数理解析研究所講究録
1065 巻 1998 年 23-27

A remark on the elementariness of Möbius groups in several dimension

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1 Introduction

In this article we consider the elementariness of Möbius groups of several dimensions which are not necessarily discrete. In two or three dimensional cases, the elementariness of Möbius groups is defined in several ways. In the theory of Kleinian groups, which are regarded as three dimensional discrete Möbius groups, Ford's definition of elementary groups is well known ([3]). However A. F. Beardon and T.Jørgensen defined independently the elementariness of Möbius groups with no assumption of discreteness ([1] and [4]). These three definitions of two or three dimensional Möbius groups can be directly extended to several dimensional groups. On the other hand G. J. Martin gave a more precise definition of elementariness of several dimensional Möbius groups ([5]). Recently A. N. Fang and Y. P. Jiang proved that Beardon's definition and Martin's definition are equivalent to each other ([2]). Here we show that Jørgensen's definition of elementarinesis is stronger than Martin's definition by constructing an example.

2 Elementary Möbius groups

For $n=2,3,4,\cdots$, let $M(B^{n})$ be the group of Möbius transformations acting on the unit ball B^{n} . For a discrete subgroup G of $M(B^{n})$, Ford's definition of the elementariness is well known.

(D-1) A discrete subgroup G is said to be F-elementary, if the limit set $\Lambda(G)$ for G consists of at most two points .

For any point $x\in Cl(B^{n})$, the closure of B^{n} in R^{n}, the orbit $G(x)$ of x is the subset of $Cl(B^{n})$ defined by

$$G(x)=\{g(x)\in Cl(B^{n}) \ \mid \ g\in G\}.$$

If there exists a point $x_{0}\in Cl(B^{n})$ so that $G(x_{0})$ is a finite set, we call that G has a finite orbit in $Cl(B^{n})$. Following two definitions are not based on discreteness.

(D-2) A subgroup G of $M(B^n)$ is said to be B-elementary, if G has a finite orbit in $Cl(B^n)$.

(D-3) A subgroup G of $M(B^n)$ is called J-elementary, if every two elements of infinite order of G have a common fixed point.

If G is discrete, we can see that these three definitions are equivalent to each other. In a view of several dimensional case, Martin gave the following definition. An element f of $M(B^n)$ is said to be an irrational rotation, if f is elliptic and ord(f), the order of f , is infinite.

(D-4) A subgroup G of $M(B^n)$ is called M-elementary, if every two elements of infinite order which are not irrational rotations have a common fixed point.

If G contains an irrational rotation, G is not discrete. An elliptic element of finite order is called a rational rotation. If G is discrete, we can easily see that definitions $(D-3)$ and $(D-4)$ are equivalent to each other. Recently A. N. Fang and Y. P. Jiang proved in [2] that $(D-2)$ and $(D-4)$ are equivalent to each other even if G is not discrete. In this note we show that the definition $(D-3)$ is essentially stronger than $(D-4)$ by constructing a group which is M-elementary, but J-elementary.

3 An example

We can construct examples for each $n \geq 4$. But it suffices to show the four dimensional case. For any matrix A, denote by A^T the transposed matrix of A. Let $R^4 = \{(x_1, x_2, x_3, x_4)^T \mid x_k \in R, k = 1, 2, 3, 4\}$ be the four dimensional Euclidean space and regarded as a direct product

$$R^4 = R_1 \times R_2 \times R_3 \times R_4,$$

where $R_1 = \{(x_1, 0, 0, 0)^T \mid x_1 \in R\}, R_2 = \{(0, x_2, 0, 0)^T \mid x_2 \in R\}$ and so on. The unit ball $B^4 = \{x \in R^4 \mid \ \| x \| < 1\}$ with the metric $ds^2 = dx^2/(1-|x|^2)^2$ is a model of the four dimensional hyperbolic space. Any element $g \in M(B^4)$ extends to a conformal automorphism of $Cl(B^4)$ and consequently have a fixed point in B^4 or on its boundary ∂B^4. We denote F_g by the set of fixed points of $g \in M(B^4)$ in $\overline{R^4} = R^4 \cup \{\infty\}$. Here we define an orthogonal matrix T_0 by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos\theta_0 & -\sin\theta_0 & 0 \\ 0 & \sin\theta_0 & \cos\theta_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\theta_0 \in R$ and $\theta_0/2\pi$ is irrational. This matrix is regarded as an element of $M(B^4)$. Since $\theta_0/2\pi$ is an irrational number, then $\mathrm{ord}(T_0)$ is infinite. So it follows that T_0 is an irrational rotation. Note that T_0 fixes every point of R_4 and invert a hyperbolic geodesic $R_1 \cap B^4$.

