数理解析研究所講究録
1067 巻 1998 年 135-140

ERROR ESTIMATES OF THE REAL INVERSION FORMULAS OF THE LAPLACE TRANSFORM(abstract)

K. AMANO, S. SAITOH AND M. YAMAMOTO

群馬大工 天野 一男 (Kazuo Amano)
群馬大工 斎藤 三郎 (Saburou Saitoh)
東大数理 山本 昌宏 (Masahiro Yamamoto)

INTRODUCTION AND RESULTS

For any $q > 0$, we let L_q^2 be the class of all square integrable functions with respect to the measure $t^{1-2q}dt$ on the half line $(0, \infty)$. Then we consider the Laplace transform

$$[\mathcal{L}F](x) = \int_0^\infty F(t)e^{-xt}dt \quad (x > 0)$$

for $F \in L_q^2$. Then we have

Proposition 1 ([2, 5]). *For any fixed $q > 0$ and for any function $F \in L_q^2$, put $f = \mathcal{L}F$. Then the inversion formula*

$$F(t) = s - \lim_{N\to\infty} \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx \qquad (t > 0) \tag{1}$$

is valid, where the limit is taken in the space L_q^2 and the polynomials $P_{N,q}$ are given by the formulas

$$P_{N,q}(\xi) = \sum_{0\le\nu\le n\le N} \frac{(-1)^{\nu+1}\Gamma(2n+2q)}{\nu!(n-\nu)!\Gamma(n+2q+1)\Gamma(n+\nu+2q)}\xi^{n+\nu+2q-1} \times \left\{ \frac{2(n+q)}{n+\nu+2q}\xi^2 - \left(\frac{2(n+q)}{n+\nu+2q} + 3n + 2q \right) \xi + n(n+\nu+2q) \right\}.$$

Moreover the series

$$\sum_{n=0}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n[xf'(x)]|^2 x^{2n+2q-1}dx \tag{2}$$

converges and the truncation error is estimated by the inequality

$$\left\| F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx \right\|_{L_q^2}^2 \le \sum_{n=N+1}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \left|\partial_x^n[xf'(x)]\right|^2 x^{2n+2q-1}dx. \tag{3}$$

Some characteristics of the strong singularity of the polynomials $P_{N,1}(\xi)$ and some effective algorithms for the real inversion formula in Proposition 1 are examined by J. Kajiwara and M. Tsuji [3, 4] and K. Tsuji [6]. Furthermore they gave numerical experiments by using computers.

In connection with the integral in (2) we have

Proposition 2 ([5], Chapter 5). *Let $q > 0$ be arbitrary and let $F \in L^2_q$. For the Laplace transform $\mathcal{L}F = f$, we have the isometrical identity*

$$\int_0^\infty |F(t)|^2 t^{1-2q} dt = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n [xf'(x)]|^2 x^{2n+2q-1} dx. \tag{4}$$

Moreover the image $f = \mathcal{L}F$ belongs to the Bergman-Selberg space $H_q(R^+)$ on the right half complex plane $R^+ = \{Re\ z > 0\}$ admitting the reproducing kernel

$$K_q(z, \overline{u}) = \frac{\Gamma(2q)}{(z+\overline{u})^{2q}}$$

and comprising analytic functions on R^+. For $q > \frac{1}{2}$, we can characterize

$$\begin{aligned} H_q(R^+) = &\{f : f \quad \text{analytic on} \quad R^+, \\ &\frac{1}{\Gamma(2q-1)\pi} \iint_{R^+} |f(z)|^2 (2x)^{2q-2} dx dy < \infty\} \end{aligned}$$

and for $q = \frac{1}{2}$

$$\begin{aligned} H_{\frac{1}{2}}(R^+) = &\{f : f \quad \text{analytic on} \quad R^+, \\ &\lim_{x \to +0} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x+iy)|^2 dy < \infty\}. \end{aligned}$$

Moreover for any $q > 0$, we have the representation of the norm in $H_q(R^+)$

$$\|f\|^2_{H_q(R^+)} = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n (xf'(x))|^2 x^{2n+2q-1} dx. \tag{5}$$

Now we can state our main results.

