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<th>$\overline{\partial}$-PROBLEMS AND SOME APPLICATIONS (Reproducing Kernels and their Applications)</th>
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Kyoto University
0. Preliminaries.

Let $D$ be a bounded domain in $\mathbb{C}^n$ with $C^1$ boundary. We denote by $C^{1,\infty}(\partial D \times D)$ the space of all functions in $C^1(\partial D \times D)$ which are $C^\infty$ in the second variable. A $(1,0)$-form $W = \sum_{j=1}^{n} w_j(\zeta, z) d\zeta_j$ is called a generating form with coefficients in $C^{1,\infty}(\partial D \times D)$ if $W$ satisfies the following conditions (1) and (2):

1. $w_j(\zeta, z) \in C^{1,\infty}(\partial D \times D)$.
2. $\sum_{j=1}^{n} w_j(\zeta, z)(\zeta_j - z_j) = 1$.

We define $\beta = |\zeta - z|^2$, $B = \frac{\partial \beta}{\beta}$, $I = [0,1]$.

The homotopy form on $(\partial D \times I) \times D$ associated to $W$ is defined by

$$\hat{W}(\zeta, \lambda, z) = \lambda W(\zeta, z) + (1 - \lambda)B(\zeta, z).$$

Cauchy-Fantappié kernel $\Omega_q(\hat{W})$ of order $q$ generated by $\hat{W}$ is defined by

$$\Omega_q(\hat{W}) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \left( \begin{array}{c} n-1 \\ q \end{array} \right) \hat{W} \wedge (\bar{\partial} \hat{W})^{n-q} \wedge (\partial \hat{W})^q, \quad 0 \leq q \leq n-1.$$ 

$\Omega_q(W)$ is defined in the same way, with $W$ instead of $\hat{W}$. We define $K_q = \Omega_q(B)$.

Then we have the Cauchy-Fantappié integral formula(cf. Range[22]):

**Theorem 1.** For $1 \leq q \leq n$, define the linear operator

$$T_q^W : C_{0,q}(\overline{D}) \to C_{0,q-1}(D)$$

by

$$T_q^W f = \int_{\partial D \times I} f \wedge \Omega_{q-1}(\hat{W}) - \int_D f \wedge K_{q-1},$$

and set $T_0^W = T_{n+1}^W = 0$. Then the following holds:

(a) For $k = 0, 1, \cdots, \infty$, if $f \in C_{0,q}^k(D) \cap C_{0,q}(\overline{D})$, then $T_q^W f \in C_{0,q-1}^k(D)$.

(b) For $0 \leq q \leq n$, if $f \in C_{0,q}^1(\overline{D})$, then

$$f = \int_{\partial D} f \wedge \Omega_q(W) + \bar{\partial} T_q^W f + T_q^{W+1} f \quad \text{on} \quad D.$$
Remark. If $W = \sum_{j=1}^{n} w_j(\zeta, z) d\zeta_j$ is holomorphic in $z$, then $\Omega_q(W) = 0$ for $q \geq 1$. In this case, if $f$ is a $\bar{\partial}$-closed $(0, q)$-form, then it holds that $f = \bar{\partial}(T_q W f)$.

In 1965, Hörmander obtained $L^2$ estimates for solutions of the $\bar{\partial}$-problem in bounded pseudoconvex domains in $\mathbb{C}^n$. On the other hand, $L^p$ and Hölder estimates for solutions of the $\bar{\partial}$-problem using the above integral formula have been studied since 1970. We begin with the $\bar{\partial}$-problem in strictly pseudoconvex domains in $\mathbb{C}^n$.

1. $\bar{\partial}$-problems in bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary.

