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ERROR ESTIMATES OF THE REAL INVERSION FORMULAS
OF THE LAPLACE TRANSFORM (abstract)
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INTRODUCTION AND RESULTS

For any $q > 0$, we let $L^2_q$ be the class of all square integrable functions with respect to the measure $t^{1-2q}dt$ on the half line $(0, \infty)$. Then we consider the Laplace transform

$$[LF](x) = \int_0^\infty F(t)e^{-xt}dt \quad (x > 0)$$

for $F \in L^2_q$. Then we have

Proposition 1 ([2, 5]). For any fixed $q > 0$ and for any function $F \in L^2_q$, put $f = LF$. Then the inversion formula

$$F(t) = s - \lim_{N \to \infty} \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx \quad (t > 0)$$

is valid, where the limit is taken in the space $L^2_q$ and the polynomials $P_{N,q}$ are given by the formulas

$$P_{N,q}(\xi) = \sum_{0 \leq \nu \leq n \leq N} \frac{(-1)^{\nu+1}\Gamma(2n + 2q)}{\nu!(n-\nu)!\Gamma(n+2q+1)\Gamma(n+\nu+2q)}\xi^{n+\nu+2q-1}$$

$$\times \left\{ \frac{2(n+q)}{n+\nu+2q} \xi^2 - \left( \frac{2(n+q)}{n+\nu+2q} + 3n + 2q \right) \xi + n(n+\nu+2q) \right\}.$$

Moreover the series

$$\sum_{n=0}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n [xf'(x)]|^2 x^{2n+2q-1}dx$$

converges and the truncation error is estimated by the inequality

$$\left\| F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx \right\|^2_{L^2_q} \leq \sum_{n=N+1}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n [xf'(x)]|^2 x^{2n+2q-1}dx.$$
Some characteristics of the strong singularity of the polynomials $P_{N,1}(\xi)$ and some effective algorithms for the real inversion formula in Proposition 1 are examined by J. Kajiwara and M. Tsuji [3, 4] and K. Tsuji [6]. Furthermore they gave numerical experiments by using computers.

In connection with the integral in (2) we have

**Proposition 2** ([5], Chapter 5). Let $q > 0$ be arbitrary and let $F \in L_{q}^{2}$. For the Laplace transform $\mathcal{L}F = f$, we have the isometric identity

\[
\int_{0}^{\infty} |F(t)|^{2} t^{1-2q} dt = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + 2q + 1)} \int_{0}^{\infty} |\partial_{x}^{n} (xf'(x))|^{2} x^{2n+2q-1} dx.
\]

Moreover the image $f = \mathcal{L}F$ belongs to the Bergman-Selberg space $H_{q}(R^{+})$ on the right half complex plane $R^{+} = \{\text{Re } z > 0\}$ admitting the reproducing kernel

\[
K_{q}(z, \overline{u}) = \frac{\Gamma(2q)}{(z + \overline{u})^{2q}}
\]

and comprising analytic functions on $R^{+}$. For $q > \frac{1}{2}$, we can characterize

\[
H_{q}(R^{+}) = \{f : f \text{ analytic on } R^{+}, \frac{1}{\Gamma(2q-1)\pi} \iint_{R^{+}} |f(z)|^{2} (2\pi)^{2q-2} dx dy < \infty\}
\]

and for $q = \frac{1}{2}$

\[
H_{\frac{1}{2}}(R^{+}) = \{f : f \text{ analytic on } R^{+}, \lim_{x \to +0} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x + iy)|^{2} dy < \infty\}.
\]

Moreover for any $q > 0$, we have the representation of the norm in $H_{q}(R^{+})$

\[
\|f\|_{H_{q}(R^{+})}^{2} = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + 2q + 1)} \int_{0}^{\infty} |\partial_{x}^{n} (xf'(x))|^{2} x^{2n+2q-1} dx.
\]

Now we can state our main results.

**Theorem 1.** We assume that

\[
\max \left(\frac{1}{2}, 2q - 1\right) < \alpha < 1,
\]
and
\[ \alpha \leq \beta < q + \frac{\alpha}{2}. \]

If \( f \in H_q(R^+) \) and
\[ f(z)z^\beta \in H_{q+\frac{\alpha}{2} - \beta}(R^+), \]
then the following error estimate holds
\[ |F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx| = t^{q-1+\frac{\alpha}{2}}O\left(N^{\frac{1-2\alpha}{4}}\right) \]
as \( N \to \infty. \)

Next we give a sufficient condition for \( F \) whose Laplace transform satisfies (8).

**Theorem 2.** Let us assume (7). We further assume
\[ q + \frac{\alpha}{2} > 1. \]
If
\[ F \in C^2[0, \infty), \]
\[ F(0) = F'(0) = 0, \]
and
\[ F'(t) = O(t^{-\delta}), \quad t > 0 \]
for
\[ 2 - q - \frac{\alpha}{2} < \delta < 1, \]
then (8) holds.

