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Notes on Discrete Subgroups of $PU(1,2;\mathbf{C})$ with Heisenberg Translations

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0. Recently Parker, Basmajian and Miner have independently given some conditions for a subgroup of $PU(1,2;\mathbf{C})$ to be non-discrete. In this paper we show that under some conditions Parker's theorem leads to some Basmajian and Miner's result.

1.To introduce Parker's theorem and Basmajian-Miner's theorem, we need some definitions and notation. Let $\mathbf{C}$ be the field of complex numbers. Let $V=V^{1,2}(\mathbf{C})$ denote the vector space $\mathbf{C}^3$, together with the unitary structure defined by the Hermitian form

$$\widetilde{\Phi}(z^*,w^*)=-(\overline{z_0^*}w_1^*+\overline{z_1^*}w_0^*)+\overline{z_2^*}w_2^*$$

for $z^*=(z_0^*,z_1^*,z_2^*), w^*=(w_0^*,w_1^*,w_2^*)$ in V.

An automorphism g of V, that is a linear bijection such that $\widetilde{\Phi}(g(z^*),g(w^*))=\widetilde{\Phi}(z^*,w^*)$ for z^*,w^* in V, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,2;\mathbf{C})$. Set $PU(1,2;\mathbf{C})=U(1,2;\mathbf{C})/(center)$. An element g in $PU(1,2;\mathbf{C})$ acts on the Siegel domain

$$H^2=\{w=(w_1,w_2)\in\mathbf{C}^2 \mid Re(w_1)>\frac{1}{2}|w_2|^2\}$$

and its boundary ∂H^2. Denote $H^2\cup\partial H^2$ by $\overline{H^2}$. We define a new coordinate system in $\overline{H^2}-\{\infty\}$. To $q=(w_1,w_2)\in\overline{H^2}-\{\infty\}$ we can correspond the 3-tuple $(k,t,w_2)\in(\mathbf{R}^+\cup\{0\})\times\mathbf{R}\times\mathbf{C}$, where $k=Re(w_1)-\frac{1}{2}|w_2|^2$ and $t=Im(w_1)$. This 3-tuple $(k,t,w_2)_H$ is called the $H-coordinates$ of q. For simplicity, we use $(t_1,w')_H$ for $(0,t_1,w')_H$.

The *Cygan metric* $\rho(p,q)$ for $p=(k_1,t_1,w')_H$ and $q=(k_2,t_2,W')_H$ is given by

$$\rho(p,q)=|\{\frac{1}{2}|W'-w'|^2+|k_2-k_1|\}+i\{t_1-t_2+Im(\overline{w'}W')\}|^{\frac{1}{2}}.$$

We note that this Cygan metric ρ is a generalization of the *Heisenberg metric* δ in ∂H^2 (see [7]). Let $f=(a_{ij})_{1\le i,j\le 3}\in PU(1,2;\mathbf{C})$ with $f(\infty)\ne\infty$. We define the *isometric sphere* I_f of f by

$$I_f=\{w=(w_1,w_2)\in\overline{H^2} \mid |\tilde{\Phi}(W,Q)|=|\tilde{\Phi}(W,f^{-1}(Q))|\},$$

where $Q=(0,1,0)$, $W=(1,w_1,w_2)$ in V (see [3]). It follows that the isometric sphere I_f is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f=\sqrt{1/|a_{12}|}$, that is,

$$I_f = \left\{ z = (k,t,w') \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C} \mid \rho(z, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

Remark 1.1. In $PU(1,1;\mathrm{C})$, our radius of isometric sphere is the square root of the usual one.

We have the same formulae as in Möbius transformations (see [3]).

Proposition 1.2. *Let g and h be elements with $g(\infty) \neq \infty$ and $h(\infty) \neq \infty$. Then:*

(1) $R_{gh} = \dfrac{R_g R_h}{\delta(g^{-1}(\infty), h(\infty))}$.

(2) $R_h^2 = \delta((gh)^{-1}(\infty), h^{-1}(\infty))\delta(g^{-1}(\infty), h(\infty))$.

Now we are ready to state Parker's Theorem.