Theorem 1. *We assume that*

$$\max\left(\frac{1}{2}, 2q-1\right) < \alpha < 1, \tag{6}$$

and

(7) $$\alpha \leq \beta < q + \frac{\alpha}{2}.$$

If $f \in H_q(R^+)$ *and*

(8) $$f(z)z^\beta \in H_{q+\frac{\alpha}{2}-\beta}(R^+),$$

then the following error estimate holds

(9) $$\left| F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx \right| = t^{q-1+\frac{\alpha}{2}} o\left(N^{\frac{1-2\alpha}{4}}\right)$$

as $N \to \infty$.

Next we give a sufficient condition for F whose Laplace transform satisfies (8).

Theorem 2. *Let us assume (7). We further assume*

(10) $$q + \frac{\alpha}{2} > 1.$$

If

(11) $$F \in C^2[0, \infty),$$

(12) $$F(0) = F'(0) = 0,$$

and

(13) $$F'(t) = O(t^{-\delta}), \quad t > 0$$

for

(14) $$2 - q - \frac{\alpha}{2} < \delta < 1,$$

then (8) holds.

Note that from (12) and (13)

(15) $$\lim_{t\to\infty} e^{-xt}F(t) = \lim_{t\to\infty} e^{-xt}F'(t) = 0, \quad x > 0.$$

Finally, we characterize F whose Laplace transform satisfies (8).

Theorem 3. *If $f = \mathcal{L}F$ satisfies (8), then there exists $h \in L^2_{q+\frac{\alpha}{2}-\beta}$ such that (7) is true and*

(16) $$F(t) = \int_0^t h(x)(t-x)^{\beta-1}dx.$$

A real inversion formula for the Laplace transform is known (eg. Widder [7], page 386), which is different from ours. However it seems that no error estimates in the truncation are known.

PRELIMINARIES

First we shall give

Lemma. *If $f \in C^\infty(0,\infty)$ and*

(17) $$I_{q,\alpha}(f) := \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n[xf'(x)]|^2 x^{2n+2q-1+\alpha}dx < \infty,$$

for fixed

(18) $$\max\left(\frac{1}{2}, 2q-1\right) < \alpha,$$

then

(19) $$\begin{aligned}\Bigg| \sum_{n=N+1}^{\infty} & \frac{1}{n!\Gamma(n+2q+1)} \\ & \times \int_0^\infty \partial_x^n[xf'(x)]\partial_x^n(x\partial_x(e^{-tx}))x^{2n+2q-1}dx\Bigg| \\ = & t^{\frac{\alpha-2q}{2}} o\left(N^{\frac{1-2\alpha}{4}}\right),\end{aligned}$$

as $N \to \infty$.

CONCLUDING REMARKS

(1) The conditions (12) and (13) are not essential if we know $F(0)$ and $F'(0)$, and we can assume that

(20) $$|F(t)|, |F'(t)| \leq O(e^{kt}) \quad \text{for} \quad t > 0 \quad \text{with} \quad k > 0.$$

In fact, we set

(21) $$\tilde{F}(t) = (F(t) - F(0) - F'(0)t)e^{-2kt}, \quad t > 0.$$

Then $\tilde{F}$ satisfies (12) and (13).

On the other hand,

$$(\mathcal{L}\tilde{F})(z) = f(z+2k) - \frac{F(0)}{z+2k} - \frac{F'(0)}{(z+2k)^2}. \tag{22}$$

Thus we first apply Theorems 1 and 2 to this function (22) so that we can obtain approximations $\tilde{F}_N(t)$ for $\tilde{F}(t)$:

$$|\tilde{F}(t) - \tilde{F}_N(t)| = t^{q-1+\frac{\alpha}{2}} o(N^{\frac{1-2\alpha}{4}}). \tag{23}$$

We set

$$\hat{F}_N(t) = \tilde{F}_N(t)e^{2kt} + F(0) + F'(0)t, \quad \text{for} \quad t > 0. \tag{24}$$

Then we have

$$|F(t) - \hat{F}_N(t)| = e^{2kt}|\tilde{F}(t) - \tilde{F}_N(t)| = e^{2kt}t^{q-1+\frac{\alpha}{2}} o(N^{\frac{1-2\alpha}{4}}). \tag{25}$$

Thus we can obtain error estimates in any finite interval in t, which however breakes as $t \to \infty$.

