Notes on Discrete Subgroups of $PU(1, 2; \mathbb{C})$
with Heisenberg Translations

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0. Recently Parker, Basmajian and Miner have independently given some conditions for a subgroup of $PU(1, 2; \mathbb{C})$ to be non-discrete. In this paper we show that under some conditions Parker’s theorem leads to some Basmajian and Miner’s result.

1. To introduce Parker’s theorem and Basmajian-Miner’s theorem, we need some definitions and notation. Let $\mathbb{C}$ be the field of complex numbers. Let $V = V^{1, 2}(\mathbb{C})$ denote the vector space $\mathbb{C}^3$, together with the unitary structure defined by the Hermitian form

$$\tilde{\Phi}(Z^*, w^*) = -(Z_0^* - w_1^* + Z_{12}^* w_2^*) + z_2^* w_2^*$$

for $z^* = (z_0^*, z_1^*, z_{12}^*), w^* = (w_0^*, w_1^*, w_2^*)$ in $V$.

An automorphism $g$ of $V$, that is a linear bijection such that $\tilde{\Phi}(g(Z^*), g(w^*)) = \tilde{\Phi}(z^*, w^*)$ for $z^*, w^*$ in $V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1, 2; \mathbb{C})$. Set $PU(1, 2; \mathbb{C}) = U(1, 2; \mathbb{C})/(center)$. An element $g$ in $PU(1, 2; \mathbb{C})$ acts on the Siegel domain

$$H^2 = \{w = (w_1, w_2) \in \mathbb{C}^2 \mid Re(w_1) > \frac{1}{2} |w_2|^2\}$$

and its boundary $\partial H^2$. Denote $H^2 \cup \partial H^2$ by $\overline{H^2}$. We define a new coordinate system in $\overline{H^2} - \{\infty\}$. To $q = (w_1, w_2) \in \overline{H^2} - \{\infty\}$ we can correspond the 3-tuple $(k, t, w) \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R} \times \mathbb{C}$, where $k = Re(w_1) - \frac{1}{2} |w_2|^2$ and $t = Im(w_1)$. This 3-tuple $(k, t, w_{12})_H$ is called the $H$-coordinates of $q$. For simplicity, we use $(t_1, w')_H$ for $(0, t_1, w')_H$.

The Cygan metric $\rho(p, q)$ for $p = (k_1, t_1, w_{12})_H$ and $q = (k_2, t_2, W')_H$ is given by

$$\rho(p, q) = |\{\frac{1}{2} |W' - w'|^2 + |k_2 - k_1|\} + i(t_1 - t_2 + Im(W'W'))|^\frac{1}{2}.$$

We note that this Cygan metric $\rho$ is a generalization of the Heisenberg metric $\delta$ in $\partial H^2$ (see [7]). Let $f = (a_{ij})_{1 \leq i, j \leq 3} \in PU(1, 2; \mathbb{C})$ with $f(\infty) \neq \infty$. We define the isometric sphere $I_f$ of $f$ by

$$I_f = \{w = (w_1, w_2) \in \overline{H^2} \mid |\tilde{\Phi}(W, Q)| = |\tilde{\Phi}(W, f^{-1}(Q))|\},$$

where $Q = (0, 1, 0), W = (1, w_1, w_2) \in V$ (see [3]). It follows that the isometric sphere $I_f$ is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,
$I_f = \left\{ z = (k, t, w') \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R} \times \mathbb{C} \mid \rho(z, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}$.

Remark 1.1. In $PU(1,1;\mathbb{C})$, our radius of isometric sphere is the square root of the usual one.

We have the same formulae as in Möbius transformations (see [3]).

Proposition 1.2. Let $g$ and $h$ be elements with $g(\infty) \neq \infty$ and $h(\infty) \neq \infty$. Then:

1. $R_{gh} = \frac{R_g R_h}{\delta(g^{-1}(\infty), h(\infty))}$
2. $R_h^2 = \delta((gh)^{-1}(\infty), h^{-1}(\infty)) \delta(g^{-1}(\infty), h(\infty))$.