**Theorem 2.** (Henkin[10], Ramirez[19]) Suppose $D \subset \mathbb{C}^n$ is strictly pseudoconvex with $C^\infty$ boundary. There are a neighborhood $U$ of $\partial D$, positive constants $\delta, c$ and $\gamma$, and a function $g \in C^\infty(U \times D_\delta)$ with the following properties:

(i) $g(\zeta, z)$ is holomorphic in $z$ on $D_\delta$.
(ii) $g(\zeta, \zeta) = 0$ for $\zeta \in U$.
(iii) $\text{Reg}(\zeta, z) > 0$ for $(\zeta, z) \in U \times D_\delta$ with $r(\zeta) - r(z) + c|\zeta - z|^2 > 0$.
(iv) On $\{((\zeta, z) \in U \times D_\delta : |\zeta - z| \leq \gamma\}$ there is a function $A \in C^\infty(U \times D_\delta)$ with $|A(\zeta, z)| \geq \frac{2}{3}$, so that $g = FA$, where $F$ is the Levi polynomial.

Using Hefer's theorem, there are functions $g_j \in C^\infty(U \times D_\delta)$, with $g_j(\zeta, \cdot) \in O(D_\delta)$ such that

$$g(\zeta, z) = \sum_{j=1}^{n} g_j(\zeta, z)(\zeta_j - z_j) \quad \text{on} \quad U \times D_\delta$$

We define

$$W^{HR} = \sum_{j=1}^{n} \frac{g_j(\zeta, z)}{g(\zeta, z)} d\zeta_j.$$ 

Then $W^{HR}$ is called the Henkin-Ramirez generating form. Using the above Henkin-Ramirez generating form, the following theorem was obtained (cf. Henkin[10], Kerzman[13], Lieb[16], Henkin-Romanov[11], Grauert-Lieb[9], Range-Siu[25]).

**Theorem 3.** Let $D \subset \mathbb{C}^n$ be strictly pseudoconvex with smooth boundary. For $1 \leq q \leq n$, there are linear operators

$$\hat{S}_q : L^1_{0,q}(D) \rightarrow L^1_{0,q-1}(D)$$

and a constant $C$ with the following properties:

(i) $\|\hat{S}_q f\|_{L^p(D)} \leq C\|f\|_{L^p(D)}$ for $1 \leq p \leq \infty$.
(ii) $\|\hat{S}_q f\|_{A_{1/2}(D)} \leq C\|f\|_{L^\infty(D)}$.
(iii) For $k = 0, 1, 2, \ldots$, if $f \in L^1_{0,q}(D) \cap C^k(D)$, then $\hat{S}_q f \in C^k_{0,q-1}(D)$.
(iv) If $f \in C^1_{0,q}(D) \cap L^1_{0,q}(D)$ and $\bar{\partial}f = 0$, then $\bar{\partial}(\hat{S}_q f) = f$ on $D$. 


Krantz[14] obtained the optimal Lipschitz and $L^p$ estimates for $\bar{\partial}$ in strictly pseudoconvex domains:

**Theorem 4.** Let $D$ be a bounded strictly pseudoconvex domain with $C^5$ boundary. Let $A_{(0,1)}^\infty(D)$ be the space of all $\bar{\partial}$-closed $(0,1)$-forms $f$ whose coefficients are $C^\infty$ in $\overline{D}$. Then there is a linear operator

$$H : A_{(0,1)}^\infty(D) \to C^\infty(D)$$

satisfying $\bar{\partial}Hf = f$. Moreover $Hf$ satisfies

(i) $||Hf||_{L^\infty((2n+2)/(2n+1))} \leq A_\varepsilon ||f||_{L^1}$ for small enough $\varepsilon > 0$

(ii) if $1 < p < 2n+2$, then $||Hf||_{L^p} \leq A_p ||f||_{L^p}$,

(iii) if $2n+2 < p \leq \infty$, then $||Hf||_{\Lambda_{\alpha}} \leq A_p ||f||_{L^p}$, where $\frac{1}{\Lambda} = \frac{1}{p} - \frac{1}{2n+2}$.