(2) Since a typical member of the Bergman-Selberg space $H_q(R^+)$ is the reproducing kernel $K_q(z, \overline{u})$, we see that typical functions f satisfying (17) are given by

$$f(z) = \frac{z^{-\beta}}{(z+\overline{u})^{2q+\alpha-2\beta}}, \quad \text{Re}\, u > 0 \tag{26}$$

for α and β satisfying (7). From the identities (16) and

$$K_{q+\frac{\alpha}{2}-\beta}(z, \overline{u}) = \int_0^\infty e^{-tz} e^{-t\overline{u}} t^{2q+\alpha-2\beta-1} dt,$$

we see that the Laplace transform of the functions

$$\int_0^t e^{-x\overline{u}} x^{2q+\alpha-2\beta-1}(t-x)^{\beta-1} dx, \quad \text{Re}\, u > 0, \ \beta > 1 \tag{27}$$

satisfies the property (17).

(3) As functions F where $f = \mathcal{L}F$ satisfies the conditions in Theorem 1, we consider Dirichlet series

$$F(t) = \sum_{k=1}^{\infty} C_k t^{\gamma-1} e^{-a_k t} \quad (a_k > 0, \gamma \geq 1), \tag{28}$$

where

$$\sum_{k=1}^{\infty} |C_k| a_k^{q-\gamma} < \infty, \quad \sum_{k=1}^{\infty} |C_k| a_k^{q+\frac{\alpha}{2}-\gamma} < \infty, \quad \gamma > q + \frac{\alpha}{2} > 0. \tag{29}$$

Then $F \in L_q^2$ and $f = \mathcal{L}F$ satisfies (8) for β satisfying (7).

ACKNOWLEDGEMENT

This research was partially supported by the Japanese Ministry of Education, Science, Sports and Culture; Grant-in-Aid Scientific Research, Kiban Kenkyuu (A)(1), 10304009.

REFERENCES

1 M. Abramowitz and I. A. Stegun Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables Dover Publications New York 1972

2 D.-W. Byun and S. Saitoh A real inversion formula for the Laplace transform Zeitschrift für Analysis und ihre Anwendungen 12 1993 597-603

3 J. Kajiwara and M. Tsuji Program for the numerical analysis of inverse formula for the Laplace transform Proceedings of the Second Korean-Japanese Colloquium on Finite and Infinite Dimensional Complex Analysis 1994 93-107

4 J. Kajiwara and M. Tsuji, Inverse formula for Laplace transform Proceedings of the 5th International Colloquium on Differential Equations, pp.163-172 VSP-Holland 1995

5 S. Saitoh Integral Transforms, Reproducing Kernels and Their Applications Pitman Research Notes in Mathematics Series, 369, Addison Wesley Longman UK 1997

6 K. Tsuji An algorithm for sum of floating point numbers without rounding error In Abstracts of the Third International Colloquium on Numerical Analysis Bulgaria 1995

7 D. V. Widder The Laplace Transform Princeton University Press Princeton 1946

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING, GUNMA UNIVERSITY, KIRYU 376-8515, JAPAN

E-mail address: kamano@math.sci.gunma-u.ac.jp

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING, GUNMA UNIVERSITY, KIRYU 376-8515 JAPAN

E-mail address: ssaitoh@eg.gunma-u.ac.jp

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, TOKYO 153-8914, JAPAN

E-mail address: myama@ms.u-tokyo.ac.jp