Now we are ready to state Parker's Theorem.

Theorem 1.3 ([9]). Let $g$ be a Heisenberg translation with the form

$$g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \alpha \\ a & 0 & 1 \end{pmatrix},$$

where $Re(s) = \frac{1}{2}|a|^2$. Let $f$ be any element of $PU(1,2;\mathbb{C})$ with isometric sphere of radius $R_f$. If

$$R_f^2 > \delta(gf^{-1}(\infty), f^{-1}(\infty)) \delta(gf(\infty), f(\infty)) + 2|a|^2,$$

then the group $\langle f, g \rangle$ generated by $f$ and $g$ is not discrete.

Remark 1.4. Suppose that $g$ is a vertical Heisenberg translation. As $a = 0$, this theorem is equivalent to the result in [5] and [6].

Let

$$B_r = \{ z \in \partial H^2 \mid \rho(z,0) = \delta(z,0) < r \},$$

and let $\overline{B}^c_s = \partial H^2 \cup \{\infty\} - B_s$. For $0 < r < 1$, the pair of open sets $(B_r, \overline{B}_1^c)$ is said to be stable with respect to a set $S$ of elements in $PU(1,2;\mathbb{C})$ if for any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}_1^c.$$

A loxodromic element $f$ has a unique complex dilation $\lambda(f)$ such that $|\lambda(f)| > 1$. Let $S(r, \varepsilon(r))$ denote the family of loxodromic elements $f$ with fixed points in $B_r$ and $\overline{B}_1^c$, and satisfying $|\lambda(f) - 1| < \varepsilon(r)$. 
For positive real numbers \( r \) with \( \frac{1}{\sqrt{3 + \sqrt{3}} - \sqrt{2}} = 0.549... \), we define \( \epsilon(r) \) by

\[
(*) \quad \epsilon(r) = \text{sup}\{ |\lambda(f) - 1| < \epsilon \},
\]

where \( \lambda(f) \) satisfies the inequalities below

\[
|\lambda(f) - 1| < \sqrt{2 + \left( \frac{1 - (3 + |\lambda(f) - 1|)r^2}{1 - 2r^2} \right)^2 \frac{(1 - 3r^2)}{(1 - r^2)^2} - \sqrt{2},}
\]

\[
|\lambda(f)| < \frac{1 - 2r^2}{r^2}.
\]

We show the graph of \( \epsilon(r) \) below.

![Graph of \( \epsilon(r) \)](image)

If \( r < \frac{1}{\sqrt{3 + \sqrt{3} - \sqrt{2}} \} \) and \( \epsilon < \epsilon(r) \), a pair of non-negative numbers \( (r, \epsilon) \) is called a **stable basin point**.

For four points \( q_1, q_2, q_3, q_4 \) in \( \partial H^2 \), define the **real cross ratio** \( ||(q_1, q_2, q_3, q_4)|| \) by

\[
||(q_1, q_2, q_3, q_4)|| = \frac{\delta^2(q_3, q_1)\delta^2(q_4, q_2)}{\delta^2(q_4, q_1)\delta^2(q_3, q_2)}.
\]

Note that this real cross ratio is invariant under \( PU(1, 2; \mathbb{C}) \).

We shall state Basmajian-Miner's result.

**Theorem 1.4 ([1])**. Fix a stable basin point \( (r, \epsilon) \). Let \( g \) be a parabolic element with fixed point \( \infty \). If \( f \) is a loxodromic element with fixed points \( 0 \) and \( q \) satisfying \( |\lambda(f) - 1| < \epsilon \).

If \( ||(0, q, g(0), g(q))|| < r^4 \), then the group \( \langle f, g \rangle \) generated by \( f \) and \( g \) is not discrete.
2. In this section we show that Parker's Theorem leads to Basmajian-Miner's theorem under some conditions. First we treat a simple case.