For $i \in \{1, \cdots, N\}$, we denote by $D_i$ a strictly pseudoconvex open sets in $\mathbb{C}^n$ with $C^2$ boundary. Let $\rho_i$ be a defining function for $D_i$. For sufficiently small $\delta_i > 0$ we denote $V_i^\delta = \{-\delta < \rho_i(z) < \delta\}$. We assume that for $1 \leq i_1 < i_2 < \cdots < i_l \leq N$, $d\rho_{i_1}, d\rho_{i_2}, \cdots, d\rho_{i_l}$ are $\mathbb{R}$-linearly independent at all points of $V_{i_1}^\delta \cap V_{i_2}^\delta \cap \cdots \cap V_{i_l}^\delta$. We set $D = \bigcap_{i=1}^N D_i$. Then Menini[17] proved the following:

**Theorem 5.** Let $f \in L^p_{0,q}(D)(1 \leq q \leq n, 1 \leq p \leq \infty)$ be $\bar{\partial}$-closed. Then there exists a kernel $K$ such that if one defines on $D$, $T_qf(z) = c_{q,n} \int_D f(\zeta) \wedge K(\zeta, z)$ then $\bar{\partial}(T_qf) = f$. Moreover

(i) for $1 \leq p < \infty$,

$$T_q : L^p_{(0,q)}(D) \to L^r_{0,q-1}(D)$$

is a bounded linear operator where $\frac{1}{r} = \frac{1}{p} + \frac{1}{1+\eta}$, $0 \leq \eta < \frac{1}{2n-1+2\inf(N_0,n-1)}$, where $N_0$ is the maximal number of the common intersections,

(ii) for $p = \infty$,

$$T_q : L^p_{(0,q)}(D) \to \Lambda^{1/2-\varepsilon}_{(0,q-1)}(D)$$

is a bounded linear operator for any $\varepsilon > 0$.

2. $\bar{\partial}$-problems in $q$-convex domains in a complex manifold.

**Theorem 6.** (Fischer-Lieb[7]) Let $X$ be a complex manifold and let $D \subset X$ be a strongly $q$-convex domain (in the sense of Andreotti-Grauert) with $C^3$ boundary. Then there exists a constant $K$ with the following properties:
For each $\overline{\partial}$-closed $(0, r)$-form $\beta$ on $D$ with $r \geq q$ there exists a $(0, r - 1)$-form $\alpha$ on $D$ with $\overline{\partial}\alpha = \beta$ and $|\alpha| \leq K|\beta|$.

Let $X$ be an $n$-dimensional complex manifold. $D \Subset X$ is called a strictly $q$-convex $C^2$ intersection if there exists a finite number of real $C^2$ functions $\rho_1, \cdots, \rho_N$ in a neighborhood $U$ of $\overline{D}$ such that

$$D = \{ z \in U : \rho_j(z) < 0 \text{ for } 1 \leq j \leq N \}$$

and the following condition is fulfilled: if $z \in \partial D$ and $1 \leq k_1 < \cdots < k_l \leq N$ with $\rho_{k_1}(z) = \cdots = \rho_{k_l}(z) = 0$, then

$$d\rho_{k_1}(z) \wedge \cdots \wedge d\rho_{k_l}(z) \neq 0$$

and, for all $\lambda_1, \cdots, \lambda_l \geq 0$ with $\lambda_1 + \cdots + \lambda_l = 1$, the Levi form at $z$ of the function

$$\lambda_1 \rho_{k_1} + \cdots + \lambda_l \rho_{k_l}$$

has at least $q + 1$ positive eigenvalues. $D$ is called completely $q$-convex if there exists a real $C^2$ function $\varphi$ on $D$ whose Levi form has at least $q + 1$ positive eigenvalues at each point in $D$ and such that

$$\{ z \in D : \varphi(z) < C \} \subset \subset D \text{ for all } C > 0.$$ 

Let $E$ be a holomorphic vector bundle over $X$. Denote by $B_{n,r}^\beta(D, E), \beta \geq 0, r = 0, 1, \cdots, n$, the Banach space of $E$-valued continuous $(n, r)$-forms $f$ on $D$ such that

$$\sup_{z \in D} ||f(z)||[\text{dist}(z, \partial D)]^\beta < \infty,$$

and denote by $C_{n,r}^\alpha(\overline{D}, E), 0 \leq \alpha \leq 1, r = 0, 1, \cdots, n$, the Banach space of $E$-valued $(n, r)$-forms which are Hölder continuous with exponent $\alpha$ on $\overline{D}$. In this setting, Laurent-Thiébaut-Leiterer[15] proved the following:

**Theorem 7.** Let $D \Subset X$ be a strictly $q$-convex $C^2$ intersection and completely $q$-convex. Then:

(i) If $0 \leq \beta < \frac{1}{2}$, then there exist linear operators

$$T_r : B_{n,r}^\beta(D, E) \cap \ker d \rightarrow \cap_{0 < \epsilon \leq 1/2 - \beta} C_{n,r-1}^{1/2 - \beta - \epsilon}(D, E),$$

$n - q \leq r \leq n$, which are compact as operators from $B_{n,r}^\beta(D, E) \cap \ker d$ to each $C_{n,r-1}^{1/2 - \beta - \epsilon}(D, E), 0 < \epsilon \leq 1/2 - \beta$, and such that

$$dT_r f = f.$$
for all \( n-q \leq r \leq n \) and \( f \in B_{n,r}^\beta(D, E) \cap \ker d \).

(ii) If \( 1/2 \leq \beta < 1 \), then there exist linear operators

\[
T_r : B_{n,r}^\beta(D, E) \cap \ker d \rightarrow \bigcap_{\epsilon > 0} B_{n,\beta,\epsilon}^{1/2}(D, E),
\]

\( n-q \leq r \leq n \), which are compact as operators from \( B_{n,r}^\beta(D, E) \cap \ker d \) to each \( B_{n,r-1}^{\beta+1/2}(D, E) \), \( \epsilon > 0 \), and such that

\[
dT_r f = f
\]

for all \( n-q \leq r \leq n \) and \( f \in B_{n,r}^\beta(D, E) \cap \ker d \).

3. \( \overline{\partial} \)-problems in bounded weakly pseudoconvex domains in \( \mathbb{C}^n \).

In the case of weakly pseudoconvex domains there are several results in \( \mathbb{C}^2 \).

Theorem 8. (Range[21]) Let \( D \subset \mathbb{C}^2 \) be a bounded convex domain with real analytic boundary. Then there are positive constants \( \alpha \) and \( K \) such that for every bounded \( \overline{\partial} \)-closed \( f \in C_{0,1}^{1}(\overline{D}) \) there is \( u \in C^{1}(D) \) such that \( \overline{\partial}u = f \) and

\[
|u(z) - u(z')| \leq K\|f\|_{L^\infty(D)}|z-z'|^{\alpha}, \quad z, z' \in D.
\]

Theorem 9. (Show[26]) Let \( D \) be a pseudoconvex domain in \( \mathbb{C}^2 \) of uniform strict type \( m \). Let \( f \) be a continuous \((0,1)\)-form on \( \overline{D} \) and \( \overline{\partial} f = 0 \), then there exists a function \( u \in \Lambda_{1/m}(\overline{D}) \) such that \( \overline{\partial}u = f \) and \( u \) satisfies the following estimates:

(i) \( \|u\|_{L^1(D)} \leq c(\|f\|_{L^1(D)} + \|f\|_{L^1(\partial D)}) \),

(ii) if \( p = 1 \), then \( \|u\|_{L^p(\partial D)} \leq c\|f\|_{L^1(\partial D)} \) for every small \( \epsilon > 0 \),

(iii) if \( 1 < p < m+2 \), then \( \|u\|_{L^p(\partial D)} \leq c_p\|f\|_{L^p(\partial D)} \) where \( \frac{1}{p} = \frac{1}{m} - \frac{1}{m+2} \),

(iv) if \( p = m+2 \), then \( \|u\|_{L^q(\partial D)} \leq c_p\|f\|_{L^p(\partial D)} \) for all \( q < \infty \),

(v) if \( m+2 < p \leq \infty \), then \( \|u\|_{\Lambda_{1/(m+2)/mp}(\partial D)} \leq c_p\|f\|_{L^p(\partial D)} \),

(vi) \( \|u\|_{\Lambda^p_1(\partial D)} \leq c_p\|f\|_{L^p(\partial D)} \) for every \( 1 \leq p \leq \infty \).

