let $(\phi, \mathrm{dev}) : (\pi_1(M), \tilde{M}) \to (\mathrm{PO}(n+1,1;\mathbb{K}), S^{(n+1)|K|-1})$ be the developing pair, and put $\Gamma = \rho(\pi_1(M))$. For a group $H \subset \mathrm{PO}(n+1,1;\mathbb{K})$, the limit set $L(H)$ in $S^{(n+1)|K|-1}$ is defined to be the boundary of the closure of the orbit $H \cdot w$ for a point $w \in \mathbb{H}^{n+1}_{\mathbb{K}}$. (Compare [3].) First of all we can restate Theorems 2.2.1, 2.3.1 and Proposition 2.3.2 of [9] by using the above proposition 3.

**Theorem 6** *Let* $M$ *be a closed conformally flat n-manifold. Suppose that the holonomy group* $\Gamma$ *leaves invariant a geometric m-subsphere* $S^m$ *for* $0 \leqq m \leqq n-1$.

*(i) If* $m \leqq n-3$, *then* $\mathrm{dev} : \tilde{M} \longrightarrow S^n - L(\Gamma)$ *is diffeomorphic.*

*(ii) If* $m = n-2$ *then according to whether* $L(\Gamma)$ *is a proper subset of* $S^{n-2}$ *or* $L(\Gamma) = S^{n-2}$, $\mathrm{dev} : \tilde{M} \longrightarrow S^n - L(\Gamma)$ *is diffeomorphic or* $\mathrm{dev} : \tilde{M} \longrightarrow \mathbb{H}^{n-1}_{\mathbb{R}} \times \mathbb{R}^1$ *is diffeomorphic.*

*Here* $\mathbb{H}^{n-1}_{\mathbb{R}} \times \mathbb{R}^1$ *is conformally equivalent to the universal covering space of the Riemannian manifold* $\mathbb{H}^{n-1}_{\mathbb{R}} \times S^1$ *of nonpositive sectional curvature with the group of isometries* $\mathrm{PO}(n-1,1) \times \mathrm{O}(2)$. *(Note that* $\Gamma$ *is contained in* $\mathrm{PO}(n-1,1) \times \mathrm{O}(2)$ *but not necessarily discrete in it.)*

*(iii) If* $m = n-1$, *then* $M$ *or its two-fold covering decomposes into the union* $M_+ \cup M_{\mathbb{R}} \cup M_-$ *composed of complete hyperbolic n-manifolds with ideal boundaries from* $M_\pm$ *and the union boundary components in* $M_{\mathbb{R}}$.

Finally we continue the same argument to spherical $CR$ manifolds and spherical pseudo-quaternionic manifolds to yield the following result stated in the beginning.

**Theorem 7** *Let* $M$ *be a compact* $(\mathrm{PO}(n+1,1;\mathbb{K}), S^{(n+1)|K|-1})$*-manifold in dimension* $(n+1)|K|-1$ *where* $\mathbb{K} = \mathbb{C}, \mathbb{F}$. *Suppose that the holonomy group* $\Gamma$ *leaves a geometric subsphere* $S^{(m+1)|K|-1}$ $(0 \leqq m \leqq n-1)$. *Then the restriction of the developing map*

$$\mathrm{dev} : \tilde{M} \longrightarrow S^{(n+1)|K|-1} - L(\Gamma) \quad \textit{is a covering map.}$$

***Proof.*** First note that $S^{(m+1)|K|-1}$ is a closed proper subset of $S^{(n+1)|K|-1}$. According to whether $m \leqq n-2$ for the case (1) of $\mathbb{C}$ or $m \leqq n-1$ for the case of $\mathbb{K}$ of Corollary 5, dev : $\tilde{M} \longrightarrow \mathrm{dev}(\tilde{M})$ is injective. In particular, $\Gamma$ acts properly discontinuously on $\mathrm{dev}(\tilde{M})$. So the developing image misses the limit set $L(\Gamma)$. The developing map reduces to the following: dev : $\tilde{M} \longrightarrow S^{(n+1)|K|} - L(\Gamma)$. Moreover, as $\Gamma$ acts properly discontinuously and freely on $S^{(n+1)|K|} - L(\Gamma)$, choosing a Riemannian metric on the orbit space $(S^{(n+1)|K|} - L(\Gamma))/\Gamma$ if necessary, we conclude that dev : $\tilde{M} \longrightarrow S^{(n+1)|K|} - L(\Gamma)$ is a covering map and hence a diffeomorphism.