Theorem 2.1. Fix a stable basin point \((r, \varepsilon)\). Let \(g\) be a Heisenberg translation with the form

\[
g = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \bar{a} \\
a & 0 & 1
\end{pmatrix},
\]

where \(Re(s) = \frac{1}{2}|a|^2\). Let \(f\) be a loxodromic element with fixed points \(a_f = (0,0)\) and \(b_f = (it,0)\) \((t > 0)\) such that \(|\lambda(f) - 1| < \varepsilon\). If \(|[a_f, b_f, g(a_f), g(b_f)]| < r^4\), then the group \(<f,g>\) generated by \(f\) and \(g\) is not discrete.

Lemma 2.2 immediately leads to

Corollary 2.3. Fix a stable basin point \((r, \varepsilon)\). Let \(f\) and \(g\) be the same elements as in Theorem 2.1. If \(\delta(0, g(0)) > \frac{\delta(0, q)}{r^2} (1 + r^2 + \sqrt{1 + r^2})\), then the group \(<f,g>\) generated by \(f\) and \(g\) is not discrete.

When the condition on fixed points of a loxodromic element is weakened, we obtain

Theorem 2.4. Fix a stable basin point \((r, \varepsilon)\), where \(r < 0.48\). Let \(g\) be the same element as in Theorem 2.1. Let \(f\) be a loxodromic element with fixed point \(0\) and \(q \neq \infty\) satisfying \(|\lambda(f) - 1| < \varepsilon\). If \(\delta(0, q) > \frac{\delta(0, g(0))}{r^2} (1 + r^2 + \sqrt{1 + r^2})\), then the group \(<f,g>\) generated by \(f\) and \(g\) is not discrete.

For our proof of Theorem 2.4, we need

Proposition 2.5. Let \(f\) be a loxodromic element with the attracting fixed point \(a_f\) and the repelling fixed point \(b_f\). Then:

1. \(|[f(z), z, b_f, a_f]| = |\lambda(f)|^2\) for any \(z \in \partial H^2\).

2. \(\delta(f(z), f(w)) = \frac{\delta(a_f, g(a_f))}{r^2} (1 + r^2 + \sqrt{1 + r^2})\delta(z, w)\) for \(z, w \in \partial H^2\).

3. \(R_f^2 = \delta(f, f^{-1}(\infty))\delta(a_f, f^{-1}(\infty)) = \delta(b_f, f(\infty))\delta(b_f, f(\infty))\).

4. \(\frac{\delta(a_f, f^{-1}(\infty))}{\delta(b_f, f^{-1}(\infty))} = \frac{\delta(a_f, f(\infty))}{\delta(b_f, f(\infty))} = |\lambda(f)|\).

5. \(\delta(a_f, f(\infty)) = \delta(b_f, f^{-1}(\infty)) = R_f|\lambda(f)|^{-\frac{1}{2}}\).

6. \(\delta(a_f, f^{-1}(\infty)) = \delta(b_f, f(\infty)) = R_f|\lambda(f)|^{\frac{1}{2}}\).

7. \(R_f(|\lambda(f)|^{\frac{1}{2}} - |\lambda(f)|^{-\frac{1}{2}}) \leq \delta(a_f, b_f) \leq R_f(|\lambda(f)|^{\frac{1}{2}} + |\lambda(f)|^{-\frac{1}{2}})\).

Remark 2.6. If \(f\) is an element of \(PU(1,1; \mathbb{C})\), then

\[R_f(|\lambda(f)| - |\lambda(f)|^{-1})^{\frac{1}{2}} = \delta(a_f, b_f)\]

But this is not true for an element of \(PU(1,2; \mathbb{C})\).
3. The details of this paper will be published elsewhere.

References

8. S. Kamiya, Discrete Subgroups of $PU(1,2;C)$ with Heisenberg Translations, Proceedings of the Fifth International Colloquium on Finite or Infinite Dimensional Complex Analysis, Beijing, 137-140, (1997).

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