Theorem 10. (Range[23]) Let \( D \) be a smoothly bounded pseudoconvex domain in \( \mathbb{C}^2 \) of finite type \( m \), and let \( f \in C_{0,1}^{1}(\overline{D}) \) be \( \overline{\partial} \)-closed. Then for every \( \eta > 0 \) there is a solution \( u^{(\eta)} \) of \( \overline{\partial}u = f \) on \( D \) which satisfies

\[
|u^{(\eta)}(z) - u^{(\eta)}(w)| \leq C_\eta\|f\|_{L^\infty}|z-w|^{1/m-\eta}
\]

for \( z, w \in D \).
Theorem 11. (Polking[18]) Let $D \Subset \mathbb{C}^2$ be convex with $C^2$ boundary. Then there is an integral solution operator $T$ for $\overline{\partial}$ on $D$ such that $\|Tf\|_{L^p(D)} \leq C_p \|f\|_{L^p(D)}$ for all $1 < p < \infty$.

Theorem 12. (Range[24]) Let $D \Subset \mathbb{C}^2$ be convex with $C^2$ boundary. Then there is an integral solution operator $T$ for $\overline{\partial}$ on $D$ such that

(i) $|Tf|_{\Lambda(D)} \leq C_{\alpha}|f|_{\Lambda(D)}$ for all $f$ with $\overline{\partial}f = 0$ and all $\alpha > 0$.
(ii) $\|Tf\|_{BMO(D)} \leq C \|f\|_{L^\infty(D)}$.

Now we study the uniform and $L^p$ estimates for solutions of the $\overline{\partial}$-problem in pseudoconvex domains which may be of infinite type.

Let $\Psi \in C^2([0,1])$ be a real valued function satisfying

(A) $\Psi(0) = 0$ and $\Psi(1) = 1$.
(B) $\Psi'(t) > 0$, $0 < t \leq 1$.
(C) $\Psi'(t) + t\Psi''(t) > 0$, $0 < t \leq 1$.
(D) There exists $\tau \in (0,1)$ such that $\Psi''(t) > 0$, $0 < t < \tau$.

Define

$$D_\Psi = \{z \in \mathbb{C}^n : |z_j| < 1, j = 1, \ldots, n, \sum_{j=1}^{n-1}|z_j|^2 + \Psi(|z_n|^2) < 1\}.$$

For $\alpha > 0$, define $\Psi_\alpha(t) = e^{-1/t^\alpha}$. Then $\Psi_\alpha$ satisfies all conditions (A)-(D). In this case the domain $D_\Psi$ is not of finite type.

Theorem 13. (Adachi-Cho[2]) Let $f \in L^p_{0,q}(D_\Psi)$, $1 \leq p \leq \infty$, be $\overline{\partial}$-closed. If $\int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds < \infty$, then there is a solution $u$ of $\overline{\partial}u = f$ on $D_\Psi$ such that $\|u\|_{L^p(D_\Psi)} \leq c(p) \|f\|_{L^p(D_\Psi)}$.

where the constant $c(p)$ is independent of $f$.

Remark. In case $n = 2$ and $p = \infty$, Theorem 13 was obtained by Verdera[28].

4. $\overline{\partial}$-problems in ellipsoids.

Define

$$D_1 = \{z \in \mathbb{C}^n : \sum_{i=1}^{n}|z_i|^{m_i} < 1\},$$

$$D_2 = \{z = (z_1, \ldots, z_n) : \sum_{i=1}^{n}(x_i^{l_i} + y_i^{m_i}) < 1, z_j = x_j + iy_j\},$$

where $m_i, l_i$ are positive even integers. We set

$$k_1 = \sup_{1 \leq i \leq n}\{m_i\}, \quad k_2 = \sup_{1 \leq i \leq n}\{\inf\{l_i, m_i\}\}.$$

We have the following:

$$\int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds < \infty$$
Theorem 14. (Range[20]) For each $\alpha < 1/k_1$, there exists a constant $C_\alpha$ such that for every bounded, $\partial$-closed $(0,1)$-form $f$ on $D_1$, there exists a $\alpha$-Hölder continuous function $u$ on $D_1$ such that $\partial u = f$ and $||u||_{\Lambda_\alpha(D_1)} \leq C_\alpha ||f||_{L^\infty(D_1)}$.