Now, let $M$ be a spherical $CR$ manifold of dimension $2n+1$ such that $\Gamma$ leaves a geometric subsphere $S^{2n-1}$ $(m = n-1)$. By the case (2) of $\mathbb{C}$ of Corollary 5, dev : $\tilde{M} - \mathrm{dev}^{-1}(S^{2n-1}) \longrightarrow S^{2n+1} - S^{2n-1}$ is a covering map where $\pi_1(S^{2n+1} - S^{2n-1}) = \mathbb{Z}$. Suppose that $\mathrm{dev}^{-1}(S^{2n-1}) \neq \emptyset$. Since dev is a local homeomorphism, $\mathrm{dev}_* : \pi_1(\tilde{M} - \mathrm{dev}^{-1}(S^{2n-1})) \longrightarrow \pi_1(S^n - S^{n-2}) \approx \mathbb{Z}$ is onto. Hence dev : $\tilde{M} - \mathrm{dev}^{-1}(S^{2n-1}) \longrightarrow S^{2n+1} - S^{2n-1}$ is diffeomorphic. By the same argument as above, dev : $\tilde{M} \longrightarrow S^{2n+1} - L(\Gamma)$ is a diffeomorphism. Especially $L(\Gamma)$ is a proper subset of $S^{2n-1}$ in this case. If $\mathrm{dev}^{-1}(S^{2n-1}) = \emptyset$, then dev : $\tilde{M} \longrightarrow S^{2n+1} - S^{2n-1}$ is a covering map. In this case, $\Gamma \subset \mathrm{P}(\mathrm{O}(n,1;\mathbb{C}) \times \mathrm{O}(1;\mathbb{C})) = \mathrm{U}(n,1)$. There exists an exact sequence $S^1 \rightarrow \mathrm{U}(n,1) \longrightarrow \mathrm{PU}(n,1)$ where $S^1 = \mathcal{Z}(n,1;\mathbb{C})$ is the center. Let $\mathrm{U}(n,1)^{\sim}$ be the lift of $\mathrm{U}(n,1)$ corresponding to $S^1$. Then dev maps $\tilde{M}$ onto the universal covering space $X$ of $S^{2n+1} - S^{2n-1}$ for which $\pi$ maps isomorphically onto the subgroup $\tilde{\Gamma}$ lying in $\mathrm{U}(n,1)^{\sim}$. We obtain a compact Lorentz space form of negative constant curvature $\tilde{\Gamma}\backslash\mathrm{U}(n,1)^{\sim}/\mathrm{U}(n)$ diffeomorphic to $M$. Then we know that $\tilde{\Gamma}$ admits a central extension: $\mathbb{Z} \rightarrow \tilde{\Gamma} \xrightarrow{\nu} \Gamma$ for which $\nu$ maps $\tilde{\Gamma}$ *discretely* onto $\Gamma$ of $\mathrm{U}(n,1)$. Compare [14]. Therefore $\Gamma$ acts properly discontinuously on $S^{2n+1} - L(\Gamma)$. Since $L(\Gamma) \subset S^{2n-1}$, choosing a $\Gamma$-invariant Riemannian metric on $S^{2n+1} - L(\Gamma)$, we can show that dev : $\tilde{M} \longrightarrow S^{2n+1} - L(\Gamma)$ is a covering map. As a consequence, $L(\Gamma) = S^{2n-1}$.