Note that from (12) and (13)
\[ \lim_{t \to \infty} e^{-xt}F(t) = \lim_{t \to \infty} e^{-xt}F'(t) = 0, \quad x > 0. \]

Finally, we characterize \( F \) whose Laplace transform satisfies (8).
Theorem 3. If $f = LF$ satisfies (8), then there exists $h \in L^2_{q+\alpha-\beta}$ such that (7) is true and

\[(16) \quad F(t) = \int_0^t h(x)(t-x)^\alpha-1\,dx.\]

A real inversion formula for the Laplace transform is known (e.g. Widder [7], page 386), which is different from ours. However it seems that no error estimates in the truncation are known.

PRELIMINARIES

First we shall give

Lemma. If $f \in C^\infty(0, \infty)$ and

\[(17) \quad I_{q,\alpha}(f) := \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n[f(x)]|^2x^{2n+2q-1+\alpha} \,dx < \infty,\]

for fixed

\[(18) \quad \max(\frac{1}{2}, 2q-1) < \alpha,\]

then

\[(19) \quad \left| \sum_{n=N+1}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \partial_x^n[f(x)]\partial_x^n(x\partial_x(e^{-tx}))x^{2n+2q-1} \,dx \right| = t^{\frac{\alpha-2q}{2}} o(N^{1-2\alpha}),\]

as $N \to \infty$.

CONCLUDING REMARKS

(1) The conditions (12) and (13) are not essential if we know $F(0)$ and $F'(0)$, and we can assume that

\[(20) \quad |F(t)|, |F'(t)| \leq O(e^{kt}) \quad \text{for} \quad t > 0 \quad \text{with} \quad k > 0.\]

In fact, we set

\[(21) \quad \tilde{F}(t) = (F(t) - F(0) - F'(0)t)e^{-2kt}, \quad t > 0.\]
Then $\tilde{F}$ satisfies (12) and (13).

On the other hand,

\begin{equation}
(\mathcal{L}\tilde{F})(z) = f(z+2k) - \frac{F(0)}{z+2k} - \frac{F'(0)}{(z+2k)^2}.
\end{equation}

Thus we first apply Theorems 1 and 2 to this function (22) so that we can obtain approximations $\tilde{F}_N(t)$ for $\tilde{F}(t)$:

\begin{equation}
|\tilde{F}(t) - \tilde{F}_N(t)| = t^{q-1+\frac{\alpha}{2}} o(N^{\frac{1-2\alpha}{4}}).
\end{equation}

We set

\begin{equation}
\hat{F}_N(t) = \tilde{F}_N(t)e^{2kt} + F(0) + F'(0)t, \quad \text{for } t > 0.
\end{equation}

Then we have

\begin{equation}
|F(t) - \hat{F}_N(t)| = e^{2kt}|\tilde{F}(t) - \tilde{F}_N(t)| = e^{2kt}t^{q-1+\frac{\alpha}{2}} o(N^{\frac{1-2\alpha}{4}}).
\end{equation}

Thus we can obtain error estimates in any finite interval in $t$, which however breaks as $t \to \infty$.

(2) Since a typical member of the Bergman-Selberg space $H_q(R^+)$ is the reproducing kernel $K_q(z, \overline{u})$, we see that typical functions $f$ satisfying (17) are given by

\begin{equation}
f(z) = \frac{z^{-\beta}}{(z + \overline{u})^{2q+\alpha-2\beta}}, \quad \text{Re } u > 0
\end{equation}

for $\alpha$ and $\beta$ satisfying (7). From the identities (16) and

\[ K_{q+\frac{\alpha}{2}-\beta}(z, \overline{u}) = \int_0^\infty e^{-tx}e^{-t\overline{u}x}x^{2q+\alpha-2\beta-1}dx, \quad \text{Re } u > 0, \beta > 1
\]

we see that the Laplace transform of the functions

\begin{equation}
\int_0^t e^{-z\overline{u}x}x^{2q+\alpha-2\beta-1}(t-x)^{\beta-1}dx, \quad \text{Re } u > 0
\end{equation}

satisfies the property (17).

(3) As functions $F$ where $f = \mathcal{L}F$ satisfies the conditions in Theorem 1, we consider Dirichlet series

\begin{equation}
F(t) = \sum_{k=1}^\infty C_k t^{\gamma-1}e^{-a_k t} \quad (a_k > 0, \gamma \geq 1),
\end{equation}

where

\begin{equation}
\sum_{k=1}^\infty |C_k|a_k^{\gamma-\gamma} < \infty, \quad \sum_{k=1}^\infty |C_k|a_k^{\frac{q}{2}+\frac{\alpha}{2}-\gamma} < \infty, \quad \gamma > q + \frac{\alpha}{2} > 0.
\end{equation}
Then $F \in L_q^2$ and $f = \mathcal{L}F$ satisfies (8) for $\beta$ satisfying (7).

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