Now we define two spheres S_1 and S_2 by

$$S_1 = \{x \in R^4 \mid \ \| x - \sqrt{5}/2e_1 \| = 1/2\}$$
$$S_2 = \{x \in R^4 \mid \ \| x + \sqrt{5}/2e_1 \| = 1/2\}$$

,where $e_1 = (1,0,0,0)^T$. Then S_1 (resp. S_2) is a three dimensional sphere centered at $\sqrt{5}/2e_1$ (resp. $-\sqrt{5}/2e_1$) and orthogonal to $S^3 = \partial B^4$. Let f be a hyperbolic transformation which maps $Ext(S_1)$, the exterior of S_1, to $Int(S_2)$, the interior of S_2. We define two irrational rotations by

$$g_1 = T_0, \quad \text{and} \quad g_2 = f \circ T_0 \circ f^{-1}.$$

Then we have that $F_{g_1} = \overline{R_4}, F_{g_2} = f(\overline{R_4}), F_{g_1} \cap F_{g_2} = \emptyset$ and g_1 and g_2 invert a geodesic $R_1 \cap B^4$. In fact we can easily see $g_k(R_1 \cap B^4) = R_1 \cap B^4, g_k(e_1) = -e_1, g_k(-e_1) = e_1$ and $g_k^2|_{R_1} = Id$, the identity transformation for $k = 1,2$. Let G be the group generated by g_1, g_2. Since G contains irrational rotations g_1, g_2, then G is a non-discrete subgroup of $M(B^4)$. Two elements g_1, g_2 of infinite order have not a common fixed point. So we conclude that G is not J-elementary. Now we prove that G is M-elementary. To show this result, we need the following proposition.

PROPOSITION *If a non trivial element* $g \in M(B^n)$ *inverts a hyperbolic line* σ, g *is elliptic.*

PROOF Let $\zeta_1, \zeta_2 \in \partial B^n$ be the end points of σ. Since $g(\sigma) = \sigma, g(\zeta_1) = \zeta_2$ and $g(\zeta_2) = \zeta_1$, then $g^2(\zeta_k) = \zeta_k$ and $g^2(\sigma) = \sigma$ for $k = 1,2$. Any parabolic transformation cannot fix two distinct points. So g^2 is loxodromic or elliptic. Suppose that g^2 is loxodromic. Since g^2 fixes ζ_1, ζ_2, then σ is the axis of g^2. Hence g is loxodromic and fixes ζ_1 and ζ_2. It contradicts that g inverts σ. So g is elliptic.

REMARK

If a non-trivial element g inverts a hyperbolic line, g is an elliptic element of order two in two or three dimensional cases. But it is not valid when the dimension is greater than three. For example, T_0 invert a hyperbolic line $\sigma = R_4 \cap B^4$, but T_0 is an irrational rotation, not an elliptic element of order two.

Now we can see that every element g of G fixes $e_1, -e_1$ or exchange e_1 and $-e_1$. In any case g^2 fixes e_1 and $-e_1$. So G does not contain any parabolic element. If g is loxodromic, Proposition yields that g fixes e_1 and $-e_1$. Hence we conclude that any element of infinite order which is not an irrational rotation and fixes e_1 and $-e_1$ must be a loxodromic element. In fact such a loxodromic element exists in G. To show this, we need to define the element h by

$$h = g_1 \circ g_2 = T_0 \circ f \circ T_0 \circ f^{-1}.$$

We show that h is loxodromic. Since

$$h(\pm e_1) = T_0 \circ f \circ T_0 \circ f^{-1}(\pm e_1) = T_0 \circ f \circ T_0(\pm e_1) = T_0 \circ f(\mp e_1) = T_0(\mp e_1) = \pm e_1,$$

h fixes e_1 and $-e_1$. Suppose that h is an elliptic element. Every elliptic element has a fixed point in B^4. So there exists a point x_0 in B^4 so that $h(x_0) = x_0$. Let g_0 be a Möbius transformation which maps x_0 to the origin 0. Now we set $\tilde{h} = g_0 \circ h \circ g_0^{-1}$. Then the element $\tilde{h}$ fixes three points $0, g_0(e_1)$ and $g_0(-e_1)$. For any distinct points $\zeta_1, \zeta_2 \in Cl(B^4)$, we denote $\sigma(\zeta_1, \zeta_2)$ by the geodesic with the end points ζ_1, ζ_2. Since $\tilde{h}$ is an orthogonal matrix, every point in $\sigma(0, g_0(e_1))$ and $\sigma(0, g_0(-e_1))$ is fixed by $\tilde{h}$. It follows that the eigenspace of $\tilde{h}$ with the eigenvalue 1 contains $\sigma(0, g_0(e_1)), \sigma(0, g_0(-e_1))$ and so $\sigma(g_0(e_1), g_0(-e_1))$. Hence we conclude that $\tilde{h}(x) = x$ for any $x \in g_0(\sigma(e_1, -e_1))$. Therefore h fixes every point in $\sigma(e_1, -e_1) = R_1 \cap B^4$. But it cannot occur. To prove this fact, it suffices to show $h(0) \neq 0$. Note that the orthogonal transformation T_0 exchange two spheres S_1 and S_2. Since $0 \in Ext(S_2)$, then $f^{-1}(0) \in Int(S_1)$ and we have $T_0 \circ f^{-1}(0) \in Int(S_2)$. Note that $Int(S_2)$ is contained in $Ext(S_1)$. So $f \circ T_0 \circ f^{-1}(0) \in Int(S_2)$ and we have

$$h(0) = T_0 \circ f_0 \circ T_0 \circ f^{-1}(0) \in Int(S_1)$$

and so $h(0) \neq 0$. Since the origin 0 is contained in $\sigma(e_1, -e_1)$, then it contradicts the fact every point in $\sigma(e_1, -e_1)$ is fixed by h. So we conclude that the element h is loxodromic. Hence any two elements of infinite order which are not irrational rotations have same fixed point e_1 and $-e_1$. It means that G is M-elementary.

References

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