□

**Corollary 8** *Let $M$ be a compact* $(\mathrm{PO}(n+1,1;\mathbb{K}), S^{(n+1)|K|-1})$*-manifold in dimension* $(n+1)|K|-1$ *where* $\mathbb{K} = \mathbb{C}, \mathbb{F}$. *Suppose that the holonomy group* $\phi(\pi_1(M))$ *leaves a geometric subsphere* $S^{(m+1)|K|-1}$ $(0 \leqq m \leqq n-1)$.

- ($\mathbb{C}$) *If $L(\Gamma) \subset S^{2n-3}$ at most, or $L(\Gamma)$ is a proper subset of $S^{2n-1}$, then* dev : $\tilde{M} \longrightarrow S^{2n+1} - L(\Gamma)$ *is a diffeomorphism. When $L(\Gamma) = S^{2n-1}$,* dev : $\tilde{M} \longrightarrow S^{2n+1} - S^{2n+1}$ *is a covering map.*

- ($\mathbb{F}$) *If $L(\Gamma)$ is contained in $S^{4n-1}$ at most, then* dev : $\tilde{M} \longrightarrow S^{4n+3} - L(\Gamma)$ *is a diffeomorphism.*

When the limit set of a (generalized) Schottky group $\Gamma$ of $\mathrm{PO}(n+1,1;\mathbb{K})$ is embedded in the small geometric subsphere, we can state the following result (Compare [10].)

**Corollary 9** *Let $M$ be a compact* $(\mathrm{PO}(n+1,1;\mathbb{K}), S^{(n+1)|K|-1})$*-manifold in dimension* $(n+1)|K|-1$ *where* $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{F}$. *Suppose that the limit set $\Lambda$ of the holonomy group $\phi(\pi_1(M))$ is a proper subset of $S^{n-2} = \partial\mathbb{H}_{\mathbb{R}}^{n-1}$, $S^{2n-1} = \partial\mathbb{H}_{\mathbb{C}}^{n}$ or $S^{4n-1} = \partial\mathbb{H}_{\mathbb{F}}^{n}$ respectively.*

*Then $M$ is* $(\mathrm{PO}(n+1,1;\mathbb{K}),S^{(n+1)|K|-1})$*-equivalent to the orbit space* $(S^{(n+1)|K|-1}-\Lambda)/\phi(\pi_1(M))$.

## 3 Horospherical geometry

The $\mathrm{PO}(n+1,1;\mathbb{K}),S^{(n+1)|K|-1})$ - structure restricted to the sphere $S^{(n+1)|K|-1}$ with one point removed is called the horospherical geometry. If $\{\infty\}$ is the point at infinity, then $S^{(n+1)|K|-1}-\{\infty\}$ is isomorphic to the nilpotent Lie group $\mathcal{H}$ where $\mathrm{Im}\,\mathbb{K}\to\mathcal{H}\xrightarrow{\nu}\mathbb{K}^n$ is a central group extension. In particular, if $\mathbb{K}=\mathbb{R}$, then $\mathcal{H}=\mathbb{R}^n$ is the vector space and if $\mathbb{K}=\mathbb{C},\mathbb{F}$, then the center $\mathrm{Im}\,\mathbb{K}$ is the vector space isomorphic to $\mathbb{R},\mathbb{R}^3$ respectively. Denote by $\mathrm{Sim}(\mathcal{H})$ the stabilizer of $\mathrm{PO}(n+1,1;\mathbb{K})$ at $\{\infty\}$. Since the maximal noncompact amenable Lie group of $\mathrm{O}(n+1,1;\mathbb{K})$ (viewed as the noncompact symmetric space of rank 1) is isomorphic to the semidirect product $\mathcal{H}\rtimes(\mathrm{O}(n;\mathbb{K})\times\mathbb{K}^*)$ where $\mathbb{K}^*$ is the multiplicative group, $\mathrm{Sim}(\mathcal{H})$ is isomorphic to the quotient group $\mathcal{H}\rtimes(P(\mathrm{O}(n;\mathbb{K})\times\mathrm{O}(1;\mathbb{K}))\times\mathbb{R}^+)$. More precesicely, according to whether $\mathbb{K}=\mathbb{R},\mathbb{C}\,\mathbb{F}$, $\mathrm{Sim}(\mathcal{H})$ is $\mathbb{R}^n\rtimes(\mathrm{O(n)}\times\mathbb{R}^+)$, $\mathcal{N}\rtimes(\mathrm{U}(n)\times\mathbb{R}^+)$, or $\mathcal{M}\rtimes(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)\times\mathbb{R}^+)$.