Theorem 1.3 ([9]). *Let g be a Heisenberg translation with the form*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \overline{a} \\ a & 0 & 1 \end{pmatrix},$$

where $Re(s) = \frac{1}{2}|a|^2$. Let f be any element of $PU(1,2;\mathrm{C})$ with isometric sphere of radius R_f. If

$$R_f^2 > \delta(gf^{-1}(\infty), f^{-1}(\infty)\delta(gf(\infty), f(\infty)) + 2|a|^2,$$

then the group $< f, g >$ generated by f and g is not discrete.

Remark 1.4. Suppose that g is a vertical Heisenberg translation. As $a = 0$, this theorem is equivalent to the result in [5] and [6].

Let

$$B_r = \{z \in \partial H^2 \mid \rho(z,0) = \delta(z,0) < r\},$$

and let $\overline{B}_s^c = \partial H^2 \cup \{\infty\} - B_s$. For $0 < r < 1$, the pair of open sets $(B_r, \overline{B}_{1/r}^c)$ is said to be *stable* with respect to a set S of elements in $PU(1,2;\mathrm{C})$ if for any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}_{1/r}^c.$$

A loxodromic element f has a unique complex dilation $\lambda(f)$ such that $|\lambda(f)| > 1$. Let $S(r, \varepsilon(r))$ denote the family of loxodromic elements f with fixed points in B_r and $\overline{B}_{1/r}^c$, and satisfying $|\lambda(f) - 1| < \varepsilon(r)$.

For positive real numbers r with $r < 1/\sqrt{3+\sqrt{3}-\sqrt{2}} = 0.549...$, we define $\varepsilon(r)$ by

$$(*) \qquad \varepsilon(r) = sup\{|\lambda(f) - 1| < \varepsilon\},$$

where $\lambda(f)$ satisfies the inequalites below

$$|\lambda(f) - 1| < \sqrt{2 + \left(\frac{1-(3+|\lambda(f)-1|)r^2)}{1-2r^2}\right)^2 \left(\frac{1-3r^2}{1-r^2}\right)^2} - \sqrt{2},$$

$$|\lambda(f)| < \frac{1-2r^2}{r^2}.$$

We show the graph of $\varepsilon(r)$ below.

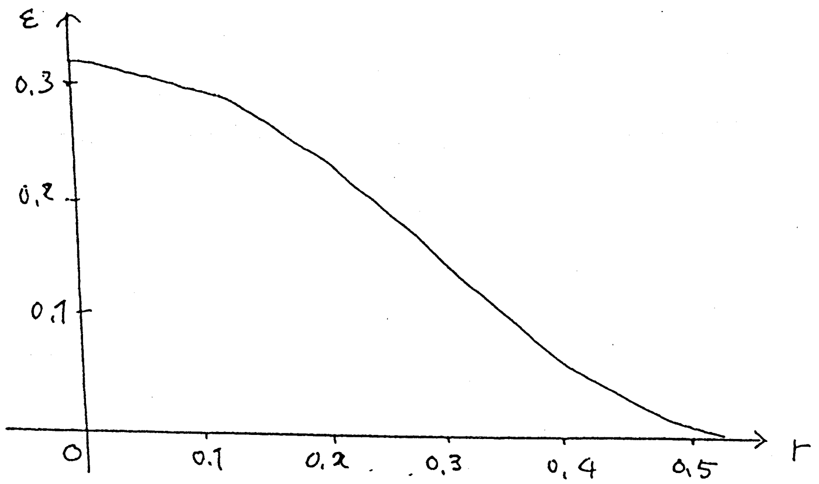

figure 1.

If $r < 1/\sqrt{3+\sqrt{3}-\sqrt{2}}$ and $\varepsilon < \varepsilon(r)$, a pair of non-negative numbers (r, ε) is called a *stable basin point.*

For four points q_1, q_2, q_3, q_4 in ∂H^2, define the *real cross ratio* $|[q_1, q_2, q_3, , q_4)]|$ by

$$|[q_1, q_2, q_3, , q_4]| = \frac{\delta^2(q_3, q_1)\delta^2(q_4, q_2)}{\delta^2(q_4, q_1)\delta^2(q_3, q_2)}.$$

Note that this real cross ratio is invariant under $PU(1,2;\mathrm{C})$.