Theorem 15. (Diederich-Fornaess-Wiegerinck[5]) There exists a constant $C$ such that for every bounded, $\partial$-closed $(0,1)$-form $f$ on $D_2$, there exists a $\alpha = 1/k_2$-Hölder continuous function $u$ on $D_2$ such that $\partial u = f$ and $||u||_{\Lambda_\alpha(D_2)} \leq C ||f||_{L^\infty(D_2)}$.

Remark. Diederich, Fornaess and Wiegerinck pointed out in their paper that Theorem 14 is also true in case $\alpha = 1/k_1$.

Theorem 16. (Chen-Krantz-Ma[4]) Let $D_1$ be the complex ellipsoid defined above. Then for every $\partial$-closed $(0,1)$-form $f$ with coefficients in $L^p(D_1)$, there exists a function $u$ on $D_1$ such that $\partial u = f$, and $u$ satisfies the following estimates:

(i) if $p = 1$, then $\mu\{|u| > t\} \leq C\{||f||_{L^p(D_1)}^{1/2}\}^\lambda$ for all $t > 0$, where $\lambda = \frac{k_1n+2}{k_1n+1}$,
(ii) if $1 < p < k_1n+2$, then $||u||_{L^q(D_1)} \leq C_p ||f||_{L^p(D_1)}$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{k_1n+2}$,
(iii) if $p = k_1n+2$, then $||u||_{L^q(D_1)} \leq C_p ||f||_{L^p(D_1)}$ for all $q < \infty$,
(iv) if $p > k_1n+2$, then $||u||_{\Lambda_\alpha(D_1)} \leq C_p ||f||_{L^p(D_1)}$, where $\alpha = \frac{1}{k_1} - (n + \frac{2}{k_1})\frac{1}{p}$.

We give the results obtained by Fleron[8]. Ho[12] obtained similar results in the case where $D$ is a complex ellipsoid.

Theorem 17. (Fleron[8]) Let $1 \leq q \leq n - 1$. Let $D$ be a real or complex ellipsoid. Suppose that $\Delta q$ is the maximal order of contact of the boundary of the ellipsoid $D$ with $q$-dimensional complex linear spaces. Then there are linear operators $T_q : C_{(0,q)}(\overline{D}) \to C_{(0,q-1)}(D)$ satisfying the following:

(i) if $f \in C^1_{(0,q)}(\overline{D})$ and $\partial f = 0$, then $\partial(T_q f) = f$ on $D$,
(ii) there is a constant $c$ such that $|T_q f(z) - T_q f(z')| \leq c ||f||_{L^\infty(D)} |z - z'|^{\frac{1}{\Delta q}}$ for $z, z' \in D$.

Now we give the following optimal $L^p$ estimates for solutions of the $\partial$-problem in ellipsoids.

Theorem 18. (Adachi[1]) Let $m$ be the maximal order of contact of the boundary of the complex ellipsoid $D$ with $q$-dimensional complex linear subspaces. Let $p \geq 1$. Then for every $\partial$-closed $(0,q)$-form $f$ with coefficients in $L^p(D)$, there exists a function $u$ on $D$ such that $\partial u = f$, and $u$ satisfies the following estimates:

(i) if $p = 1$, then $\mu\{|u| > t\} \leq C\{||f||_{L^p(D_1)}^{1/2}\}^\lambda$ for all $t > 0$, where $\lambda = \frac{mn+2}{mn+1}$,
(ii) if $1 < p < mn+2$, then $||u||_{L^q(D)} \leq C_p ||f||_{L^p(D)}$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{mn+2}$,
(iii) if $p = mn+2$, then $||u||_{L^q(D)} \leq C_p ||f||_{L^p(D)}$ for all $s < \infty$,
(iv) if $p > mn+2$, then $||u||_{\Lambda_\alpha(D)} \leq C_p ||f||_{L^p(D)}$, where $\alpha = \frac{1}{m} - (n + \frac{2}{m})\frac{1}{p}$.
Theorem 19. (Adachi[1]) Let $m$ be the maximal order of contact of the boundary of the real ellipsoid $D$ with $q$-dimensional complex linear subspaces. Let $p \geq 1$. Then for every $\overline{\partial}$-closed $(0,q)$-form $f$ with coefficients in $L^p(D)$, there exists a function $u$ on $D$ such that $\overline{\partial}u = f$ and $u$ satisfies the following estimates:

(i) if $p = 1$, then $||u||_{L^{(1-p)}(D)} \leq c||f||_{L^1(D)}$, where $\gamma = \frac{mn+2}{mn+1}$ and $\varepsilon$ is any small number,

(ii) if $1 < p < mn+2$, then $||u||_{L^s(D)} \leq c||f||_{L^p(D)}$, where $s < q_0$ and $q_0$ satisfies $\frac{1}{q_0} = \frac{1}{p} - \frac{1}{mn+2}$,

(iii) if $p = mn + 2$, then $||u||_{L^s(D)} \leq C_p||f||_{L^p(D)}$ for all $s < \infty$,

(iv) if $p > mn + 2$, then $||u||_{\Lambda_\alpha(D)} \leq C_p||f||_{L^p(D)}$, where $\alpha = \frac{1}{m} - (n + \frac{2}{m})\frac{1}{p}$.

5. Applications of the $\overline{\partial}$-problem.

A. Uniform approximation of holomorphic functions.

Theorem 20. (Kerzman[13]) Let $D \Subset \mathbb{C}^n$ be a strongly pseudoconvex domain with smooth boundary. There exists an open set $\hat{D} \Subset \mathbb{C}^n, D \subset \overline{D} \Subset \hat{D}$, which has the following properties:

(a) Any continuous function $u : \overline{D} \rightarrow \mathbb{C}$ which is holomorphic in $D$ can be uniformly approximated on $\overline{D}$ by holomorphic functions $\hat{u}$ defined on $\hat{D}$.

(b) Let $u : D \rightarrow \mathbb{C}$ be holomorphic and assume $u \in L^p(D), 1 \leq p \leq \infty$. Then there exists a sequence of holomorphic functions $\hat{u}_n : \hat{D} \rightarrow \mathbb{C}$ such that (b1), (b2) and (b3) below hold:

(b1) $\hat{u}_n \rightarrow u$ uniformly on compact subsets of $D$ when $n \rightarrow \infty$,

(b2) $||\hat{u}_n||_{L^p(D)} \leq K||u||_{L^p(D)}, \quad 1 \leq p \leq \infty$,

(b3) if $p < \infty$, then $||\hat{u}_n - u||_{L^p(D)} \rightarrow 0$ when $n \rightarrow \infty$, where $K$ is independent of $n, p$ and $u$.

B. Vanishing cohomology theorems.

Theorem 21. (Kerzman[13], Lieb[16]) Let $D \Subset \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary. Let $\mathcal{F}$ and $\mathcal{B}$ be the sheaf of germs of holomorphic functions in $D$ which are continuous on $\overline{D}$ and the sheaf of germs of holomorphic functions in $D$ which are bounded in $D$, respectively. Then we have

$$H^q(\overline{D}, \mathcal{F}) = H^q(\overline{D}, \mathcal{B}) = 0 \quad \text{for} \quad q \geq 1.$$
Theorem 22. (Show[27]) Let $D$ be a real ellipsoid in $\mathbb{C}^n$. Given any analytic variety of complex dimension $(n-1)$ such that $M$ is the zero sets of an analytic function on $D$ of finite area, there exists a function $h$ belonging to the Nevanlinna class such that $M$ is the zero sets of $h$.

Theorem 23. (Arlebrick[3]) Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^2$ with $C^3$ boundary. If $X$ is a positive divisor of $D$ with finite area and the canonical cohomology class of $X$ in $H^2(D, \mathbb{Z})$ is zero, then there exists a bounded holomorphic function that defines $X$.

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