A representation $\rho:\Gamma\to\mathrm{PO}(n+1,1;\mathbb{K})$ is said to be *amenable* if the closure of the image $\overline{\rho(\Gamma)}$ in $\mathrm{PO}(n+1,1;\mathbb{K})$ lies in the maximal amenable Lie subgroup of $\mathrm{PO}(n+1,1;\mathbb{K})$. We note the following result.

**Theorem 10** *Let $M$ be a compact $(n+1)|K|-1$ dimensional* $(\mathrm{PO}(n+1,1;\mathbb{K}),S^{(n+1)|K|-1})$*-manifold. If the holonomy group is amenable, then $M$ is finitely covered by the sphere $S^{(n+1)|K|-1}$, a Hopf manifold $S^1\times S^{(n+1)|K|-2}$ or a nilmanifold $\mathcal{H}/\Gamma$.*

The horospherical geometry $(\mathrm{Sim}(\mathbb{R}^n),\mathbb{R}^n)$ is said to be a similarity geometry. The above theorem was first proved by Fried [7] when $M$ is a compact similarity manifold (*i.e.*, $(\mathrm{Sim}(\mathbb{R}^n),\mathbb{R}^n)$-manifold). In general, the theorem for a compact conformally flat manifold with amenable holonomy has been seen in [19], [18], [11]. For the Heisenberg similarity geometry $(\mathrm{Sim}(\mathcal{N}),\mathcal{N})$, the theorem is proved by Miner [19], and for the pseudo-quaternionic Heisenberg similarity geometry $(\mathrm{Sim}(\mathcal{M}),\mathcal{M})$, proved by Kamishima [13].

The idea of proof for $\mathbb{K}=\mathbb{C},\mathbb{F}$ in [13] is to examine the Carnot-Carathéodory structure on $(\mathrm{PO}(n+1,1;\mathbb{K}),S^{(n+1)|K|-1})$-manifold for $\mathbb{C}$ or $\mathbb{F}$ respectively. We verify that the restricted $(\mathrm{PO}(n+1,1;\mathbb{K}),S^{(n+1)|K|-1})$- structure gives a certain Carnot-Carathéodory structure (a codimension 1 or 3 - bundle $B$) on $\mathcal{H}$. In fact the projection $\nu$ in the above extension maps $B$ isomorphically onto the tangent space of $\mathbb{K}^n$ at each point. Moreover, the restriction to $B$ of the left invariant metric on $\mathcal{H}$ coincides with the complex (resp. quaternionic) euclidean metric on $\mathbb{K}^n$. We then apply the Fried's *incompleteness* argument to the Carnot-Carathéodory structure, which gives the desired result. Using Theorem 10, we obtain the following result, which has been indicated by Kulkarni and Pinkall [17] and proved in [11] for the conformally flat case.

**Theorem 11** *Let $M$ be a compact* $(\mathrm{PO}(n+1,1;\mathbb{K}),S^{(n+1)|K|-1})$*-manifold. If the developing map* dev *is not surjective, then* $\mathrm{dev}:\tilde{M}\longrightarrow\mathrm{dev}(\tilde{M})$ *is a covering map.*