We shall state Basmajian-Miner's result.

Theorem 1.4 ([1]). *Fix a stable basin point (r, ε). Let g be a parabolic element with fixed point ∞. If f is a loxodromic element with fixed points 0 and q satisfying $|\lambda(f) - 1| < \varepsilon$. If $|[0, q, g(0), g(q)]| < r^4$, then the group $< f, g >$ generated by f and g is not discrete.*

2. In this section we show that Parker's Theorem leads to Basmajian-Miner's theorem under some conditions. First we treat a simple case.

Theorem 2.1. *Fix a stable basin point* (r,ε). *Let* g *be a Heisenberg translation with the form*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \overline{a} \\ a & 0 & 1 \end{pmatrix},$$

where $Re(s) = \frac{1}{2}|a|^2$. *Let* f *be a loxodromic element with fixed points* $a_f = (0,0)$ *and* $b_f = (it, 0)$ $(t > 0)$ *such that* $|\lambda(f) - 1| < \varepsilon$. *If* $|[a_f, b_f, g(a_f), g(b_f)]| < r^4$, *then the group* $< f, g >$ *generated by* f *and* g *is not discrete.*

Lemma 2.2 immediately leads to

Corollary 2.3. *Fix a stable basin point* (r,ε). *Let* f *and* g *be the same elements as in Theorem 2.1. If* $\delta(a_f, b_f) > \frac{\delta(a_f, g(a_f))}{r^2}(1 + r^2 + \sqrt{1+r^2})$, *then the group* $< f, g >$ *generated by* f *and* g *is not discrete.*

When the condition on fixed points of a loxodromic element is weakened, we obtain

Theorem 2.4. *Fix a stable basin point* (r,ε), *where* $r < 0.48$. *Let* g *be the same element as in Theorem 2.1. Let* f *be a loxodromic element with fixed point 0 and* $q (\neq \infty)$ *satisfying* $|\lambda(f) - 1| < \varepsilon$. *If* $\delta(0, q) > \frac{\delta(0, g(0))}{r^2}(1 + r^2 + \sqrt{1+r^2})$, *then the group* $< f, g >$ *generated by* f *and* g *is not discrete.*

For our proof of Theorem 2.4, we need

Proposition 2.5. *Let* f *be a loxodromic element with the attracting fixed point* a_f *and the repelling fixed point* b_f. *Then:*

(1) $|[f(z), z, b_f, a_f]| = |\lambda(f)|^2$ *for any* $z \in \partial H^2$.

(2) $\delta(f(z), f(w)) = \frac{R_f^2}{\delta(z, f^{-1}(\infty))\delta(w, f^{-1}(\infty))}\delta(z, w)$ *for* $z, w \in \partial H^2$.

(3) $R_f^2 = \delta(a_f, f^{-1}(\infty))\delta(b_f, f^{-1}(\infty)) = \delta(a_f, f(\infty))\delta(b_f, f(\infty))$.

(4) $\frac{\delta(a_f, f^{-1}(\infty))}{\delta(b_f, f^{-1}(\infty))} = \frac{\delta(b_f, f(\infty))}{\delta(a_f, f(\infty))} = |\lambda(f)|$.

(5) $\delta(a_f, f(\infty)) = \delta(b_f, f^{-1}(\infty)) = R_f|\lambda(f)|^{-\frac{1}{2}}$.

(6) $\delta(a_f, f^{-1}(\infty)) = \delta(b_f, f(\infty)) = R_f|\lambda(f)|^{\frac{1}{2}}$.

(7) $R_f(|\lambda(f)|^{\frac{1}{2}} - |\lambda(f)|^{-\frac{1}{2}}) \leq \delta(a_f, b_f) \leq R_f(|\lambda(f)|^{\frac{1}{2}} + |\lambda(f)|^{-\frac{1}{2}})$.

Remark 2.6. If f is an element of $PU(1,1;\mathbf{C})$, then

$$R_f(|\lambda(f)| - |\lambda(f)|^{-1})^{\frac{1}{2}} = \delta(a_f, b_f).$$

But this is not true for an element of $PU(1,2;\mathbf{C})$.

3. The details of this paper will be published elsewhere.

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