***Proof.*** Given a developing pair $(\phi, \mathrm{dev})$, denote by $\partial\mathrm{dev}(\tilde{M})$ the boundary of the developing image in $S^{(n+1)|K|-1}$. If $\partial\mathrm{dev}(\tilde{M})$ consists of one point, say $\{\infty\}$, then the holonomy group $\phi(\pi_1(M)) = \Gamma$ leaves $\{\infty\}$ fixed. So, the representation is amenable by the definition. Appllying Theorem 1, the developing map is a homeomorphism onto its image. Suppose that $\partial\mathrm{dev}(\tilde{M})$ contains more than one point. By the minimal property, the limit set $L(\Gamma) \subset \partial\mathrm{dev}(\tilde{M})$ so that $\mathrm{dev}(\tilde{M}) \subset S^{(n+1)|K|-1} - L(\Gamma)$. If $\Gamma$ is discrete in $\mathrm{PO}(n+1, 1; \mathbb{K})$, then $\Gamma$ acts properly discontinuously on the domain of discontinuity $\Omega = S^{(n+1)|K|-1} - L(\Gamma)$. Therefore there is a $\Gamma$-invariant Riemannian metric on $\Omega$ (cf. [15], [23]). As $M$ is compact, dev is a covering map onto its image $\Omega$. Let $\bar{\Gamma}^0$ be the identity component of the closure of $\Gamma$. If $\bar{\Gamma}^0$ is compact, then it fixes the unique point in $\mathbb{H}_{\mathbb{K}}^{n+1}$ or a totally geodesic subspace $\mathbb{H}_{\mathbb{K}}^{m+1}$ pointwisely $(0 \leqq m \leqq n-1)$. If $\bar{\Gamma}^0$ is noncompact, then it follows from the theorem of [3] that $\bar{\Gamma}^0$ leaves invariant a totally geodesic subspace $\mathbb{H}_{\mathbb{K}}^{m+1}$ $(0 \leqq m \leqq n-1)$. As $\bar{\Gamma}^0$ is normal in $\bar{\Gamma}$, $\bar{\Gamma}$ has the unique fixed point or leaves invariant $\mathbb{H}_{\mathbb{K}}^{m+1}$ in each case. Thus either $\Gamma$ is contained in the maximal compact group $P(\mathrm{O}(n+1; \mathbb{K}) \times \mathrm{O}(1; \mathbb{K}))$ or it leaves invariant $S^{(m+1)|K|-1}$. In the former case, $M$ will be covered by the sphere $S^{(n+1)|K|-1}$. Suppose $\Gamma$ leaves invariant a positive dimensional geometric subsphere $S^{(m+1)|K|-1}$. Let $\mathbb{K} = \mathbb{C}, \mathbb{F}$. If $M$ is a compact $(\mathrm{PO}(n+1, 1; \mathbb{K}), S^{(n+1)|K|-1})$-manifold, then Theorem 7 implies that $\mathrm{dev} : \tilde{M} \longrightarrow S^{(n+1)|K|} - L(\Gamma)$ is a covering map.

Consider the case that $M$ is a closed $n$-dimensional conformally flat $(\mathrm{PO}(n+1, 1), S^n)$-manifold. In this case, $\Gamma$ leaves $S^m$ invariant $(0 \leqq m \leqq n-1)$, or $\Gamma \subset \mathrm{PO}(n+1)$. If $m \leqq n-2$, then $\mathrm{dev} : \tilde{M} \longrightarrow S^n - L(\Gamma)$ is a covering map by Theorem 6.

Let $m = n-1$. The holonomy group $\Gamma \subset \mathrm{PO}(n, 1)$ leaves invariant $S^{n-1}$. If $\Gamma$ is discrete, then $\Gamma$ acts properly discontinuously on $S^n - L(\Gamma)$. The same argument as above implies that $\mathrm{dev} : \tilde{M} \longrightarrow S^n - L(\Gamma)$ is a covering map.

Let $S^\ell$ be a geometric subsphere of $S^{n-1} = \partial\mathbb{H}_{\mathbb{R}}^n$. Suppose that $\bar{\Gamma}^0$ is nontrivial and compact. Then $\bar{\Gamma}^0$ fixes $S^\ell$ for some $\ell$ or stabilizes a unique point inside $\mathbb{H}_{\mathbb{R}}^n$. The latter case implies that $M$ is a spherical space form so $\Gamma$ is finite, which contradicts that $\bar{\Gamma}^0$ is nontrivial. If $\ell < n-1$, the result follows by the preceding argument because $\bar{\Gamma}$ leaves $S^\ell$ invariant. On the other hand, if $\bar{\Gamma}^0$ fixes $S^{n-1}$, it must fix the whole sphere $S^n$, hence $\bar{\Gamma}^0 = \{1\}$ by effectivity. Thus, $\bar{\Gamma}^0$ is noncompact, again by the theorem of [3], $\bar{\Gamma}^0$ is transitive on a totally geodesic subspace $\mathbb{H}_{\mathbb{R}}^{\ell'+1}$ and so $\bar{\Gamma}$ leaves invariant the geometric subsphere $S^{\ell'}$. As above, only $\ell' = n-1$ is necessary to check. Then note that $L(\Gamma) = L(\bar{\Gamma}^0) = S^{n-1}$. As $S^n - L(\Gamma)$ consists of two components of hyperbolic spaces and $\mathrm{dev}(\tilde{M}) \subset S^n - L(\Gamma)$, this implies that $\mathrm{dev} : \tilde{M} \longrightarrow \mathbb{H}_{\mathbb{R}}^n$. By Corollary 5, dev is a homeomorphism onto $\mathbb{H}_{\mathbb{R}}^n$. As a matter of fact, $\Gamma$ would be discrete, which contradicts the above hypothesis. So the case that $\bar{\Gamma}^0$ is nontrivial does not occur. This completes the proof.

$\square$

**Proof of Theorem A** (Compare [13].)

We may assume that $\pi \subset \mathrm{PSp}(m, 1)$ $(2 \leqq m \leqq n)$. Let $\rho : \pi \longrightarrow \mathrm{PSp}(n+1, 1)$ be the holonomy representation. Considering the Zariski closure of $\rho(\pi)$ in $\mathrm{PSp}(n+1, 1)$ and by the classification [3] of connected subgroups in $\mathrm{PSp}(n+1, 1)$, we see that $\rho(\pi)$ is conjugate

to a subgroup of an almost direct product $K \cdot H$ of the compact Lie subgroup $K$ with a noncompact semisimple Lie subgroup $H$, or conjugate to a subgroup of an amenable Lie subgroup in $\mathrm{PSp}(n+1,1)$. Let $P : K \cdot H \longrightarrow PH$ be the projection onto the semisimple Lie group $PH$ for which $PH$ has no compact factor and no center. If $\rho(\pi) \subset K \cdot H$, then we can assume that $PH$ is the smallest semisimple connected group containing $P \circ \rho(\pi)$ and so $P \circ \rho(\pi)$ is Zariski dense in $PH$. Then the Corlettes' superrigidity says that $P \circ \rho$ extends to a continuous homomorphism $\varphi : \mathrm{PSp}(m,1) \longrightarrow PH$ for $m \geqq 2$. It is easy to see that $\varphi$ is onto. Since $\mathrm{PSp}(m,1)$ has no normal subgroup, $\varphi : \mathrm{PSp}(m,1) \longrightarrow PH$ is an isomorphism. As $PH \subset \mathrm{PSp}(n+1,1)$, $PH$ must be conjugate to $\mathrm{PSp}(m,1)$ by the classification of connected Lie groups from [3]. Then $PH$ leaves invariant a geometric sphere $S^{4m-1}$ and so does $K \cdot H$. In particular, $\rho(\pi)$ leaves $S^{4m-1}$ invariant so that $L(\rho(\pi)) \subset S^{4m-1}$. Since $2 \leqq m \leqq n$, applying Corollary 8 yields that $\mathrm{dev} : \tilde{M} \longrightarrow S^{4n+3} - L(\rho(\pi))$ is homeomorphic. As $M$ is compact, $L(\rho(\pi)) = S^{4m-1}$. We obtain that $M$ is pseudo-quaternionically isomorphic to $S^{4n+3} - S^{4m-1}/\rho(\pi)$.
On the other hand, if $\rho(\pi)$ is amenable, then dev is homeomorphic by Theorem 10, which implies that $\pi$ would be virtually nilpotent. This is impossible from our hypothesis. This proves Theorem A.

□

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